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Operator *o* and analysis of harmonic distortion

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***Abstract*—It has been shown that the description of mildly nonlinear circuits with the use of an operator *o* introduced by Meyer and Stephens in their paper published more than forty years ago was flawed. The problem now with their incorrect and imprecise definition is that it is still replicated in one or another form, as, for example, in publications of Palumbo and Pennisi on harmonic distortion calculation in integrated CMOS amplifiers or an article of Shrimali and Chatterjee on nonlinear distortion analysis of a three-terminal MOS-based parametric amplifier. Here, we discuss the versions of *o* operator presented in the works mentioned above and show points, where mistakes were committed. Also, we derive the correct forms of nonlinear circuit descriptions that should be used.**

***Keywords*—Operator o, descriptions of mildly nonlinear circuits in the frequency-domain, nonlinear distortion, Volterra series.**

I. Introduction

In a short conference paper [1], the author of this article pointed out faulty formulations of the so-called operator (operation) *o*. Here, this subject is continued referring to as the recent publications [2-6] and didactic materials for students published on a website [7], in which the above operator, in one or another form, is used. We do this because an incorrect and imprecise definition of the above operator, that was introduced by Meyer and Stephens in their paper [8] published more than forty years ago, is still replicated. In this paper, we revisit the definitions of *o* operator and their usage in the works mentioned above and show points, where mistakes were committed. Finally, we derive the correct forms of nonlinear circuit descriptions exploiting the Volterra series for different sets of circuit inputs (voltages, currents); this has been promised to do in [1].

The remainder of this paper is organized as follows. In the next section, we try first of all to understand the real meaning of an imprecise definition of the operator *o* presented in [2-5], [9]. Next, using the relationships existing between the Volterra series based methods of nonlinear analysis and the approach exploiting the balance of harmonics and phasors [2-5], [9], we show that the above definition is partly erroneous. We derive a correct expression defining the operator *o* needed in the latter method. In section II, we present also an useful interpretation of the expressions derived that let us better understanding of the assumptions underlying the meaning of analysis of weakly (mildly) nonlinear circuits. In the next section, we show that it not possible to replace the operator *o* by an ordinary multiplication [6] in a general formulation of the Volterra series containing the *o* operation [8]. This is allowable, as we show here, only in one specific case in which the input signal is a single harmonic. For this case, the form of the expansion presented in [6] is corrected accordingly. The paper ends with some concluding remarks.

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II. Imprecise Meaning of Operator *o* in Works of Palumbo and Coworkers

In [3-5], [9], the definition of an operator *o* has been formulated as follows: "Let

$$x(t) = X_1 \exp(j2\pi f_s t) + X_2 \exp(j2\pi 2 f_s t) + \\ + X_3 \exp(j2\pi 3 f_s t) \tag{1}$$

be a complex valued signal consisting of three harmonics: the fundamental of frequency f_s, the second, and third one that is applied to a weakly nonlinear circuit. In (1), X_1, X_2, and X_3, mean generally complex amplitudes of the above harmonics. Further, assume that a weakly nonlinear circuit has a strictly transferring character. That is it can be fully described by some input-output type relations.

Then, the signal at circuit output will be expressed (exactly or approximately) by

$$y(t) = x(t) o \Big[a_1(f_s) + a_2(f_s) x(t) + \\ + a_3(f_s) \big(x(t)\big)^2 \Big], \tag{2}$$

where the operator "*o*" means that the functions which appear within the square brackets must be evaluated at the frequency of the incoming signal. This operator must be used whenever we evaluate the output of a nonlinear block."

The coefficients $a_1(f_s)$, $a_2(f_s)$, and $a_3(f_s)$ occurring in (2) were named in [3] "the nonlinearity coefficients", but in [5] "the first (linear), second-, and third-order nonlinearity transfer functions", respectively.

Observe that the above definition is not mathematically clear and highly imprecise. So, its application in the analysis of weakly nonlinear circuits can lead to errors. One example of such an error has been presented in [10].

Before proceeding further with the above definition, compare it however first with the definition of an operator *o* presented by Meyer and Stephens in [8]. Referring to Narayanan [11], Meyer and Stephens claim therein that there exists a mixed (time-frequency) form of the Volterra series representation. Using it, we can relate, after them, the output signal $y(t)$ of a mildly nonlinear circuit with its input signal $x(t)$ in the following way

$$y(t) = A_1(f) o x(t) + A_2(f_1, f_2) o \big(x(t)\big)^2 + \\ + A_3(f_1, f_2, f_3) o \big(x(t)\big)^3 + \ldots, \tag{3}$$

where $A_1(f)$, $A_2(f_1, f_2)$, and $A_3(f_1, f_2, f_3)$ mean the nonlinear transfer functions of the circuit considered of the first-, second-, and third-order, respectively; they are called the Volterra coefficients in [8]. Obviously, these transfer functions are the one-, two-, and three-dimensional Fourier transforms of the corresponding nonlinear circuit impulse responses of the first-, second-, and third-order [12], accordingly. About the operator *o*, Meyer and Stephens say in [8] that "the operator sign indicates that the magnitude and phase of each term in $(x(t))^n$, $n = 1, 2, 3, \ldots$, is to be changed by the magnitude and phase of $A_n(f_1, f_2, \ldots, f_n)$".

What are the similarities between the representations given by (2) and (3), and the operator *o* used in them ? First, they are unclear and imprecise. Second, the form of expressions (2) and (3) is similar, resembling a third degree polynomial of a variable *x*. Third, they represent a mixed (time-frequency) descriptions. Fourth, they try to express the magnitude and phase changes in the circuit output signal due to its nonlinear behavior.

Now, what are the differences between them ? First, (2) and (3) represent models with different input signal sets. Namely, (2) is valid only for signals of the form given by (1). In contrast to this, (3) is claimed to be more general, valid for any signals. Second, the symbol *o* used in both (2) and (3) does not mean the same. Concerning (2), it is impossible to define the operator *o* mathematically, relying upon its descriptive definition given in [3-5]. But, the situation seems to be better in the case of Meyer and Stephens definition because, as shown in [1], their *o* operator can be identified with the convolution operation. However, it has slightly different meanings in the consecutive components on the right-hand side of (3). That is it means successively the one-, two-, and three-dimensional convolution integrals, for more details, see [1].

Now, we come back to the discussion of the description given by (2). To start, we recall a result from [13] that the coefficients $a_1(f_s)$, $a_2(f_s)$, and $a_3(f_s)$ occurring in (2) can be expressed by the Volterra series based nonlinear transfer functions [12] of a circuit considered. Then, the following equalities: $a_1(f_s) = A^{(1)}(f_s)$, $a_2(f_s) = A^{(2)}(f_s, f_s)$, and $a_3(f_s) = A^{(3)}(f_s, f_s, f_s)$ hold. In these identities, $A^{(1)}(f_s)$, $A^{(2)}(f_s, f_s)$, and $A^{(3)}(f_s, f_s, f_s)$ mean the circuit nonlinear transfer functions of the corresponding orders as defined beneath (3), in which successively the following substitutions of arguments: $f = f_s$, $f_1 = f_2 = f_s$, and $f_1 = f_2 = f_3 = f_s$ have been carried out.

It follows from the above that the representation given by (2) can be alternatively written in the terminology of the Volterra series as

$$y(t) = x(t)o\left[A_1(f_s) + A_2(f_s, f_s)x(t) + \right.$$
$$\left. + A_3(f_s, f_s, f_s)(x(t))^2\right]. \tag{4}$$

This suggests to check using the Volterra series whether the above relation is really correct or not. And to this end, we will describe a weakly nonlinear circuit by a Volterra series in an operator form [14] and restrict ourselves to the first three components in it. That is we will use the following

$$\begin{aligned} y(t) &= \left((A_1) + (A_2)(\cdot)^2 + (A_3)(\cdot)^3\right)(x)(t) = \\ &= (A_1)(x)(t) + (A_2)((x)(t))^2 + \\ &+ (A_3)((x)(t))^3 = \int_{-\infty}^{\infty} a_1(\tau)x(t-\tau)d\tau + \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 + \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_3(\tau_1,\tau_2,\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)d\tau_1 d\tau_2 d\tau_3 , \end{aligned} \tag{5}$$

where the functions $a_1(\tau)$, $a_2(\tau_1, \tau_2)$, and $a_3(\tau_1, \tau_2, \tau_3)$ of the corresponding time variables are the nonlinear circuit impulse responses of the first-, second-, and third-order [12], respectively. So, they have nothing to do with the coefficients $a_1(f_s)$, $a_2(f_s)$, and $a_3(f_s)$ in (2), which are functions of a frequency variable. The former are of course related with the circuit nonlinear transfer functions $A_1(f)$, $A_2(f_1, f_2)$, and $A_3(f_1, f_2, f_3)$ via the multidimensional Fourier transforms. Further, the definitions of the Volterra operators A_1, A_2, and A_3 of the first-, second-, and third-order, respectively, follow directly from (5) as the corresponding multidimensional convolutions. Finally, note that we use here the same names for the Volterra operators as well as for the nonlinear transfer functions defined before; this will however cause no confusion.

Substituting (1) into (5) gives

$$\begin{aligned} y(t) &= (A_1)\big(X_1 \exp(j2\pi f_s t) + \\ &+ X_2 \exp(j2\pi 2 f_s t) + X_3 \exp(j2\pi 3 f_s t)\big) \\ &+ (A_2)\big(X_1 \exp(j2\pi f_s t) + \\ &+ X_2 \exp(j2\pi 2 f_s t) + X_3 \exp(j2\pi 3 f_s t)\big)^2 + \\ &+ (A_3)\big(X_1 \exp(j2\pi f_s t) + \\ &+ X_2 \exp(j2\pi 2 f_s t) + X_3 \exp(j2\pi 3 f_s t)\big)^3 . \end{aligned} \tag{6}$$

Using relationships existing between components of the Volterra series expressed in the time domain and, on the other hand, in the frequency multidimensional domains (which were published, for example, in [12] or [14]), the following generic results

$$(A_1)\left(X_x \exp\left(j2\pi f_x t\right)\right) = A_1\left(f_x\right) X_x \exp\left(j2\pi f_x t\right) \tag{7a}$$

$$\begin{aligned}&(A_2)\left(X_x X_y \exp\left(j2\pi f_x t\right)\exp\left(j2\pi f_y t\right)\right) = \\ &= A_2\left(f_x, f_y\right) X_x X_y \exp\left(j2\pi f_x t\right)\exp\left(j2\pi f_y t\right) = \\ &= A_2\left(f_x, f_y\right) X_x X_y \exp\left(j2\pi\left(f_x + f_y\right)t\right)\end{aligned} \tag{7b}$$

$$\begin{aligned}&(A_3)\left(X_x X_y X_z \exp\left(j2\pi f_x t\right)\exp\left(j2\pi f_y t\right)\cdot\right. \\ &\left.\cdot\exp\left(j2\pi f_z t\right)\right) = A_3\left(f_x, f_y, f_z\right) X_x X_y X_z \cdot \\ &\cdot\exp\left(j2\pi f_x t\right)\exp\left(j2\pi f_y t\right)\exp\left(j2\pi f_z t\right) = \\ &= A_3\left(f_x, f_y, f_z\right) X_x X_y X_z \exp\left(j2\pi(f_x + f_y + f_z)t\right)\end{aligned} \tag{7c}$$

can be easily derived. In (7), multiplications of the single tone signals with the amplitudes X_x, X_y, X_z and the corresponding frequencies f_x, f_y, f_z occur.

In the next step, we carry out all the multiplications indicated by the quadratic $(\cdot)^2$ and cubic $(\cdot)^3$ terms in (6). Then, we choose appropriate relations from those given in (7) and apply to the components in (6). As a result, we get

$$\begin{aligned}&y(t) = A_1\left(f_s\right) X_1 \exp\left(j2\pi f_s t\right) + A_1\left(2f_s\right)\cdot \\ &\cdot X_2 \exp\left(j2\pi 2 f_s t\right) + A_1\left(3f_s\right) X_3 \exp\left(j2\pi 3 f_s t\right) + \\ &+ A_2\left(f_s, f_s\right) X_1 X_1 \exp\left(j2\pi f_s t\right)\cdot \\ &\cdot\exp\left(j2\pi f_s t\right) + A_2\left(f_s, 2f_s\right) X_1 X_2 \exp\left(j2\pi f_s t\right)\cdot \\ &\cdot\exp\left(j2\pi 2 f_s t\right) + A_2\left(2f_s, f_s\right) X_2 X_1 \exp\left(j2\pi 2 f_s t\right)\cdot \\ &\cdot\exp\left(j2\pi f_s t\right) + \\ &+\text{components containing the product frequencies} \\ &\text{greater than } 3f_s + \\ &+ A_3\left(f_s, f_s, f_s\right) X_1 X_1 X_1 \exp\left(j2\pi f_s t\right)\exp\left(j2\pi f_s t\right)\cdot \\ &\cdot\exp\left(j2\pi f_s t\right) + \\ &+\text{components containing the product frequencies} \\ &\text{greater than } 3f_s\ .\end{aligned} \tag{8}$$

Consider now again relation (4) and substitute $x(t)$ given by (1) in it. In the next step, carry out the operations indicated by the operator *o* occurring in (4) according to its definition given beneath (2). As a result, we get then

$$\begin{aligned}&y(t) = X_1 \exp\left(j2\pi f_s t\right) A_1\left(f_s\right) + X_2 \exp\left(j2\pi 2 f_s t\right)\cdot \\ &\cdot A_1\left(2f_s\right) + X_3 \exp\left(j2\pi 3 f_s t\right) A_1\left(3f_s\right) + \\ &+ X_1 \exp\left(j2\pi f_s t\right) A_2\left(f_s, f_s\right) X_1 \exp\left(j2\pi f_s t\right) + \\ &\quad + X_1 \exp\left(j2\pi f_s t\right) A_2\left(f_s, f_s\right)\cdot \\ &\quad\cdot X_2 \exp\left(j2\pi 2 f_s t\right) + X_2 \exp\left(j2\pi 2 f_s t\right)\cdot \\ &\quad\cdot A_2\left(2f_s, 2f_s\right) X_1 \exp\left(j2\pi f_s t\right) + \\ &+\text{components containing the product frequencies} \\ &\text{greater than } 3f_s + \\ &+ X_1 \exp\left(j2\pi f_s t\right) A_3\left(f_s, f_s, f_s\right) X_1 X_1 \exp\left(j2\pi f_s t\right)\cdot \\ &\cdot\exp\left(j2\pi f_s t\right) + \\ &+\text{components containing the product frequencies} \\ &\text{greater than } 3f_s\ .\end{aligned} \tag{9}$$

Comparison of (8) and (9) shows that these expressions are not identical. The fifth and sixth components in these expressions differ from each other. That is

$$\begin{aligned}&A_2\left(f_s, 2f_s\right) X_1 X_2 \exp\left(j2\pi f_s t\right)\exp\left(j2\pi 2 f_s t\right) \neq \\ &\neq X_1 \exp\left(j2\pi f_s t\right) A_2\left(f_s, f_s\right) X_2 \exp\left(j2\pi 2 f_s t\right)\end{aligned} \tag{10a}$$

and

$$\begin{aligned}&A_2\left(2f_s, f_s\right) X_2 X_1 \exp\left(j2\pi 2 f_s t\right)\exp\left(j2\pi f_s t\right) \neq \\ &\neq X_2 \exp\left(j2\pi 2 f_s t\right) A_2\left(2f_s, 2f_s\right) X_1 \exp\left(j2\pi f_s t\right)\end{aligned} \tag{10b}$$

because, generally,

$$\begin{cases} A_2\left(f_s, 2f_s\right) \neq A_2\left(f_s, f_s\right) & \text{and} \\ A_2\left(2f_s, f_s\right) \neq A_2\left(2f_s, 2f_s\right)\ , \end{cases} \tag{11}$$

respectively. Obviously, differences of similar kind will also occur between some corresponding terms in the corresponding "components containing the product frequencies greater than $3f_s$" denoted in (8) and (9). These components are not, however, analyzed here because they were omitted in the papers [2-5].

From the above comparison, we draw the conclusion that the expression (4) is erroneous, and therefore also (2). So, we conclude further that the operator *o* is not defined correctly. Moreover, it follows from the above derivations that in any approach using the Volterra series, this operator is superfluous. As we saw just before, a correct formula is that given by (8). Finally, observe also that (8) reduces to

$$\begin{aligned} y(t) \cong\, & A_1(f_s)X_1\exp(j2\pi f_st) + A_1(2f_s)\cdot \\ & \cdot X_2\exp(j2\pi 2f_st) + A_1(3f_s)X_3\exp(j2\pi 3f_st) + \\ & + A_2(f_s,f_s)X_1X_1\exp(j2\pi f_st)\exp(j2\pi f_st) + \\ & + A_2(f_s,2f_s)X_1X_2\exp(j2\pi f_st)\exp(j2\pi 2f_st) + \\ & + A_2(2f_s,f_s)X_2X_1\exp(j2\pi 2f_st)\exp(j2\pi f_st) + \\ & + A_3(f_s,f_s,f_s)X_1X_1X_1\exp(j2\pi f_st)\cdot \\ & \cdot\exp(j2\pi f_st)\exp(j2\pi f_st), \end{aligned} \tag{12}$$

when one restricts himself to consideration of only product frequencies not greater than $3f_s$.

From (12), we see that the circuit output signal is approximately also a complex-valued one consisting of three harmonics: the fundamental of frequency f_s, the second, and third one. Therefore, it can be expressed similarly as $x(t)$ by (1). That is as

$$\begin{aligned} y(t) = & Y_1\exp(j2\pi f_st) + Y_2\exp(j2\pi 2f_st) + \\ & + Y_3\exp(j2\pi 3f_st) \end{aligned}, \tag{13}$$

where Y_1, Y_2, and Y_3, mean generally complex amplitudes of the above harmonics.

Note that by re-grouping the terms in (12) with respect to the product frequencies (frequencies of harmonics) we arrive at

$$\begin{aligned} y(t) \cong\, & A_1(f_s)X_1\exp(j2\pi f_st) + \\ & + \left[A_1(2f_s)X_2 + A_2(f_s,f_s)X_1X_1\right]\exp(j2\pi 2f_st) \\ & + \left[A_1(3f_s)X_3 + A_2(f_s,2f_s)X_1X_2 + A_2(2f_s,f_s)\cdot\right. \\ & \left.\cdot X_2X_1 + A_3(f_s,f_s,f_s)X_1X_1X_1\right]\exp(j2\pi 3f_st) \quad . \end{aligned} \tag{14}$$

Next, comparison of (13) with (14) allows us to write

$$Y_1 \cong A_1(f_s)X_1 , \tag{15a}$$

$$Y_2 \cong A_1(2f_s)X_2 + A_2(f_s,f_s)X_1X_1 , \tag{15b}$$

and

$$\begin{aligned} Y_3 \cong\, & A_1(3f_s)X_3 + A_2(f_s,2f_s)X_1X_2 + \\ & + A_2(2f_s,f_s)X_2X_1 + A_3(f_s,f_s,f_s)X_1X_1X_1 \end{aligned} . \tag{15c}$$

The above formulae express the magnitude and phase changes of harmonics of the circuit output signal due to the harmonics of its input signal.

The formulae (15) are also useful for interpretation purposes. With their use, we will show now how “the transfer of harmonics takes place” in an analysis of mildly (weakly) nonlinear circuits that assumes:

A. all the nonlinearities occurring in a circuit are sufficiently well described by the Volterra series or the Taylor expansions restricted to the first three terms,

B. when the harmonics of higher order than the third one arise in circuit analysis, they are neglected.

To proceed, take into account a weakly nonlinear circuit of the above type that consists of nonlinear elements connected the one to the other in some way. Further, let these elements be of input-output type. That is their descriptions will be of this type. Furthermore, let none of their inputs be the input of the whole circuit to which a single harmonic signal is applied. So, all the fundamental, second, and third harmonics will appear at the inputs and outputs of these circuit elements. And their complex amplitudes will be related with each other by the formulae (15). In particular, see from (15a) that the fundamental harmonic will not be, approximately, affected by the circuit nonlinearity. Its amplitude will solely follow the linear relation. In contrast to this, the second and third harmonics will be affected by the circuit nonlinearity. The linear relation for the second harmonic will be affected by an additive term $A_2(f_s,f_s)X_1X_1$. In other words, we can say here that the term $A_1(2f_s)X_2$ in (15b) follows from “transferring the amplitude X_2” to the circuit element output due to the linear transfer function $A_1(2f_s)$. But, the next one, i.e. $A_2(f_s,f_s)X_1X_1$, is this circuit element own contribution. Similarly, the linear relation for the third harmonic will be influenced by an additive term that has the following form: $\left[A_2(f_s,2f_s) + A_2(2f_s,f_s)\right]X_1X_2 + A_3(f_s,f_s,f_s)X_1X_1X_1$. It can be viewed as the circuit element own contribution to its third harmonic amplitude at its output. Further, $A_1(3f_s)X_3$ is that part which follows from “transferring” the third harmonic through this element to its output due to the linear transfer function $A_1(3f_s)$.

It is worth noting also in the above context that the formulas (15) reduce to

$$Y_{1i} \cong A_1(f_s)X_{1i} , \tag{16a}$$

$$Y_{2i} \cong A_2(f_s,f_s)X_{1i}X_{1i} , \tag{16b}$$

and

$$Y_{3i} \cong A_3(f_s,f_s,f_s)X_{1i}X_{1i}X_{1i} \tag{16c}$$

for a nonlinear circuit element of which input is identical with the input of the whole circuit. This is so because we have to substitute then $X_2 \equiv 0$ and $X_3 \equiv 0$ in (15). Moreover, note that to distinguish between the two cases mentioned above we added an additional index i by Y_{ni}, $n = 1,2,3$, and X_{1i} in (16).

Concluding, we see that the interpretation of relations (15) given above presents an useful view for getting a better understanding of what is happening with the harmonics inside of a circuit under the aforementioned assumptions **A** and **B**. This is important to know because these assumptions in fact define a class of nonlinear circuits that are called weakly or mildly nonlinear circuits, independently of a mathematical tool used for their analysis, which can be a Volterra series [11-12], a perturbation method [15] or a mixed one using balance of harmonics and phasors [2-5], [9].

III. Operator *o* Becomes Ordinary Multiplication in Work of Shrimali and Chatterjee

Shrimali and Chatterjee in their paper [6] refer to as the Volterra series formulation presented in [8]. That is to that given by (3). However, their version of the Volterra series assumes a slightly different form given by

$$y(t) = A_1(f)\cdot x(t) + A_2(f_1, f_2)\cdot (x(t))^2 + \\ + A_3(f_1, f_2, f_3)\cdot (x(t))^3 + \dots \quad . \tag{17}$$

Comparison of (3) with (17) shows that an ordinary multiplication appears now in the latter instead of an operator *o*. Obviously, the above substitution is erroneous. As shown in [1], the correct form of (3) is the following

$$y(t) = F_1^{-1}\{A_1(f)F_{1f}(x(t))\} + \\ + F_2^{-1}\{A_2(f_1, f_2)F_{1f_1}(x(t))F_{1f_2}(x(t))\} + \\ + F_3^{-1}\{A_3(f_1, f_2, f_3)F_{1f_1}(x(t))\cdot \\ \cdot F_{1f_2}(x(t))F_{1f_3}(x(t))\} + \dots \quad . \tag{18}$$

And the same regards also (17). In (18), $F_i^{-1}\{\cdot\}$, $i = 1, 2, 3, \dots$, means the inverse i-dimensional Fourier transform. Moreover, $F_{1f_z}\{\cdot\}$ stands for the one-dimensional Fourier transform, in which the frequency variable is denoted as f_z, $z = 1, 2, 3, \dots$.

There exists however one particular signal for which the Volterra series assumes the mixed time-frequency form, which resembles that of (17). This is the single harmonic signal

$$x_s(t) = X_s \exp(j2\pi f_s t) \tag{19}$$

denoted here as $x_s(t)$, in which X_s stands for its (generally) complex-valued amplitude, but f_s is its frequency.

Substituting (19) into the right-hand side expansion in (5), which is the Volterra series, and after some manipulations and applying the definitions of the i-dimensional Fourier transforms, we arrive finally at

$$y(t) = X_s \exp(j2\pi f_s t) A_1(f_s) + \\ + X_s \exp(j2\pi f_s t) X_s \exp(j2\pi f_s t) A_2(f_s, f_s) + \\ + X_s \exp(j2\pi f_s t) X_s \exp(j2\pi f_s t) X_s \exp(j2\pi f_s t)\cdot \\ \cdot A_3(f_s, f_s, f_s) + \dots \quad . \tag{20}$$

Next, rearranging the terms in (20) and knowing that $x_s(t)$ is given by (19), we can rewrite (20) as

$$y(t) = A_1(f_s)x_s(t) + A_2(f_s, f_s)(x_s(t))^2 + \\ + A_3(f_s, f_s, f_s)(x_s(t))^3 + \dots \quad . \tag{21}$$

We stress that the description given by (21) is valid for only one class of input signals that are the single harmonic signals (one tone signals). But in no case, it can be used for other signals.

IV. Concluding Remarks

The material presented in this paper about the definitions of the operator *o* shows that the errors are replicated, when the mathematics used is imprecise. Moreover, we stress once again that the class of input signals for which a given circuit description is valid is its inherent part. In this context, the representation for weakly nonlinear circuits given by (21) is valid only for single harmonic signals.

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