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Network winner determination problem

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Many real-world marketplaces involve some additional constraints to be addressed during the market clearing process. This is the case of various infrastructure sectors of the economy, where market commodities are associated with some elements of the infrastructure, e.g., elements of telecommunication, power transmission or transportation network. Transactions are allowed only if the infrastructure, modeled as a flow network, is able to serve them. Determination of the best offers is possible by solving the optimization problem, so called the Winner Determination Problem (WDP). We consider a new subclass of the WDP, i.e., the Network Winner Determination Problem (NWDP). We characterize different problems in the NWDP class and analyze their computational complexity. The sharp edge of tractability for NWDP-derived problems is generally designated by integer offers. However, we show that some specific settings of the problem can still be solved in polytime. We also present some exemplary applications of NWDP in telecommunication bandwidth market and electrical energy balancing market.

Key words: network auctions, network winner determination problem (NWDP), complexity of the NWDP, MILP models, multi-commodity flow optimization, graph models

1. Introduction

The Winner Determination Problem (WDP) must be addressed in any market mechanism design as a part of market clearing procedure. A process of market clearing must be supported by a decision support system in which optimization plays a key role. A wide stream of the literature has focused on combinatorial auctions and derived WDPs. In this case, the WDP is to find an allocation of items (commodities) that a seller has, to the bids (offers) that buyers submit[1]. Common belief is that the WDP is a hard, computational complex problem. However, there are some specific cases which result in tractable WDP. Two aspects of practical importance are significant for its tractability. The first aspect,

[1] An ‘Item’ is commonly used in the literature devoted to combinatorial auctions. Since we also consider divisible items, we use the term ‘commodity’ alternatively. Also, being conscious of the meaning differences on various markets, we use the term ‘offer’ as a synonym to ‘bid’ in this paper.

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which has not gained much interest in the literature so far, is a divisibility of commodities and consequently the offers' continuity. Energy, water, telecommunication bandwidth are examples of a variety of commodities that can be traded as divisible objects. The second aspect relates to additional conditions that must be taken into account in many real-world marketplaces. Market participants' preferences, technological production processes or limited transportation capabilities are a potential source of such constraints. There is a stream of researches that focus on a maximization of economical value under constraints expressed by the bidders, e.g., substitutability of commodities [1–3]. However, in the last decades many infrastructure sectors of the economy have been transformed or have started transformation into the open markets. On the infrastructure market, the traded commodities must be transported from the sellers to the buyers with the use of some infrastructure. Therefore, transactions are allowed only, if the infrastructure is able to serve them, which limits the free trade on the market. A natural way to incorporate infrastructure constraints into the WDP is to apply a graph model, i.e., a flow network that models real resources and commodities flow [4].

The WDP under flow network may be considered on the grounds of classical combinatorial auctions. Special structures related to paths in a network can be achieved by standard bidding language, i.e., XOR operator [5]. A bidder may submit each offer for each possible bundle of commodities that create a path, and then combine these offers with XOR operator. However, due to the necessity of enumeration all possible paths, the combinatorial auction with XOR bidding language is not a satisfactory tool in practice. To illustrate the problem, let us consider the network presented in Figure 1. A seller offers links c_1 to c_8 and a buyer is interesting in buying any path starting in node s and ending in t. When combinatorial auction is applied to find allocation, then the buyer is forced to enumerate all possible paths and each path is a composition (bundle) of links. All possible paths must be arranged in one offer by combing the paths with XOR operator, e.g., $\{c1,c3,c7\} \oplus \{c1,c4,c8\} \oplus \{c2,c5,c4,c8\} \oplus \{c2,c6,c8\}$, where $\oplus$ is the logical exclusive OR operator. Moreover, in some cases a buyer could not be even informed about the structure of flow network or may not be interested in analysis of all possible paths. Therefore, the combinatorial auction-based approach cannot be regarded as simple and succinct. A complete graph with $|V|$ vertices is the worst case that involves enumeration of $(|V|-2)!$ paths. What a buyer needs is just a connection between s and t.

Since combinatorial auctions based on XOR bidding languages are inefficient in terms of expressing the preferences, we define the *network auction* as an allocation and pricing mechanism that enables bidding the commodity bundles isomorphic to some graph objects: vertices, edges, paths, trees, etc. In the network auction, each buy or sell offer corresponds to some element or group of elements in a given oriented graph $G = (V,E)$, where V is a set of vertices and E is a set of edges. A specific correspondence between commodities, of-

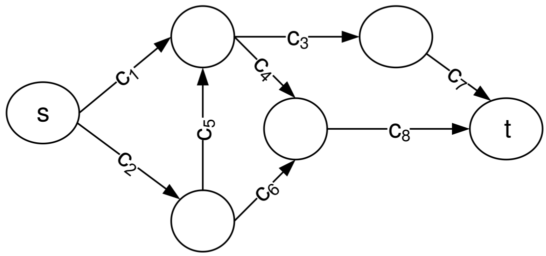

Figure 1: Example of flow network in case of auction for path from *s* to *t*

fers and graph elements is characterized by well defined network auction setting. For each network auction setting, a special variant of the WDP must be solved. This creates a subclass of WDPs, denoted as the Network Winner Determination Problem (NWDP), which covers all problems for every specific network auction setting. For instance, regarding the example we introduced earlier in the paper, we propose a very simple solution that only involves delivery of source and destination nodes of required path instead of paths enumeration needed in combinatorial auction. Having these nodes, the task is to find a flow in the network from the source to the destination that is revenue-maximizing. This exemplary problem can be modeled as a special case of the NWDP, referred to as Arc-oriented Network Winner Determination Problem further in this paper.

While the transparency and fairness are of much importance in auctions, finding the exact solution of the WDP is crucial. Unfortunately, many markets with side constraints appear to be intractable [3]. We have outlined the concept of the NWDP for the first time in [6], where the security constrained versions are considered. We considered an impact of side constraints related to security issues on numerical complexity of the problem. Comparing to [6], in this paper we analyze the NWDP class more formally in node- and arc-oriented cases and we provide deeper discussion of tractability of the basic formulation and some specific cases. Our contribution is in formulation and variant analysis of optimization problem that is the most important part of decision support system for market clearing processes. We analyze the complexity of the NWDP in several cases. First, we consider binary and continuous versions of the problem for single-item (the seller has one item of each commodity to sell). Both cases, node- and arc-oriented, are considered in capacitated and uncapacitated versions. Then, we analyze multi-item problems. Again, we consider several well-defined versions of the multi-item problem. Even though the problem is defined over the flow network, there is a sharp edge between tractability and intractability determined by the integrality of offers. However, the positive result we show in the paper is that there are some special cases in terms of offers structures, that are tractable in case of integer offers. Finally, we show some exemplary applications which involve solving the network WDP-like problems.

2. Related literature

The Winner Determination Problem is mostly recognized in the context of combinatorial auctions. There is a wide surge of research devoted to complexity of the WDP. Classical formulation of the WDP assumes only one instance of each traded item and consequently, it is equivalent to the weighted set packing problem and can also be transformed into the weighted stable set problem or the maximum weighted clique problem [7]. Moreover, general formulation of the WDP is also inapproximable [8]. Approximation algorithms yield results far from optimal solutions and they also have negative effects on other properties of auctions, e.g., strategy-proofness. Different algorithms for solving the classical WDP are discussed in [5].

It is also worth noting that explicite specification of valuations for all subsets of items is inconvenient and impractical. Several bidding languages have been developed to express valuations more succinctly and in more convenient way. OR and XOR bidding languages are the most important and often referred to in the context of WDP complexity. For more information on bidding languages in combinatorial auctions see [9] and for divisible commodities see [10].

Due to NP-hardness of the WDP, a stream of researches is aiming at reformulation of the problem to make the modified WDP problem easier to solve, e.g., by imposing some restrictions or by relaxing selected constrains. There are two main approaches to restrictive redefinition of the WDP: restrictions on the bundles on which bids can be submitted, or restrictions on the bid values. Müller has shown that, if there is a complete ordering on the items and the bidders bid only on consecutive items according to this ordering, then the coefficients matrix is totally unimodular [11]. The WDP can also be transformed to the maximum-weight matching problem and solved in $O(m^3)$ if each bid has at most two items [7]. Another structure of the WDP that guarantees its tractability is related to perfectness property of intersection graph related to subset of items allowed for submission [11]. Interval bids are an example of assumption under which the WDP can be modeled as perfect intersection graph. They assume that items can be ordered in a way that each bid relates only to consecutive items. Rothkopf et al. considered nested bids which is a special case of interval bids [7]. Vires and Vohra have proved that the WDP with OR bidding language, denoted by WDP_{OR}, is tractable for bids on neighbor items on a circle (circular-arc graph) [12]. Dynamic programming algorithm was also proposed to solve the WDP when bid sets are restricted to a subtree of tree, whose nodes are identical to items [11]. Lehman et al. have proved that even in case of restrictive constraints, i.e., every bid with value equal to 1 or every bidder submitting only one bid or every item contained in exactly two bids, the WDP remains NP-hard [8]. On the other hand, they mentioned two special cases that are easy to solve: a) homogenous items with decreasing marginal valuations (by simple sorting the items), b) unit demand valuations (by transformation to assignment problem). Contizer et al. showed that the WDP is solvable in polynomial time, if interactions among bids can be represented by the structured item graph [13]. Gottlob and Greco showed that WDP is tractable if bidder interactions can be represented with hypergraphs having bounded hypertree width [14].

Restrictions on accepted bid structure can also be introduced to the problem as side constraints. Sandholm and Suri have discussed the impact of different side constraints on complexity of the WDP [15]. Some of the constraints bring negative result and the WDP remains NP-complete, for instance, the maximum number of winners or constraints imposed by XOR-of-OR bidding language. Unfortunately, most of side constraints do not alleviate the complexity of WDP [16, 17].

Another way for transformation the WDP into tractable problem is to introduce some restrictions on bidders' valuations [11]. Linear relaxation of the WDP has an integer optimal solution if for each bidder, the valuation function satisfies the substitutes condition. Another example of tractable case, considered by Müller, is additive bid values with convex discounts [11]. Under this assumption, the problem can be transformed into the min-cost flow problem.

Instead of restrictions one can also consider relaxation of the original WDP which can also be intrinsic to the problem in some cases. The WDP_{OR} becomes polynomial if bids can be accepted partially. However, when some of side constraints are included, the problem can become NP-complete again.

Tennenoltz introduced another technique for identification the auction settings that can be solved in polynomial time [18]. He used different reductions to b-matching problem to prove tractability of several variants of the WDP. He proved that additive quantity-constrained multi-item auction and sub-additive quantity-constrained multi-item auction with bundles of size two, are tractable. He also introduced other settings that are tractable: almost-additive auctions, subadditive symmetric bids, super-additive symmetric bids for triplets. Kothari et al. showed that in case of exchange of k different items, the related WDP can be solved in polynomial time if k bids can be accepted partially and valuation functions are superadditive (e.g., bids on nonexclusive combinations of items) [19]. However, in case of subadditive valuations (e.g., bids on exclusive combinations of items), the WDP remains NP-complete, even when all bids can be accepted partially.

Loker and Larson considered complexity of the WDP assuming that sizes of some of the input parameters are fixed, which is recognized as parameterized complexity in the literature [20]. Finding a subset of parameters of the WDP, which would result in fixed-parameter tractability, is similar to the attempts of restriction imposed on the WDP. Loker and Larson showed that parametrization of total number of bids leads to fixed-parameter tractability, however parametrization the number of bids per agent does not. The parametrization of so called bid graph class corresponds to the results of Rothkopf [7]. Loker and Larson proved that the WDP is fixed-parameter tractable if bid graph is an interval graph, but it is fixed-parameter intractable when bid graph is chordal graph. They also considered a class of bid graphs built of disconnected components. When each of component has limited number of vertices, then the WDP becomes fixed-parameter tractable. This assumption is equivalent to the division of items into several groups and forbidding the bids for bundles mixing items from different groups.

Another dimension of WDP's complexity is property of free disposal of items, which is considered in [21]. In this paper, the authors showed that without free disposal, even finding a feasible solution is NP-complete for a combinatorial auction, reverse combinatorial auction or combinatorial exchange. One of the most interesting results of this paper is finding that reverse combinatorial auction with free disposal can be approximated while combinatorial auction does not.

We need to mention that auctions of items related to flow networks have been considered in a few papers, especially in the context of communication network. Typical assumption in these works is that links in a communication network are equivalent to items, and a buyer wants to purchase a set of links creating a path between two given nodes or forming a spanning tree. While the underlying problems, i.e., shortest path or minimal spanning tree, are tractable, corresponding WDP is also tractable (see, e.g., [22]). For instance, Hershberger and Suri considered auction of shortest path between two points [23]. They proved that even the need of sensitive analysis of WDP, which is required in case of Vickrey-Clarke-Groves mechanism, does not change the complexity of the WDP. A combinatorial network auction is considered by Tennenoltz in [18]. He considered a tree of items in which set of nodes is isomorphic to the set of items and any bid can be submitted only for a bundle that creates a path in the tree. Resulting WDP is proved to be tractable. It is worth noting that this problem formulation is simplified in comparison to formulation of networked auction we provide further in the paper, since it does not allow for submission more than one exclusive paths.

Nisan [9] has mentioned *network valuations* as a special case of valuations in discussion of bidding languages. He provided a short example of selling edges and buying paths. However, there is no formal definitions of the problem and no discussion concerning complexity of this case. We discuss the complexity of the NWDP in the context of security related constraints in [6]. We showed that so called MAX-WINNER constraint on maximum number of winning bidders in certain subset of bidders makes the NWDP problem NP-hard. However, constraints on min/max volume traded in total or per bidder do not affect the computational complexity. In this paper we systemize and provide a formal analysis of the Network Winner Determination Problem.

3. Network winner determination problem

3.1. Problem definition

In the literature, the Winner Determination Problem is usually defined in the context of combinatorial auctions. In classical formulation of the WDP, e.g., see [8], it is assumed that there is one seller and many buyers, and each commodity can be allocated to at most one buying offer.

Definition 1 *(The Winner Determination Problem, WDP) The seller has a set of commodities, $\mathscr{C} = \{1,2,\ldots,C\}$, to sell. Buyers submit a set of offers (bids) $\mathscr{B} = \{1,2,\ldots,B\}$. An offer $m \in \mathscr{B}$ is a tuple $\langle S_m, e_m \rangle$, where $S_m \subseteq \mathscr{C}$ is a bundle of demanded commodities, and $e_m \geqslant 0$ is an offered unit price. The Winner Determination Problem (WDP) is to find an allocation of commodities to buying offers which belongs to a set of feasible allocations $\mathscr{X}$ and is revenue-maximizing.*

The revenue is defined as $\sum_{m\in\mathscr{B}} e_m x_m$, where vector $x = (x_m)$ represents an allocation and x_m is 1 if offer m is winning and 0 in other case. Set of feasible allocations $\mathscr{X}$ is defined by bidding languages [8]. Without any additional requirements, it is OR language assumed by default, which does not restrict to any combination of bids. This problem is known to be NP-hard, since the set packing problem can be reduced to the WDP_{OR} [7]. Also for other, widely considered, bidding languages, the problem remains NP-hard [8].

In network auctions, each commodity, as well as each offer, is associated with a subset of vertices. Any allocation induces flow of commodities in the network that must be feasible. Therefore, the Winner Determination Problem must be replaced by extended version of the problem that takes into account flow constraints. This leads us to the Network Winner Determination Problem. The concept of the Network Winner Determination problem has been informally introduced in [6]. Below we provide more formal definition for the first time.

Definition 2 *(The Network Winner Determination Problem, NWDP) The Network Winner Determination Problem is the Winner Determination Problem under assumption that feasible allocation of commodities to buying offers belongs to $\mathscr{X} \cap \mathscr{X}^G$, where $\mathscr{X}^G$ is a set of allocations that induce feasible flows in flow network G. Let g and h be functions that map each commodity and offer, respectively, to a set of vertices in G, i.e., $g : \mathscr{C} \to V \times V \times \ldots \times V$, and $h : \mathscr{B} \to V \times V \times \ldots \times V$. Then, the set of allocations inducing feasible flows is defined as follows: $\mathscr{X}^G = \{x : F(x,g,h) \in \mathscr{F}\}$, where $\mathscr{F}$ is a set of feasible flows in G and $F(x,g,h)$ induces flow in G for allocation x and mappings g,h.*

The above definition is general, and the devil is hidden in the definition of functions g, h, and $F(x,g,h)$. The emerging cases for typical interpretation of

$F(x,g,h)$, g and h are collected in table 1. In the simplest case, mappings g and h associate one vertex to each commodity and offer, that is, $g : \mathscr{C} \to V$, $h : \mathscr{B} \to V$, and $F(x,g,h)$ transforms each allocated commodity into a source vertex and each allocated offer into a sink vertex. Each sell offer is for injection the commodity to the flow network at related vertex and each buy offer is for withdrawing the commodity from the flow network at given vertex. This special case will be referred further as Vertex-oriented Winner Determination Problem, VWDP [6]. In this problem, the flow in a network can be perceived as an additional service or a common good that serves the flow between contractors.

Table 1: Types of network auctions

demand	supply	type of the WDP
vertex	vertex	VWDP
arc	arc	combinatorial auction
2 vertices	arc	AWDP
arc	2 vertices	flipped AWDP
vertex	arc / vertex	mixed VWDP-AWDP
arc / vertex	vertex	flipped mixed VWDP-AWDP

The second case assumes that each sell and buy offer refers to a pair of vertices that belongs to E, that is, they refers to arcs in G. It can be interpreted as the seller posses some capacities of links to sell and the buyer is interested in buying capacities of particular links. Notice, that no flow is induced by an arc allocation, and this case does not essentially differ from the combinatorial auction with simple offers. As we discussed before, the combinatorial auction can be used to obtain the links that constitute required constellation, for instance, a path. However, this does not take advantages of graph model and the complexity lies in expressing buyer's preferences with a use of bidding language.

The graph nature of the problem is inherent in the next two variants. In both cases each commodity and each buy offer is associated with two vertices. However, on one side there are pairs of vertices from set E, and on the other side there are pairs from V, that do not have to belong to E. In third variant each commodity refers to an arc, while each buy offer reflects the need of path connecting two vertices. To better illustrate this, one can consider telecommunication bandwidth market, where sellers offer telecommunication links, and buyers are interested in buying the cheapest paths between certain vertices. In this case the demand is described by two vertices and can be also perceived as a virtual arc connecting these two nodes. Due to that, we refer to this case as the Arc-oriented Winner Determination Problem (AWDP) [6]. Notice, that a user does not have to bother with possible path enumeration. "Virtual" link presented by the user is implicitly converted into package of links that creates a proper path in the auction solution. Since a buyer needs any combination of commodities that creates a suitable path, the AWDP reminds a combinatorial auction. However, in this formulation a buyer does not express all possible bundles of edges that create a set of possible paths. In the fourth variant, which is symmetric to the previous one, commodities and offers are flipped, i.e., each commodity is associated with a path and each buy offer is related to an edge. Even though it can be difficult to give a meaningful interpretation to this setting, it completes the map of possible variants as flipped Arc-oriented Winner Determination Problem.

The last two cases represent some kind of mixture of the previous settings. They are also symmetric, therefore we will focus on the penultimate one. In this case, the vertex-oriented problem is extended in the way similar to arc-oriented formulation, i.e., there are some commodities related to vertices but there are also some commodities related to links. It means, that an accepted buying offer requires delivery of commodity available at node defined by the seller, but setting a route of delivery requires also buying links. We refer to this problem as mixed VWDP-AWDP. In the last case the commodities and offers are flipped, which generates flipped VWDP-AWDP.

It's worth to notice that each of the aforementioned problems can appear as reverse auction in the same way as reverse auction does in classical auction theory. 'Reversion' is related to number of agents taking part on supply and demand sides, while 'flipping' relates to the offered and requested commodities. Table 2 presents all combinations of 'reverse' and 'flipped' AWDP.

Table 2: Types of AWDP auctions due to number of agents and types of commodities. Differences to ADWP are marked in bold font

	buyers		sellers	
	commodity	# of agents	commodity	# of agents
AWDP	paths	n	arcs	1
reversed AWDP	paths	**1**	arcs	$\boldsymbol{n}$
flipped AWDP	**arcs**	1	**paths**	n
flipped reversed AWDP	**arcs**	$\boldsymbol{n}$	**paths**	**1**

Finally, after consideration the commodity and offer mappings to one or two vertices, let us make a comment on the case of more than two vertices. First of all, it brings some ambiguity to the problem in terms of offer's and commodity's interpretation. Some cases that go beyond regular paths are likely to be solved by applying one of the defined models: AWDP, VWDP, or mixed, and allowing for combination of offers using one of the bidding languages, e.g., XOR or OR. The case of minimal spanning tree is considered deeper in Section 3.5. Despite this, we believe that more interesting and valuable settings of the NWDP can be developed in future researches. We refer to all problems that satisfy definition 2 as the NWDP subclass of WDP problems.

The second variant listed in table 1, as it is classical combinatorial auction, has received great attention in the literature [12, 24]. However, the other variants have received very little attention till now. Let us define two basic problems, i.e., VWDP and AWDP, that are backbone of these variants of the NWDP.

Definition 3 *(The Arc-oriented network Winner Determination Problem, AWDP) Let the flow network be modeled as an oriented graph* $G = (V, E)$*, where* V *is a set of vertices and* E *is a set of edges. To make the notation simpler, we assume that* G *is a connected graph and it is defined by an incidence matrix* $a = [a_{ve}], v \in V, e \in E$*. Each commodity to sell is associated with an arc from* E*, that is,* $g : \mathscr{C} \to V$*, An offer* m *has the source and target vertices,* v^s_m*,* v^t_m *respectively, and mapping* h *is defined as follows:* $h : \mathscr{B} \to V \times V$*. The Arc-oriented network Winner Determination Problem (AWDP) is to find an allocation of commodities to buying offers which is revenue-maximizing under the constraints that for each wining offer* m*, a subset of assigned commodities constitutes a path from* v^s_m *to* v^t_m*,* $v^s_m \neq v^t_m$*.*

Figure 2 illustrates the graph model of the AWDP. Commodities c_1, c_2, ..., $c_{|E|}$ are denoted with solid lines that represent set E. Usually, they reflect the physical resources or transportation services, such that, each resource or service is associated with some arc in the graph G. Buy offers $m_1, m_2, \dots, m_{|B|}$ are defined by pairs of vertices which are connected with dotted lines in Figure 2. These are not edges of G, but for better understanding that can be perceived as virtual links. Any solution of the AWDP is equivalent to a flow in G and must satisfy flow conservation law, i.e., divergency must be positive at each source vertex, negative at each target vertex, and netural at any other vertex. A buy offer m requires a combination of arcs that create a path between the source v^s_m and sink v^t_m vertices. Any path can be constituted, based on commodities that the seller has, to satisfy given demand m.

One may notice, that commodities to sell are different from requested commodities in buy offers. Links and paths cannot be matched directly. However, they can be matched due to some kind of conversion between commodities to sell and demanded commodities. This conversion is performed implicitly by an auction mechanism and due to the incidence matrix a. A pair of vertices, or in other words, a virtual link, is implicitly converted into a set of possible paths connected with XOR operator.

In the Vertex-oriented network Winner Determination Problem (VWDP) the trade is related to commodities that can be injected by the seller into the network at a given node, and are taken by buyers from the specified nodes of the network.

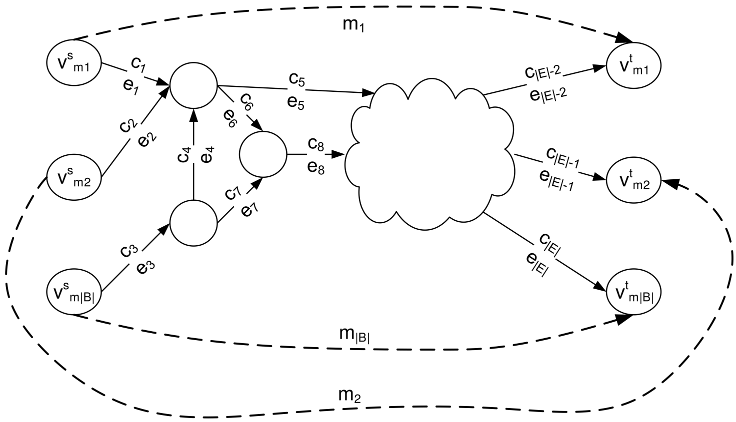

Figure 2: Graph model of the AWDP

The transportation services in the network are required to transfer the commodities from the seller to buyers. We assume, that every commodity requires the same level of resource in the network to be transferred (e.g., the same weights, sizes).

Definition 4 *(Vertex-oriented network Winner Determination Problem, VWDP) Consider the WDP and related oriented graph $G = (V, E)$. The seller has C commodities, and each commodity is located at some vertex $v \in V$. An offer m is for a commodity $c \in C$ at price e_m and located at vertex $w \in V$. The Vertex-oriented network Winner Determination Problem is to find an allocation of commodities to buying offers which is revenue-maximizing under the feasibility of commodity flow in G induced by accepted offers.*

Figure 3 illustrates a graph model of the VWDP. Commodities $c_1, \ldots, C$ flow into the flow network. The demand is defined by offers attached to vertices. A pair $\langle c, e_m \rangle$ describes and offer for commodity c at price e_m.

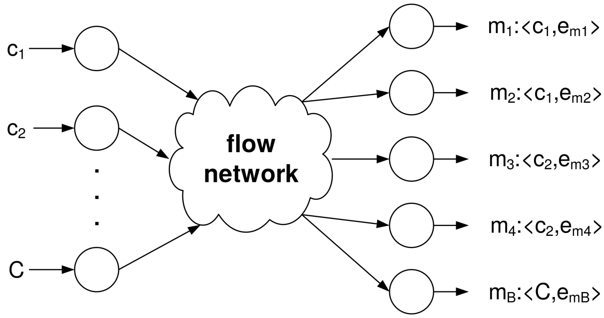

Figure 3: Graph model of the VWDP

3.2. Variants of the NWDP

Along with the standard WDP formulation, there is a commonly accepted set of assumptions. Two common assumptions are: indivisibility of offers and limitation to only one item of each commodity. We address both of them, which leads us to the more general concept of the Winner Determination Problem.

Definition 5 *(The Generalized Winner Determination Problem, GWDP) Let us consider a set of commodities to sell, $\mathscr{C} = \{1, 2, \ldots, C\}$. The seller has w_j copies of commodity $j \in \mathscr{C}$. Buyers submit a set of offers (bids) $\mathscr{B} = \{1, 2, \ldots, B\}$. Let $v_m(x_m)$ be marginal valuation in offer m if the vector of allocated commodities $x_m = (x_{1,m}, \ldots, x_{C,m})$ is assigned to offer m and belongs to the set of feasible allocations $\mathscr{X}$. The Generalized Winner Determination Problem (GWDP) is to find an allocation of commodities to buying offers which is revenue-maximizing.*

In case of continuous bid the accepted volume of a bundle can take any value below the maximum volume defined in the offer. For this case, there are also bidding languages used to express bidder valuations [10]. Notice, that the notion of bundles used in WDP formulation is replaced by the set of feasible allocations $\mathscr{X}$. If this set allows for partial allocations, then the offers and commodities are divisible. We will refer to this case as a continuous one, contrary to the binary case, in which each offer can be accepted entirely or not at all. Moreover, this formulation allows for many instances of each commodity. On the basis of GWDP definition, the Generalized Network Winner Determination Problem can be formulated.

Definition 6 *(The Generalized Network Winner Determination Problem, GN-WDP) The Generalized Network Winner Determination Problem is the GWDP problem under assumption that commodity flow induced by the allocation in the same way as in definition 2, is feasible.*

As well as specification of the GWDP leads to various formulations of the problem, the GNWDP can be reduced to different variants, particulary different variations of VWDP and AWDP. We consider three main aspects of any setting. Each aspect is related to one of the following area: offers, items, and arcs. Before we define variants of the problem, we need to introduce more notation. Let p_c and d_m be the accepted volume of sold commodity c and buy offer m, respectively. In case of AWDP, a commodity is equivalent to some arc. Therefore, p_c already defines the flow over edge associated with commodity c. To simplify the notation we will also use p_e, $e \in E$ to denote the accepted volume of commodity associated to arc e. In case of VWDP we also need to introduce f_{ce}, the flow of the commodity c over edge e induced by the solution (p_c, d_m).

Offers Offers can be either divisible (continuous) or indivisible (binary). In default formulations of VWDP and AWDP the indivisible offers are assumed, so variables d_m and p_c are binary:

$$d_m \in \{0,1\} \qquad \forall m \in \mathscr{B}, \tag{1}$$

$$p_c \in \{0,1\} \qquad \forall c \in \mathscr{C}. \tag{2}$$

In continuous NWDP, constraints (1) and (2) are replaced by the following constraints:

$$0 \leqslant d_m \leqslant 1 \qquad \forall m \in \mathscr{B}, \tag{3}$$

$$0 \leqslant p_c \leqslant 1 \qquad \forall c \in \mathscr{C}. \tag{4}$$

Notice, that no integer variant is considered in standard versions of NWDP, due to the assumption of only one copy of each commodity.

Items In multi-item problem, constraints (1) and (2) are replaced by the following constraints:

$$d_m \in \{0, d_m^{\max}\} \qquad \forall m \in \mathscr{B}, \tag{5}$$

$$p_c \in \{0, p_c^{\max}\} \qquad \forall c \in C, \tag{6}$$

where $d_m^{\max}$ is maximal volume of offer m and $p_c^{\max}$ is maximal available volume of commodity c. In case of continuous offers constraints (3) and (4) are replaced by the following constraints:

$$0 \leqslant d_m \leqslant d_m^{\max} \qquad \forall m \in \mathscr{B}, \tag{7}$$

$$0 \leqslant p_c \leqslant p_c^{\max} \qquad \forall c \in C. \tag{8}$$

In multi-item case, there is also possibility to consider integer offers. In this case, constraints are as follows:

$$d_m \in \{0, \ldots, d_m^{\max}\} \qquad \forall m \in \mathscr{B}, \tag{9}$$

$$p_c \in \{0, \ldots, p_c^{\max}\} \qquad \forall c \in C. \tag{10}$$

Arcs We consider two cases: capacitated and uncapacitated arcs. In the capacitated VWDP, the total flow over edge e is limited by the maximal flow $f_e^{\max}$, and the following constraint must be introduced to the model:

$$\sum_{c \in \mathscr{C}} f_{ce} \leqslant f_e^{\max} \qquad \forall e \in E. \tag{11}$$

In case of the AWDP, arc capacities and flow costs are already defined by offers related to arcs. However, an extra capacity constraint takes the following form:

$$p_e \leqslant f_e^{\max} \qquad \forall e \in E. \tag{12}$$

We need to add, that similarly to traditional auction designs, also network auctions can be extended to double auctions (exchanges). One sided auction is a special case of double auction, which is also called a combinatorial exchange. In an exchange, multiple buyers and sellers provide complex offers and related winner determination problem is aimed at matching offers to maximize the social welfare resulting from the matching.

3.3. Notation system in the GNWDP class

As the space of problems in GNWDP class is multidimensional and complex, we suggest to apply convenient notation for the problems which allows for quick understanding the nature of the problem and it reduces possible ambiguity. We propose field-based notation, similar to the idea of Kendall's notation in queueing theory or notation for theoretic scheduling problems. Four fields are needed to describe a network auction system: $\alpha|\beta|\gamma|\delta$, where α is related to properties of items/commodities, β reflects offer's properties, γ characterizes the flow network, and δ specifies a type of auction.

Values that we identified for each field, aim at identification the settings mentioned before in this paper. However, the dictionary of values is not closed and can be extended for each field during further research. In field α we use values `MI` and `SI` which distinguish multi- and single-item cases. For offers (field β) we need to determine their divisibility. Values `B`, `C` and `I` indicate binary (indivisible), continuous (divisible) and integer offers, respectively. In the field γ there can be more potential values separated with comma. The `cap/uncap` value informs that the network is either capacitated or uncapacitated. The `cost` value in the field γ means that some additional cost of flow appears in the network. Finally, the field δ describes an auction system, and indirectly, an objective of the auction. In this paper we mainly use VWDP and AWDP in this field. However, also reverted auctions or exchanges should be indicated in this field.

We assume the following default values for fields $\alpha - \gamma$: `SI`, `B`, `uncap`, respectively. In case of default values, a field can be left empty as well as leading empty fields can be omitted. Therefore, the system described as VWDP (or equivalently $|||$VWDP) means single-item uncapacitated vertex-oriented problem with binary offers. Default values can still be put in each field explicitly to emphasize the setting. Let us notice, that assumed order of fields facilitates reading the problem settings. For instance, MI$|B|$uncap$|$AWDP which denotes one of the most complex case, can be read as 'multi-item binary uncapacitated arc-oriented winner determination problem'. Other examples of definitions are presented in table 3.

Table 3: Exemplary notation of problems in the GNWDP class

notation	description
cap|VDWP	single item capacitated vertex-oriented problem with binary offers
MI|||VWDP	multi-item uncapacitated vertex-oriented problem with binary offers
MI|I|cap|VWDP	multi-item capacitated vertex-oriented problem with integer offers
MI|C|cap|AWDP	multi-item capacitated vertex-oriented problem with continuous offers
B|uncap,cost|AWDP	single item uncapacitated vertex-oriented problem with binary offers

Each of the problems can also appear with side constraints [3]. For instance, in [6] two types of constrains in network auction are discussed: MIN/MAX-VOLUME and MIN/MAX-WINNER. These types put constraints on a total traded volume or total number of winners in a given subnetwork, respectively. We recall this aspect to draw the whole landscape of NWDP-like problems, however, security constraint NWDP are beyond the scope of this paper. We recommend to put additional information about side constraints in front of the problem definition and separate this from the rest with a space. Therefore, the problem MIN/MAX-VOLUME MI||cap|AWDP means multi-item capacitated arc-oriented problem with binary offers and under MIN/MAX-VOLUME constraints. To distinguish problems without side constraints we refer to them as unconstrained cases.

3.4. Relationships between VWDP and AWDP

As we defined two main problems in the NWDP class, an interesting question arises: is the VWDP somehow related to the AWDP? In case of the AWDP, a buyer provides the source and target vertices, which specify a set of possible bundles of commodities, such that each bundle creates required path between the source and the destination. An offer exhibits substitutability, since any proper path can serve the buyer needs. There is also complementarity, because the links are only valuable in combinations forming proper paths, and they are worthless in other case.

The VWDP does not reflect such combinatorial aspects as the AWDP does. A flow in the network that must be constituted to let the offers become matched involves only setting up network resources, while in the AWDP it requires a combination of offers. Therefore, intuitively the VWDP brings a greater complexity and likely is more comprehensive than the AWDP. The following remark confirms the simplicity of basic variant of the VWDP.

Remark 1 *Notice, that the VWDP can be transformed into a set of trivial ASSIGNMENT problems, each problem for each commodity. This decomposition is possible, since in the VWDP, the network flow is uncapacitated and any flow is feasible. In each resulting problem a given commodity must be assigned to the most valuable offer from the set of buy offers submitted for this commodity.*

From this remark the relationship between the VWDP and the AWDP immediately results.

Remark 2 *The VWDP can be transformed into an instance of the AWDP in linear time. Each commodity can be considered separately. We construct an instance of the AWDP for a given commodity as follows. Each commodity from the VWDP is associated with some arc in the AWDP. These arcs also create paths for which buyers bid. Then, every bid in the VWDP for commodity c is transformed into similar bid in the AWDP for link* $(v^s_m; v^t_m)$. *In other words, every ASSIGNMENT problem related to the VWDP can be modeled as a trivial one-arc instance of the AWDP. Figure 4 illustrates the transformation.*

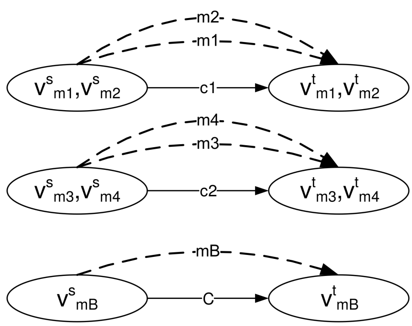

Figure 4: Model presented in Figure 3 transformed into the AWDP

Remark 3 *The capacitated VWDP can be transformed into the AWDP in linear time. Let G be a graph in the VWDP,* v_c *be a vertex in which commodity c is available, and* v_m *be a vertex associated with offer m. An instance of the AWDP with graph* G^{AWDP} *is created in the following way:*

1. *Initially,* $G^{AWDP} = G$

2. *Each arc is associated with some artificial commodity which is available at volume equal to capacity of related arc in G*

3. *For each commodity:*

   (a) *Add a new vertex* v'_c

   (b) *Add an arc* (v'_c, v_c)

4. *Each commodity c in the VWDP is transformed into a commodity associated with link* (v'_c, v_c) *and the volume as in the VWDP*

5. *Each buy offer m for a commodity c is transformed into an offer for a path connecting vertex* v'_c *with* v_m

Figure 5 illustrates the transformation.

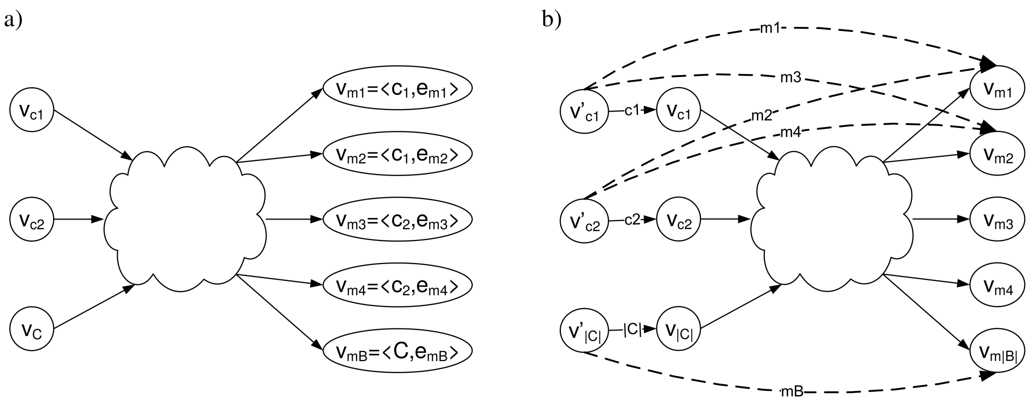

Figure 5: Transformation of (a) capacitated VWDP into (b) the AWDP

Unfortunately, due to combinatorial nature of the AWDP and assignment-like nature of the VWDP, the AWDP is not transformable into the VWDP in general. The AWDP can be modeled as the WDP with XOR-constraints, which in general is NP-complete [3].

3.5. Minimal spanning tree

Introducing the AWDP facilitates bidder's preference expression. The contractor does not have to express desirable combinations of arcs, but instead of that he expresses his preferences in terms of paths. A path can be considered as a complex commodity and it is network auction mechanism, that is responsible for conversion this commodity into a proper set of simple commodities, i.e., links. The conversion is obviously driven by a network model. Then, immediately arises the following question: are there any other graph-based structures that should become a complex commodity directly traded? We leave this question open, however, since a spanning tree is the one considered in the literature in the context of auctions [22], we will make a comment on it.

Consider an auction of links in graph $G = (V,E)$. Sellers have links $e \in E$ to sell at price c_e, and the buyer is interested in buying links that constitute a minimal spanning tree in G, where c_e is a weight of edge $e \in E$. This setting of auction can be modeled as the mixed VWDP-AWDP. More precisely, this is the problem SI|B||VWDP-AWDP, which is constructed in the following way. Graph $G = (V,E)$ creates a flow network of the problem. Commodities are naturally related to links. Sellers offer links that are binary (can be either accepted fully or rejected). The maximal volume of each offer is $|V|-1$ and price of offer e is $\frac{c_e}{|V|-1}$. Let us choose any vertex $l \in V$ as a root of the demanded spanning tree. The spanning tree will be designated by a flow of additional, artificial commodity c. This commodity is available at vertex l and the buyer is interested in acquiring one unit of commodity c in every other vertex. One seller submits an offer for $|V|-1$ units of commodity c at node l and at price 0.

Proposition 1 *The optimal solution of SI|B||VWDP-AWDP constructed as described above designates the minimal spanning tree of G.*

Proof First, notice that for every spanning tree there exists a feasible solution of the considered auction. Moreover, a graph induced by the optimal flow of the problem is connected and induced by all vertices in V, because in any feasible solution the commodity flows from vertex l to every other vertex in V. Therefore, to prove that optimal flow of the problem induces a spanning tree of G, we only need to show that there are no cycles. Let us assume that there is a cycle induced by the optimal solution of the auction. In this case at least one link in the cycle can be removed and the flow between vertices connected by this link can be redirected along the second path in the cycle. Notice, that the capacity resulting from the offered volume, which is $|V|-1$, is sufficient for this flow, since there is $|V|-1$ units of commodity in total. After removing the link, the solution remains feasible, while the total cost is lower. It means, that there are no cycles in optimal solution. Therefore, the optimal solution of the problem is a spanning tree.

Now, we need to prove that this tree is the minimal one. Since only the offers for links have non-zero costs, then the problem is formulated as follows:

$$\min \frac{c_e}{|V|-1} p_e \tag{13}$$

s.t. p_e belongs to set of flows inducing any spanning tree of G. It is binary problem, thus, p_e is either 0 or $|V|-1$. Each winning offer for a link e generates the cost c_e, while each rejected offer brings no cost. Therefore, the optimal solution of the auction is equivalent to the minimal spanning tree.

4. Complexity in the NWDP class

4.1. Complexity of AWDP-derived problems

First, let us consider AWDP-derived problems with single items.

Proposition 2 *The following problems: $C||AWDP$, $C|cap|AWDP$, $C|cost|AWDP$, and $C|cap,cost|AWDP$ (uncapacitated, capacitated without and with flow costs) can be solved in polynomial time.*

Proof Let $a = [a_{ve}], v \in V, e \in E$ be an incidence matrix describing the flow network G, which is a connected graph by the definition. A buy offer m has the source and target vertices, v^s_m, v^t_m respectively. The problem is to find an allocation of commodities to buying offers which is revenue-maximizing under the constraints that for each wining offer m a set of allocated commodities constitute a path from v^s_m to v^t_m, $v^s_m \neq v^t_m$. Problem C||AWDP can be modeled as the following linear program:

$$\max \sum_{m \in \mathscr{B}} e_m d_m, \tag{14}$$

s.t.

$$\sum_{e \in E} a_{ve} p_{em} = \begin{cases} d_m & v = v^s_m \\ 0 & v \neq v^s_m, v^t_m \\ -d_m & v = v^t_m \end{cases} \quad \forall v \in V, m \in \mathscr{B}, \tag{15}$$

$$p_e = \sum_{m \in \mathscr{B}} p_{em} \qquad \forall e \in E, \tag{16}$$

$$0 \leqslant d_m \leqslant 1 \qquad \forall m \in \mathscr{B}, \tag{17}$$

$$0 \leqslant p_e \leqslant 1 \qquad \forall e \in E, \tag{18}$$

where p_{em} is a volume of commodity related to arc e and allocated to demand m. A flow conservation is achieved due to constraint (15). In constraint (16) the total flow over arc e is calculated. Constraint (17) limits the maximal accepted volume of offers and constraint (18) assures that no more than one item can be sold. The linear programming problem was shown to be solvable in polynomial time by Khachiyan [25]. In capacitated problem the constraint (12) must be added to the model (14)–(18). The problem with external costs involves adding component $(-\sum_{e \in E} p_e c_e)$ to the objective (14), where c_e is a cost of unit commodity flow. □

Proposition 3 *Each of the following problems: AWDP, $cap|AWDP$, $cost|AWDP$, and $cap,cost|AWDP$ (uncapacitated and capacitated, without and with flow costs) is NP-complete.*

Proof The AWDP can be expressed by (14) subject to (15)–(16), and (1)–(2) (notice, that index c is isomorphic to index e in case of the AWDP). This problem is equivalent to the multi-commodity flow problem with weighted maximal flow as an objective under additional assumption of unit capacities. This problem was proved to be reducible to satisfiability problem (SAT) and, therefore, to be NP-complete, even for two commodities [26]. The capacitated version can be transformed into uncapacitated one. If capacity of a given arc is lower than 1, then this arc can be removed, since no integral amount of any offer can be allocated to it. If the capacity is higher than 1, it can be omitted since the available volume of commodity related to this arc is 1. Concerning the additional flow cost, the AWDP is a special case assuming that cost is equal 0. Therefore, AWDP can be reduced to any of the following problems: $cap|$AWDP, $cost|$AWDP, and $cap, cost|$AWDP. Thus, each of the problem is NP-complete. □

Now, let us consider multi-item AWDP-derived problems.

Proposition 4 *The following problems: $MI|C||AWDP$, $MI|C|cap|AWDP$, $MI|C|cost|AWDP$, and $MI|C|cap, cost|AWDP$ (uncapacitated, capacitated without and with flow costs) can be solved in polynomial time.*

Proof Each of the problems can be modeled as linear program. Problem MI$|C||$AWDP is an maximization problem (14) subject to (15)–(16), and (7)–(8). Capacitated problem is a modification of this formulation according to (12). Problem with additional flow cost is also simple modification of the objective (14) by adding $(-\sum_{e\in E} p_e c_e)$ component, where c_e is a cost of unit commodity flow. □

Theorem 1 *Multi-item AWDP-derived problems, i.e., $MI|||AWDP$, $MI||cap|AWDP$, $MI||cost|AWDP$, $MI||cap, cost|AWDP$ are NP-complete.*

Proof First, let us notice that MI$|||$AWDP can be reduced to any of the rest problems, because it is a special case of each of them. Therefore, it is sufficient to prove that MI$|||$AWDP is NP-complete. We reduce the KNAPSACK problem to the MI$|||$AWDP. In the knapsack problem there are B items. Each item $m \in \{1, \ldots, B\}$ has value r_m and weight w_m. W is the maximum weight that can be carried in the knapsack. We formulate an instance of MI$|||$AWDP with a flow network $G = (\{v_1, v_2\}, \{e_1 = (v_1, v_2)\})$. There are m commodities offered by B buyers for a path from v_1 to v_2 with maximal volume $d_m^{\max}$ equal to weight of item w_m, and unit price e_m equal $\dfrac{r_m}{d_m^{\max}}$. A seller has link e_1 with capacity W. The original knapsack problem has an optimal solution with total value $\hat{Q}$ if and only if the instance of MI$|||$AWDP has an optimal solution with objective value $\hat{Q}$. □

Proposition 5 *Multi-item AWDP-derived problems with integer offers, i.e., $MI|I||AWDP$, $MI|I|cap|AWDP$, $MI|I|cost|AWDP$ $MI|I|cap,cost|AWDP$ are NP-complete.*

Proof Similarly to the previous proof, let us notice that MI$|I||$AWDP can be reduced to any of the rest problems. Moreover, this is the multi-commodity flow problem which is proved to be NP-complete (see proof of proposition 3). □

4.2. Complexity of VWDP-derived problems

First, let us consider VWDP-derived problems with single items.

Proposition 6 *The following problems: $C||VWDP$, $C|cap|VWDP$, $C|cost|VWDP$, and $C|cap,cost|VWDP$ (uncapacitated, capacitated without and with flow costs) can be solved in polynomial time.*

Proof Let $a=[a_{ve}], v\in V, e\in E$ be an incidence matrix describing the flow network G, and A be the availability set, i.e., set of pairs (c,v), such that the seller has commodity c at vertex v. Let us also define a function $u: B\to C\times V$ that assigns commodity $c\in C$ and vertex $v\in V$ to an offer $m\in B$. It means, that offer m is for commodity c requested at vertex v, if $u(m)=(c,v)$. The problem is to find an allocation of commodities to buying offers which is revenue-maximizing under the constraint conserving the flow in G. We will show that each of the problems can be modeled as linear program, therefore they belongs to P complexity class. Problem C$||$VWDP can be modeled as the following linear program:

$$\max \sum_{m\in\mathscr{B}} e_m d_m, \tag{19}$$

s.t.

$$\sum_{e\in E} a_{ve} f_{ce} = \left\{ \begin{array}{ll} p_c & \text{if}(c,v)\in A \\ 0 & \text{otherwise} \end{array} \right\} - \sum_{m:u(m)=(c,v)} d_m \qquad \forall v\in V, \forall c\in C, \tag{20}$$

$$0\leqslant d_m \leqslant 1 \qquad \forall m\in\mathscr{B}, \tag{21}$$

$$0\leqslant p_c \leqslant 1 \qquad \forall c\in\mathscr{C}, \tag{22}$$

where f_{ce} is a flow of the commodity c over the edge e. Problems C$|cap|$VWDP, C$|cost|$VWDP, and C$|cap,cost|$VWDP are simple modifications of problem (19)–(22). Capacitated version is obtained by introduction constraint (11). External cost flow involves inclusion the element $-(\sum_{e\in E,c\in C} k_e f_{ce})$ in objective (19), where k_e is an external flow cost over edge e. □

Proposition 7 *Problems $VWDP$, $cost|VWDP$ are in P.*

Proof Problems $VWDP$, $cost|VWDP$ remaind the multi-commodity flow problem, which is known to be NP-complete for integer flows. However, since there is no link capacity constraint, the matrix of coefficients of problem (19)–(22) is totally unimodular as it is network matrix [27]. □

Proposition 8 *Problems $cap|VWDP$ and $cap, cost|VWDP$ are NP-complete*

Proof Problems $cap|VWDP$, $cap, cost|VWDP$ are the multi-commodity flow problem. Complexity of the multi-commodity flow problem is mentioned in the proof of proposition 3. □

Now, let us consider multi-item VWDP-derived problems.

Proposition 9 *The following problems $MI|C||VWDP$, $MI|C|cap|VWDP$, $MI|C|cost|VWDP$, and $MI|C|cap, cost|VWDP$ (uncapacitated, capacitated without and with flow costs) can be solved in polynomial time.*

Proof Each of the problems can be modeled as a linear program. Problem MI$|C||$AWDP is the maximization problem (19) subject to (20), (7), and (8). Capacitated problem is a modification of this formulation according to (11). Problem with additional flow cost is also simple modification of the objective (19) by adding $\sum_{e \in E, c \in C} k_c f_{ce}$ component, where k_c is the cost of a unit commodity flow. □

Optimization problem (19), (20), (7), and (8) remains multi-commodity flow problem without capacity constraints.

Proposition 10 *$MI|I||VWDP$ and $MI|I|cost|VWDP$ can be solved in polynomial time.*

Proof Notice, that there is no link capacity constraints. Therefore the matrix of coefficients of the problem is totally unimodular as it is network matrix [27] and the optimal solution is integer. □

Theorem 2 *$MI|I|cap|VWDP$ and $MI|I|cap, cost|VWDP$ are NP-complete.*

Proof Capacitated version of the problem includes constraint (11), which makes the problem equivalent to multi-commodity flow. In general, the multi-commodity flow has a matrix that is not totally unimodular and linear relaxation of MI$|I||$VWDP can result in fractional solution. Complexity of multi-commodity flow problem is mentioned in the proof of proposition 3. □

Table 4 collects all considered variants along with the information about their computable complexity and the way of proof.

Table 4: NWDP's complexity summary

items	offers	arcs	network	complexity	
SI	B	uncap	AWDP	NP	SAT reduction
SI	C	uncap	AWDP	P	LP
SI	B	cap	AWDP	NP	SAT reduction
SI	C	cap	AWDP	P	LP
MI	B	uncap	AWDP	NP	knapsack reduction
MI	C	uncap	AWDP	P	LP
MI	I	uncap	AWDP	NP	SAT reduction
MI	B	cap	AWDP	NP	knapsack reduction
MI	C	cap	AWDP	P	LP
MI	I	cap	AWDP	NP	SAT reduction
SI	B	uncap	VWDP	P	total unimodularity
SI	C	uncap	VWDP	P	LP
SI	B	cap	VWDP	NP	SAT reduction
SI	C	cap	VWDP	P	LP
MI	B	uncap	VWDP	NP	knapsack reduction
MI	C	uncap	VWDP	P	LP
MI	I	uncap	VWDP	P	total unimodularity
MI	B	cap	VWDP	NP	knapsack reduction
MI	C	cap	VWDP	P	LP
MI	I	cap	VWDP	NP	SAT reduction

4.3. Other tractable instances of the NWDP with integer offers

As we mentioned in the introduction, a path auction, i.e., AWDP auction, can be perceived as a combinatorial auction with XOR bidding language. Also, the problem SI|B|cap|VWDP can be considered as a combinatorial auction with XOR language. Since there are known settings of combinatorial auctions that are tractable, we apply this knowledge to discover auction settings that can be solved efficiently. This is relevant to problems proved to be NP-complete in general, i.e., SI|B|cap|VWDP, SI|B|cap|AWDP, SI|B|uncap|AWDP, MI|I|uncap|AWDP, MI|I|cap|AWDP. Since problem SI|B|cap|VWDP can be transformed into proper AWDP-derived problem in linear time (see Remark 3), we limit further consideration to AWDP-derived problems. It is worth to noting, that this approach opens a new challenging area of searching for problem structures, that allow to solve the hardest NWDP problems and algorithms that can be derived from the discovered structures. In this section we deliver the preliminary achievements.

The AWDP with limited path length Let us consider a graph $G = (V = \{v_1, v_2\}, E)$ that has several edges in the set E, each connecting v_1 and v_2. Each offer is for any path from v_1 to v_2. Therefore, only one edge from E must be allocated to an offer to constitute a path. The binary case of this problem is equivalent to a combinatorial auction with XOR language and bids for single items. This type of combinatorial auction is proved to be in P [11]. In fact, the problem can be transformed into maximal weighted flow and remains integer convex hull, even in case of integer version of the AWDP. Figure 6 illustrates this case and transformed model. Of course, the graph G can be more complex and offers may relate to different pairs of vertices, unless each offer is a path not longer than 1, i.e., there exists no other route that is longer.

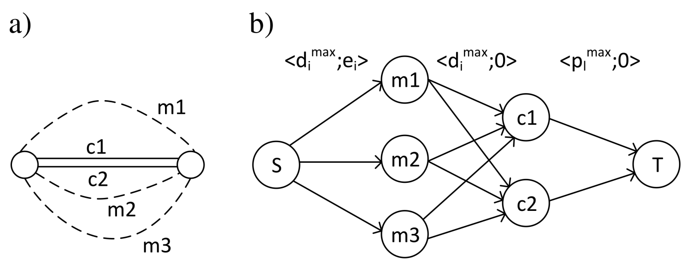

Figure 6: a) The AWDP with paths of length equal 1 and b) related max weighted flow problem

A case of path lengths limited to 1 can be extended to paths of length $\leqslant 2$, but only in case of OR language. It means that paths wanted by buyers can be of length 1 or 2, but there can only be one path between each two vertices for every buy offer. The problem can be transformed into an item graph: each link is represented by a vertex and two vertices are connected if they are covered by any buy offer (see Fig. 7). For buy offers for one link, the vertex related to the link must be doubled. In the new graph, the maximum weighted matching problem can be solved (e.g. by Edmonds' algorithm with $O(n^3)$ complexity), and therefore related SI|B|cap|AWDP, SI|B|uncap|AWDP, and MI|I|uncap|AWDP are tractable.

The OR bidding language is insufficient to express substitutability that is needed, when more than one possible path can be matched with a buy offer. Notice, that OR bidding language with dummy items can be used instead of the XOR bidding language [8]. However, adding a dummy item means lengthening the path, which leads us again to the case of original paths no longer than one.

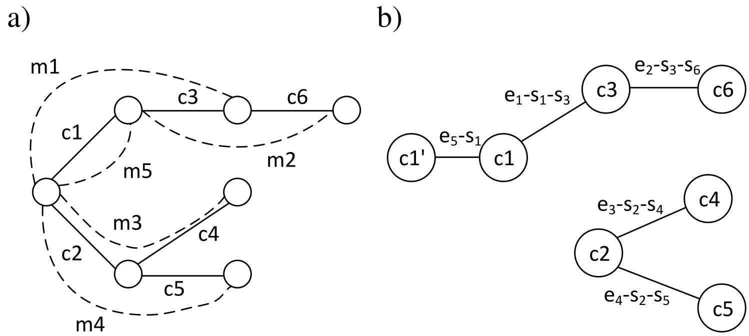

Figure 7: a) The AWDP with paths of length $\leqslant 2$ and b) related item graph

Let us consider the VWDP that is transformed into the AWDP with paths of length 1 and according to Remark 2. It means, that the VWDP is limited to only one commodity. In fact, it makes this problem equivalent to a network flow problem limited to one commodity, which becomes easy in integer case, due to total unimodularity of coefficients matrix. This is also the case of the minimal spanning tree problem considered in Section 3.5. General problem SI|B|C|VWDP-AWDP is NP-hard. However, in case of the minimal spanning tree, there is only one commodity with non-zero cost, and the problem is actually a single-commodity flow problem. This result is consistent with findings of Bikhchandani et al. [22], since they showed that the minimal spanning tree auction is tractable.

The AWDP with hierarchical structure of offers Let us assume that offer m induces only one possible path denoted by P_m. If for any two offers m and n and related paths P_m and P_n, their intersection is empty or one path is completely included in the second one, then problems SI|B|cap|AWDP and SI|B|uncap|AWDP become tractable. Each problem can be transformed into a tree, such that, if link c belongs to some node, then it also belongs to the parent node (see Fig. 8).

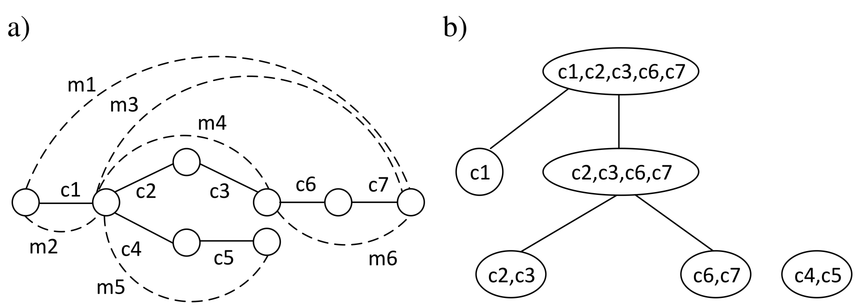

Figure 8: a) The AWDP with tree structure and b) resulting tree

Combinatorial auctions with OR language and with subtree (hierarchical) bids are known to be in P [8]. The solution can be constructed by simple depth-first traversing the tree and comparing revenue for a given node with summation of revenues for all children of this node. For instance, in the example illustrated in Figure 8, it is enough to compare the total revenue of offers m_4 and m_6 to the revenue of offer m_3 in order to decide which offers are winning in this subtree.

The AWDP with bids for consecutive items If G is an acyclic graph and links can be put into some order, such that each buy offer relates to some consecutive links, then problems SI|B|cap|AWDP and SI|B|uncap|AWDP are tractable. This case is equivalent to combinatorial auction with OR language and so called interval (or sometimes called 'linear') bids [8]. Exemplary graph is illustrated in Figure 9.

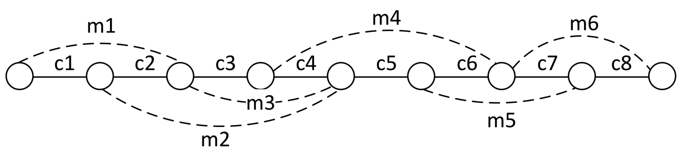

Figure 9: The AWDP with bids for consecutive items

The AWDP with tree-like network Idea of interval bids is extended to a network of goods represented by a tree in [18]. Inspired by this achievement we apply similar approach to the case of AWDP-derived problems to demonstrate that offers limited to paths in a tree make the problems computationally tractable.

Theorem 3 *If flow network G is a tree and offers can be only submitted for paths in G, then problems SI|B|cap|AWDP and SI|B|uncap|AWDP can be solved in polynomial time.*

Tennenholtz considered similar problem in which commodities are identified with vertices instead or edges, and offers are binary [18]. His proof is based on showing an isomorphism between original problem and optimal coverage of G' by circles, which can be transformed into weighted perfect matching in a bipartite graph. This approach could be also used in case of binary offers, but not in case of integer offers. Therefore, we will apply more general approach to prove the theorem.

Proof We construct the graph $G' = (V', E')$ on the basis of G and by applying the following modifications:

1. Each edge $e \in E$ has assigned weight 0.

2. For each bid m for a path from v_1^m to v_2^m we add two new vertices $v_{m'}$ and $v_{m''}$ and three edges: $e_{m'1} = (v_{m'}, v_1^m)$, $e_{2m''} = (v_2^m, v_{m''})$, and $e_{m'm''} = (v_{m'}, v_{m''})$. Edges $e_{m'1}$ and $e_{m'm''}$ have weight 0 and the edge $e_{2m''}$ has weight e_m. Each of the edges has capacity equal to $d_m^{\max}$.

3. Each vertex $v_{m'}$ is a source with in-flow equal to $d_m^{\max}$ and vertex $v_{m''}$ is a sink with out-flow equal to $d_m^{\max}$.

First, we will show that the original problem is equivalent to the maximal weighted flow problem in G'. A flow over edge $e_{2m''}$ generates revenue due to unit revenue e_m assigned to this edge. It is equivalent to accepted volume of offer m. If the in-flow at $v_{m'}$ is directed to the sink at vertex $v_{m''}$, it means that the offer m is rejected. Partially accepted offer induces flow of $d_m^{\max}$ divided over both edges $e_{m'1}$ and $e_{m'm''}$. Therefore, any solution of an auction can be encoded as a flow in G', and for any flow in G', there exists a related solution of the auction. Figure 10 illustrates the case of tree-like network and related graph G'.

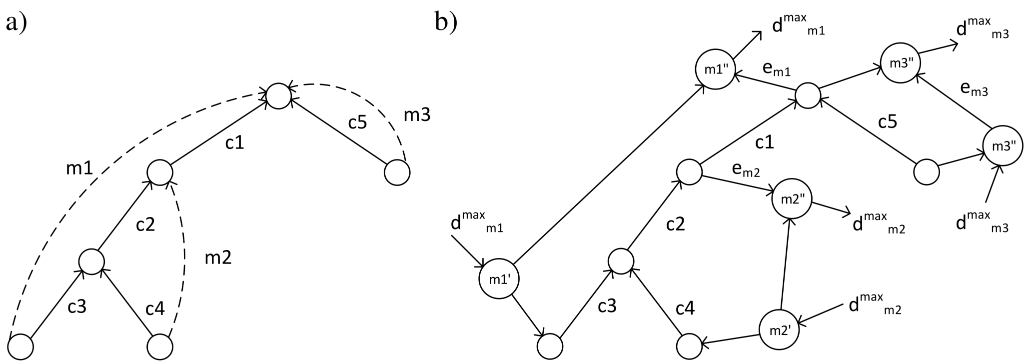

Figure 10: a) AWDP with bids for consecutive items, b) and related graph G'

One may notice, that in the optimal solution of maximal weighted flow problem the in-flow at $v_{m'}$ can be directed to $v_{n''}$, and thus it breaks the logic of the problem. However, since $v_{m''}$ is a sink vertex, there must be another accepted offer, that will be directed to $v_{m''}$ at appropriate level to conserve the flow. Such mixed flows can be always refined keeping the total flow at each edge and the objective unchanged. Let us assume that flows originated at $v_{m'}$ and $v_{n'}$ mix up at vertex v_i. The joint flow splits apart again at some vertex and flow x from v_m is directed to $v_{n''}$, while the same flow from $v_{n'}$ is directed to $v_{m''}$. It is clear that flows can be exchanged starting from the vertex v_i without affecting the total flows, as well as the objective. The refinement is illustrated in Figure 11. Note, that the refinement is not necessary to obtain the winning offers, so it does not add complexity to the problem.

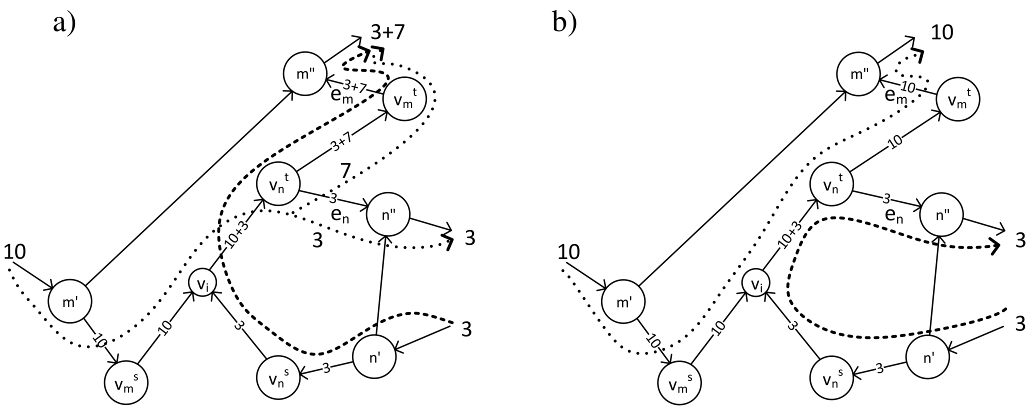

Figure 11: The flow refinement: a) flow mixed up and b) refined

Now, we need to show, that the maximal weighted flow problem in G' is tractable. Let us construct the following linear programming model:

$$\max \sum_{(i,j)\in E'} c_{ij}x_{ij}, \tag{23}$$

s.t.

$$x_{m'1} + x_{m'm''} = d_m^{\max} \qquad \forall m \in \mathscr{B}, \tag{24}$$

$$x_{m'm''} + x_{2m''} = d_m^{\max} \qquad \forall m \in \mathscr{B}, \tag{25}$$

$$\sum_{i:(j,v)\in E'} x_{jv} - \sum_{j:(v,j)\in E'} x_{vj} = 0 \qquad \forall v \in V, \tag{26}$$

where x_{ij} is a flow over edge (i, j) and c_{ij} is a unit cost of this edge. Constraints (24)–(26) represent flow conservation for source, sink and intermediate vertices, respectively. Matrix of coefficients is a network matrix which is totally unimodular which guarantees integrality of solution of the LP model. □

5. Some practical applications

Infrastructure economies which are being moved into deregulated, competitive markets are the natural areas for applications of the Network Winner Determination Problems. In this section we present some applications of the NWDP in telecommunication and energy sectors. However, the range of applications is much wider and includes: supply chain management, transportation sector (maritime, air, land, rail), time auctions, water supply, among others.

5.1. Communication bandwidth trade

The telecommunication industry deregulation that dates back to 90's, led to the development of the bandwidth markets. Originally, it was oriented on bilateral negotiations, but the new technological and conceptual opportunities have appeared, as the structure of telecommunication market became more complex and dynamic. The new efficient, flexible trading mechanisms became needed to enable the development of real competitive bandwidth markets.

Typically, telecommunication resources are conceptually organized as a layered model. For example, the Open Systems Interconnection model (OSI model) distinguishes seven layers, starting from physical layer and ending in application layer. An auction mechanism can be applied to organize an exchange of goods in one or more linked layers. Some model of an infrastructure is associated with each layer and must be addressed during the trade. At physical layer, communication ducts create the infrastructure network and resources being traded. Probably, the most natural layer for business is transport or network layer. In this case bandwidths of telecommunication links create elementary resources, that build the whole flow network. Buyers can be interested in various complex products. We will use two of them to present possible applications of the NWDP: paths in this section and virtual private network in Section 5.3.

In [28] and [29] Balancing Communication Bandwidth Trade (BCBT) model is proposed. It provides effective allocation of network resources in the sense that it maximizes global economic surplus (welfare) by matching appropriate buy and sell offers, referring not only to individual or bundled links, but also to end-to-end connections, i.e., paths. The structure of the communication network resources offered for sale is represented by multigraph $(V,\mathscr{E})$, where V is a set of network nodes (locations) and $\mathscr{E}$ denotes a set of (directed) network links, i.e., bandwidth resources offered for sale. The sell offer concerning link $e \in \mathscr{E}$ is described by $[y_e, S_e]$, where y_e is the maximal capacity the seller is willing to sell and S_e is the minimal acceptable unit price of bandwidth resource e. The assignment between sell offers and network nodes can be expressed by the incidence matrix $[a_{ve}]$, where $a_{ve} = 1$ if offer e originates in node v, -1 if e terminates in node v, and 0 otherwise.

The set of buy offers is denoted by $\mathscr{D}$. Every buy offer concerns a point-to-point bandwidth connection between a pair of specified locations in the network. The connections are unidirectional, i.e., they have source and sink nodes. The source node for buy offer $d \in \mathscr{D}$ is denoted by s_d and the sink node by t_d. There may be many demanded connections between a pair (s_d, t_d), hence $(V,\mathscr{D})$ is a multigraph. A buy offer $d \in \mathscr{D}$, concerning end-to-end connection, is described as $[h_d, E_d]$, where h_d is the maximal bandwidth capacity the buyer wants to purchase and E_d is the maximal acceptable unit price of bandwidth.

In basic version of the model it is assumed that offers can be realized partially; x_d is the variable of volume realization for buy offer d, x_e is the variable of volume realization for sell offer e. Variable x_{ed} denotes the bandwidth capacity allocated to sell offer e to serve buy offer d. The BCBT clearing model can be formulated as the following mathematical linear program:

$$\max\left(\sum_{d\in\mathscr{D}} E_d x_d - \sum_{e\in\mathscr{E}} S_e x_e\right), \tag{27}$$

s.t.

$$0 \leqslant x_d \leqslant h_d, \quad \forall d \in \mathscr{D}, \tag{28}$$

$$0 \leqslant x_e \leqslant y_e, \quad \forall e \in \mathscr{E}, \tag{29}$$

$$\sum_{d\in D} x_{ed} \leqslant x_e, \quad \forall e \in \mathscr{E}, \tag{30}$$

$$0 \leqslant x_{ed}, \quad \forall e \in \mathscr{E}, \forall d \in \mathscr{D}, \tag{31}$$

$$\sum_{e\in\mathscr{E}} a_{ve} x_{ed} = \begin{cases} x_d & v = s_d \\ 0 & v \neq s_d, t_d \\ -x_d & v = t_d \end{cases}, \forall v \in V, \forall d \in \mathscr{D}. \tag{32}$$

Constraints (28) and (29) set upper bounds on bandwidth realization of demand and supply offers. Constraint (30) sets the values of contracted nonnegative bandwidth volume on each link according to flows allocated to network paths. The flow constraints (32) are expressed by demand realization variables x_d.

The above model can be perceived as an AWDP-derived exchange (two-sided auction). But we can also directly use the model of capacitated multi-item AWDP with cost flow, i.e., MI|C|cost|AWDP. Maximal volumes and prices of sell offers need to be mapped onto link capacities and additional cost flow in the problem. Constraint (32) is equivalent to the constraint (15), constraints (29) and (30) are equivalent to constraints (17)–(18), and constraint (30) is equivalent to (16). Inequality sign in (30) is introduced only due to numerical properties and at optimal solution this constraint is satisfied at equality. According to our findings, the resulting problem is tractable, but as far as integer case is considered, it becomes NP-complete. Thus, for instance in [30], some algorithmic technics are proposed to deal with its computational complexity.

Contrary to other models of telecommunication resource auction, in the BCBT model, the buy offer d does not have to specify explicitly a bundle of inter-node links that realize the best connection between selected nodes. Therefore, the buyer does not have to guess which links to choose to obtain the allocation of demanded capacity. It is the decision of the model that allocates the most efficient links to connections. This flexibility is achieved thanks to AWDP-derived model.

The basic BCBT model can be extended in several ways, in order to take into account specific, real-world trade requirements. A variety of more general exchange models covers different market needs and requirements, such as capacity modularity [31], bidirectional bandwidth trading [32], or atomicity. These cases lead to some variants of AWDP-derived problems, including integer/binary offers, remodeling a flow network model into the case of directed flow. The complexity of each of these models is crucial for practical applications. Therefore, the knowledge of computational complexity of underlying problems is useful in determining promising directions of model development. It points out which functional requirements can be relatively easily included and which ones are hard.

5.2. Electrical energy balancing market

Electrical energy markets very well illustrate the complexity of the infrastructure market. Market balancing is being obtained by multi-step processes of the multi-commodity trade. A market operator is responsible for running a complex process that leads to demand and supply balance at every moment of time. Usually, this process is a sequence of auctions. On a single auction the operator needs to acquire an energy provided by suppliers and required to balance the system. There is one buyer, the system operator, that must acquire certain volumes of energy at given nodes of the power grid. There are many sellers of energy, who inject sold energy into the power grid at given nodes. Decisions arising from the auction models refer to the levels of generations and loads at each node of power grid. An energy is injected and taken at given nodes and it requires transmission network resources to be delivered and committed. The auction mechanism must provide the solution that guarantees feasibility of energy flow in the power grid. Therefore, the problem is equivalent to the reversed MI|C|cap|VWDP. However, there is an important feature that distinguishes an electricity market from the telecommunication bandwidth trade. It is the need for taking into account the physical properties of the power flow. While in the BCBT-derived models, both based on AWDP and VWDP, the commodity flow in the network was a result of optimization of some economic measure, on electricity auction the power flow comes from nonlinear electrical laws, including the Kirchoff laws.

At different stages of balancing process, the models of commodity flows are taken into account with different accuracy. For the auctions carried out well ahead of delivery time the limitations of the network may be in the form of restrictions on the divergence of individual nodes. For shorter horizons, the linearized models are usually considered. Typically, the power flow is incorporated into the Winner Determination Problem by means of so-called Power Transfer Distribution Factors (PTDFs). Assuming some referenced node in the network, PTDF is defined for a pair of node-line and it denotes the MW change in the power flow at given line when additional (relatively to some assumed flow) 1 MW is transferred from the given node of the network to the referenced node. Then, in the auction models, the network restrictions take the following form of constraints:

$$-\overline{P}_e \leqslant \sum_{v \in V} PTDF_{ve} P_v \leqslant \overline{P}_e \qquad \forall e \in E, \tag{33}$$

where $\overline{P}_e$ is maximal admissible load of the transmission line e and P_v denotes the divergence in the node v. Divergence at node v is a result of solving the auction model (energy injected and received at the node v) similar to the reversed VWDP model. The following model is an example of typical short-time balancing model:

$$\min_{d,p} [\sum_{j \in J} s_j p_j], \tag{34}$$

s.t.

$$\sum_{v \in V} P_v = 0, \tag{35}$$

$$P_v = \sum_{j \in J_v} p_j - \sum_{m \in M_v} d_m \qquad \forall_{v \in V}, \tag{36}$$

$$-\overline{P}_e \leqslant \sum_{v \in V} PTDF_{ve} P_v \leqslant \overline{P}_e \qquad \forall_{e \in E}, \tag{37}$$

$$0 \leqslant p_j \leqslant p_j^{\max} \qquad \forall_{j \in J}, \tag{38}$$

$$0 \leqslant d_m \leqslant d_m^{\max} \qquad \forall_{m \in M}, \tag{39}$$

where sets J and M denote the sets of generation units and demand respectively, and $J_v \subseteq J$ and $M_v \subseteq M$ are subsets of generators and demand respectively, restricted to node v. Decision variables d_m and p_j are accepted volumes of bought energy at node related to demand $m \in M$ and sell offer $j \in J$ respectively. Unit cost of energy for generator j is s_j. Feasible volumes of offers are limited by $p_j^{\max}$ for sell offer j and by $d_m^{\max}$ for buy offer m. This particular model is clearly the security-constrained reversed VWDP-like problem. Constraint (37) makes the problem differ from the clear reversed VWDP, since it is required to provide security-constrained solution. Nonetheless, all our findings for clear VWDP or reversed VWDP, can be applied also for this problem. Particulary, they can indicate potential changes in the model that would make the problem hard.

5.3. Virtual private network

Virtual Private Network (VPN) auctions are an interesting example of applications of both subclasses of the NWDP, the VWDP and the AWDP, to solve a practical problem. A family of models dedicated to the trade of telecommunication resources for the VPN is another example of models derived from the BCBT. In this problem a seller offers bandwidth for telecommunication links. Buyers want to acquire the bandwidths, that will be proper to organize VPN for their purposes. In the literature, there are two approaches to express the needs of VPN users. In the pipe-model a user needs to specify the bandwidth requirement between any two endpoints [33]. The requirements for bandwidth connections are determined between each pair of nodes belonging to the VPN and resulting model is a AWDP-like one. However, if the telecommunication network consists of a large number of nodes and number of endpoints increases, the routing uncertainty is growing and the pipe-like approach becomes inconvenient for the auctioneers. Number of pairs for which a customer needs to specify an offer increases rapidly. To solve this problem, the hose model has been proposed. In the hose model a user needs to specify the amount of traffic that can be sent to and received from the backbone network at each endpoint [33]. A user must only provide required in- and out- flow at vertices he or she is interested in. The hose model simply leads to VWDP-like problem. Let us notice, that the hose formulation suffers from uncertainty issues, and there is no superior solution for VPN problem.

As we discussed before, VWDP and AWDP are not equivalent. Therefore, derived functional features of the VPN auctions are different and they are the subject of undergoing researches. The auction models for both cases were introduced in [33], where detailed models can be found.

6. Summary

In this paper we defined the Network Winner Determination Problem, which is a new subclass of the Winner Determination Problems, such that the winning offers induce a commodity flow in a certain flow network. We focused on two general problems in the NWDP class: the Arc- and Vertex-oriented Winner Determination Problems. These problems can be expressed in terms of combinatorial auctions, however, the AWDP and the VWDP formulations significantly facilitate expressing the agent preferences. Figure 12 summarizes our findings in field of computational complexity of various variants of AWDP/VWDP problems. Even though, the sharp edge of tractability is generally designated by binary/integer offers, it turns out that uncapacitated integer VWDP is tractable. More over, some special structures of offers and flow network can make the problem easy, even if in general it is NP-hard problem. This includes hierarchical structure of the bids, bis for consecutive items, among others. One of the most interesting case is when the flow network is a tree, and offers may only cover paths along the branches connecting leafs with root of the tree.

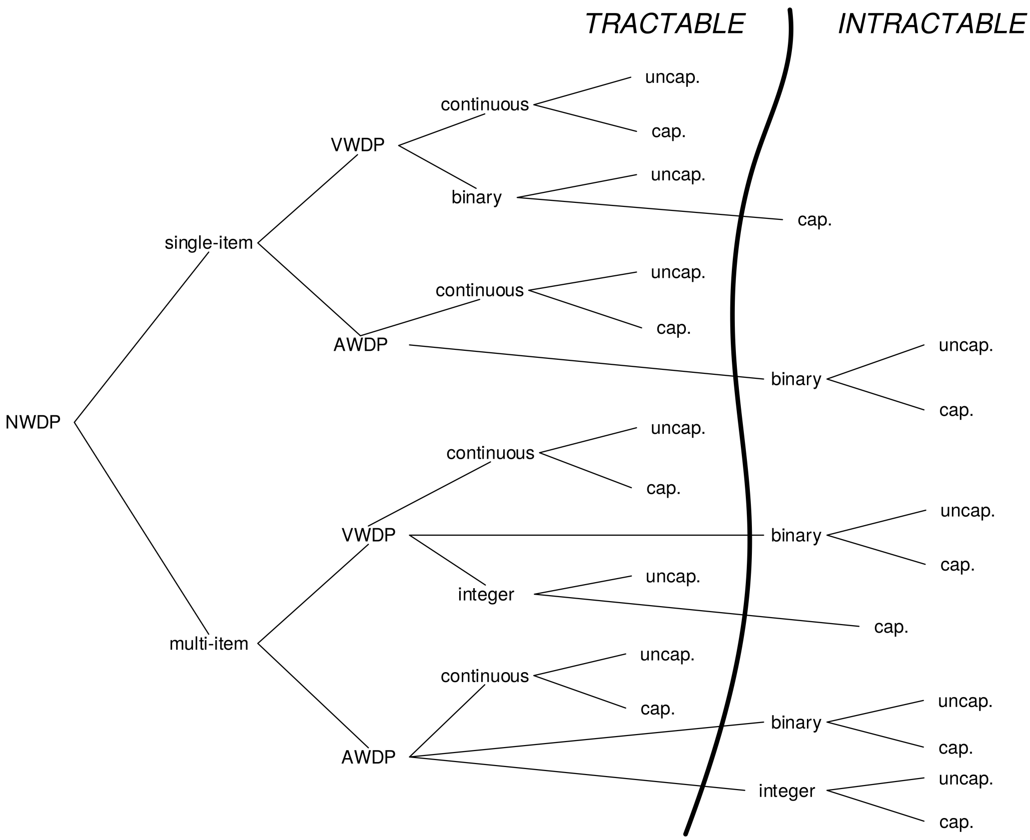

Figure 12: Variants of NWDP and the edge of tractability

Systematic approach to network auctions that we proposed, and our findings concerning the behaviour of different variants in terms of computational complexity, support designing the network auction market mechanisms. A designer may not only use VWDP- or AWDP-derived auction setting instead of tough combinatorial auction, but he or she is also aware of complexity issues, when the problem drifts into some variants recognized in this paper. Due to the abstract model, that can be applied in different infrastructure economies, the knowledge in field of auction design can be easier transferred to other fields of economy. We presented two applications areas, telecommunications and energy sectors, providing specific real-world problems.

We have also created a new area for possible future research. We have left opened interesting question if there exists other structures, that make certain variants of the problem easy. Our preliminary results in security constrained auctions are presented in [6].

References

[1] N. Nisan: Bidding and allocation in combinatorial auctions. *Proceedings of ACM Conference on Electronic Commerce*, (2000), 1–12.

[2] T. Sandholm, S. Suri, A. Gilpin and D. Levine: Winner determination in combinatorial auction generalizations. *Proceedings of the first international joint conference on Autonomous agents and multiagent systems: part 1*, AAMAS '02, (New York, NY, USA), (2002), 69–76, ACM.

[3] T. Sandholm and S. Suri: Side constraints and non-price attributes in markets. *Games and Economic Behavior*, **55**(2) (2006), 321–330. Mini Special Issue: Electronic Market Design.

[4] M. Kaleta and T. Traczyk, eds.: *Modeling Multi-commodity Trade: Information Exchange Methods*. Advances in Intelligent and Soft Computing, Springer, 2012.

[5] T. Sandholm: Algorithm for optimal winner determination in combinatorial auctions. *Artificial Intelligence*, **135**(1–2) (2002), 1–54.

[6] M. Kaleta: Security constrained network winner determination problem, *Automatyzacja procesów dyskretnych. Teoria i zastosowania*, A. Świerniak and J. Krystek, eds., pp. 111–118, Wydawnictwo Pracowni Komputerowej Jacka Skalmierskiego, 2012.

[7] M. H. Rothkopf, A. Pekec and R. M. Harstad: Computationally manageable combinational auctions. *Manage. Sci.*, **44** (1998), 1131–1147, Aug.

[8] D. Lehman, R. Müller and T. Sandholm: The winner determination problem. *Combinatorial Auctions*, P. Cramton, Y. Shoham, and R. Steinberg, eds., ch. 12, The MIT Press, 2006.

[9] N. Nisan: Bidding languages. *Combinatorial Auctions*, P. Cramton, Y. Shoham, and R. Steinberg, eds., ch. 9, The MIT Press, 2006.

[10] M. KALETA: Bidding languages for continuous auctions. *New Trends in Databases and Information Systems*, (2013), 211–220, Springer Berlin Heidelberg.

[11] R. MÜLLER: Tractable cases of the winner determination problem. *Combinatorial Auctions*, P. Cramton, Y. Shoham, and R. Steinberg, eds., ch. 13, The MIT Press, 2006.

[12] S. DE VRIES and R. V. RAKESH: Combinational auctions: A survey. *Informs Journal on Computing*, **15**(3) (2003), 284–309.

[13] V. CONITZER, J. DERRYBERRY and J. SANDHOLM: Combinatorial auctions with structured item graphs. *Proceedings of the National Conference on Artificial Intelligence (AAAI)*, (2004), 212–217, The AAAI Press.

[14] G. GOTTLOB and G. GRECO: On the complexity of combinatorial auctions: Structured item graphs and hypertree decomposition. *Proceedings of the 8th ACM Conference on Electronic Commerce*, EC '07, (New York, NY, USA), (2007), 152–161, ACM.

[15] T. SANDHOLM and S. SURI: Side constraints and non-price attributes in markets. *Games and Economic Behavior*, **55**(2) (2006), 321–330, Mini Special Issue: Electronic Market Design.

[16] M. BICHLER, J. R. KALAGNANAM and H. S. LEE: Reco: Representation and evaluation of configurable offers. In *Computational Modeling and Problem Solving in the Networked World: Interfaces in Computer Science and Operations Research*, H. K. Bhargava and N. Ye, eds., (2003), 235–258, Boston, MA: Springer US.

[17] A. DAVENPORT and J. KALAGNANAM: Price negotiations for procurement of direct inputs. Tech. Rep. RC 22078, IBM, May 2001.

[18] M. TENNENHOLTZ: Tractable combinatorial auctions and b-matching. *Artificial Intelligence*, **140**(1) (2002), 231–243.

[19] A. KOTHARI, T. SANDHOLM and S. SURI: Solving combinatorial exchanges: optimality via a few partial bids. *Proceedings of the Third International Joint Conference on Autonomous Agents and Multiagent Systems*, AAMAS (2004), 1418–1419, July.

[20] D. LOKER and K. LARSON: Parameterizing the winner determination problem for combinatorial auctions. *9th International Conference on Autonomous Agents and Multiagent Systems* (AAMAS 2010), Toronto, Canada, May 10-14, Volume 1-3, (2010), 1483–1484.

[21] T. SANDHOLM, S. SURI, A. GILPIN and D. LEVINE: Winner determination in combinatorial auction generalizations. *AAMAS '02: Proceedings of the first international joint conference on Autonomous agents and multiagent systems*, (2002), 69–76, ACM.

[22] S. BIKHCHANDANI, S. DE VRIES, J. SCHUMMER and R. V. VOHRA: Linear programming and Vickrey auctions. *Mathematics of the Internet: E-Auction and Markets*, vol. 127 of *IMA Volumes in Mathematics and its Applications*, pp. 75–116, Springer, 2002.

[23] J. HERSHBERGER and S. SURI: Vickrey pricing in network routing: Fast payment computation, *In Proc. of the 42nd IEEE Symposium on Foundations of Computer Science*, (2001), 252–259.

[24] P. CRAMTON, Y. SHOHAM and R. STEINBERG: *Combinatorial Auctions*. The MIT Press, 2006.

[25] L. G. KHACHIYAN: A polynomial algorithm in linear programming (in Russian). *Doklady Akademii Nauk SSSR*, **244** (1979), 1093–1096. English translation: Soviet Mathematics Doklady, **20**(1), (1979), 191–194.

[26] S. EVEN, A. ITAI and A. SHAMIR: On the complexity of time table and multi-commodity flow problems. *16th Annual Symposium on Foundations of Computer Science (sfcs 1975)*, (1975), 184–193, Oct.

[27] A. SCHRIJVER: *Combinatorial Optimization – Polyhedra and Efficiency*. Springer, 2003.

[28] W. STAŃCZUK, J. LUBACZ and E. TOCZYŁOWSKI: Trading transport resources of communication networks on bandwidth exchanges. *Proceedings 46th FITCE Congress – The Broadband Way to the Future*, (2007), 222–444.

[29] W. STAŃCZUK, J. LUBACZ and E. TOCZYŁOWSKI: Trading links and paths on a communication bandwidth market. *Journal of Universal Computer Science*, **14** (2008), 642–653, Mar.

[30] P. PAŁKA, K. KOŁTYŚ, E. TOCZYŁOWSKI and I. ŻÓŁTOWSKA: Model for balancing aggregated communication bandwidth resources. *Journal of Telecommunication and Information Technology*, **3** (2009).

[31] K. KOŁTYŚ, P. PAŁKA, E. TOCZYŁOWSKI and I. ŻÓŁTOWSKA: Multicommodity auction model for indivisible network resource allocation. *Journal of Telecommunication and Information Technology*, **4** (2008), 60–66.

[32] P. Kacprzak, M. Kaleta, K. Kołtyś, P. Pałka, K. Pieńkosz, E. Toczyłowski and I. Żółtowska: Multicommodity exchange model for trading bandwidth in undirected networks. *Proceedings of 14th International Telecommunications Network Strategy and Planning Symposium NETWORKS* (2010), 180–184.

[33] K. Kołtyś, K. Pieńkosz and E. Toczyłowski: A flexible auction model for virtual private networks. *Lecture Notes in Computer Science*, **6641** (2011), 97–108.