

Hybrid MSV-MGARCH Models – General Remarks and the GMSF-SBEKK Specification

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Submitted: 4.09.2016, Accepted: 6.11.2016

Abstract

The first so-called hybrid MSV-MGARCH models were characterized by the conditional covariance matrix that was a product of a univariate latent process and a matrix with a simple MGARCH structure (Engle's DCC or scalar BEKK). The aim was to parsimoniously describe volatility of a large group of assets. The proposed hybrid models, similarly as pure MSV specifications (and other models based on latent processes), required the Bayesian approach equipped with efficient MCMC simulation tools. The numerical effort has paid – the hybrid models seem particularly useful due to their good fit and ability to jointly cope with large portfolios. In particular, the simplest hybrid, now called the MSF-SBEKK model, has been successfully used in many applications. However, one latent process may be insufficient in the case of a highly heterogeneous portfolio. Thus, in this study we discuss a general hybrid MSV-MGARCH model structure, showing its basic characteristics that explain greater flexibility of such hybrid structure with respect to the corresponding MGARCH class. From the empirical perspective, we advocate the GMSF-SBEKK specification, which uses as many latent processes as there are relatively homogeneous groups of assets. We present full Bayesian inference for such models, with the use of an efficient MCMC simulation strategy. The approach is used to jointly model volatility on very different markets. Joint modelling is formally compared to individual modelling of volatility on each market.

Keywords: Bayesian econometrics, multivariate volatility models, MGARCH processes, MSV processes, financial markets, commodity markets

JEL Classification: C11, C32, C51, C58

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1 Introduction

In recent years there has been growing interest in the analysis of not only financial markets, but also commodity markets - in particular oil and natural gas, gold, silver and copper, and other raw materials crucial for economic development; see Marimoutou, Raggad and Trabelsi (2009). There are studies of interrelations and shock transmissions among these markets; see Vo (2011). The importance of joint analysis of many different markets lies in globalization and the integration of commodity and financial markets. Also, investors seek new opportunities to make profits and diversify their risk. Thus, it is more and more important to propose new tools that enable simple (but appropriate and efficient) modelling of prices and returns on many markets, in order to describe and measure their volatilities and relationships.

Most of n -variate volatility models used in financial econometrics belong to either the MGARCH or MSV (multivariate stochastic volatility or variance) class, or are based on copulas; see, e.g., Bauwens, Laurent and Rombouts (2006), Tsay (2005), Pajor (2010), Jondeau and Rockinger (2006), Patton (2006, 2012), Doman (2011), Hafner and Manner (2012), and Almeida and Czado (2012). Only some of them are simple, practical tools for analysing large portfolios, e.g. the Scalar BEKK (SBEKK) model and, in particular, the Dynamic Conditional Correlation (DCC) structure of Engle (2002). Both cases represent the MGARCH class and in each we can use variance targeting and approximate methods to estimate the parameter vector of dimension growing with the portfolio size; the remaining parameters, requiring more numerical effort, form a vector of fixed dimension irrespective of the number of assets. Latent AR(1) processes, used in the MSV class to describe volatility, are very efficient in dealing with outliers and, thus, in modelling tail behaviour. Since such modelling is crucial for any risk assessment, the MSV class should be kept under consideration in spite of the fact that MSV structures with many latent processes are too complicated to be practical in highly dimensional problems. Easier way of modelling was proposed by Osiewalski and Pajor (2007) through a hybrid model, based on Engle's DCC structure and the simplest MSV structure, the Multiplicative Stochastic Factor (MSF, or Stochastic Discount Factor, SDF) specification. However, the MSF-DCC (previously called SDF-DCC) model was still too complex and, thus, Osiewalski (2009) and Osiewalski and Pajor (2009) proposed the MSF-SBEKK hybrid models.

The simplest MSF-SBEKK model, based on a multivariate Gaussian white noise, proved quite flexible in modelling portfolios of high dimension; see Osiewalski and Pajor (2009, 2010), and Pajor and Osiewalski (2012). It also successfully competes with pure MGARCH and MSV specifications in terms of the marginal data density value, which is the natural Bayesian measure of model fit; see Pajor (2010). The MSF-SBEKK model has been already used to analyse the problem of missing observations in individual daily returns within multivariate framework as well as to examine the

presence of a long-run relation between two markets; see Osiewalski and Osiewalski (2012b, 2013).

The simplicity of the MSF-SBEKK specification is its clear virtue. However, it means that volatilities of all individual time series are driven by one common latent stochastic process. If we model different markets or a heterogeneous portfolio we would prefer to use as many latent processes as there are relatively homogeneous groups of markets or assets. This idea was adopted in some of our earlier works, see Osiewalski and Osiewalski (2012a, 2011), then gradually elaborated as a generalisation of the MSF-SBEKK model and discussed at many conferences. It was formally presented in Osiewalski (2015), where statistical and numerical details as well as new empirical illustrations were worked out. This paper is a distillation of our earlier works, which were written in Polish, with adding a more mature and holistic perspective on the hybrid models (that can rely on several latent processes).

In order to summarize the research on hybrid specifications, we aim in this paper at describing foundations and general features of MSV-MGARCH models, and we present the generalised version of the n -variate MSF-SBEKK specification - the n -variate GMSF-SBEKK model that uses k latent processes for k groups of assets or markets, where $k < n$. Our approach to statistical inference is fully Bayesian for both fundamental and practical reasons. As regards the latter, it is quite easy to analyse models with latent processes using Bayesian statistical framework equipped with MCMC simulation tools; see, e.g., Pajor (2010).

It is important to stress that the hybrid MSV-MGARCH specifications have been proposed on purely empirical grounds, as *ad hoc* extensions that are flexible and well-fitted to the analysed data. Any formal properties of the underlying hybrid multivariate stochastic processes (like covariance stationarity) have not been proven yet. In this paper we do not present any such results either. However, we discuss some deeper links between MSV-MGARCH and pure MGARCH or MSV models, in particular through their forms of the conditional n -variate sampling density given past observations. For example, we explain the source of additional flexibility of the MSF-MGARCH counterpart of a MGARCH model with the same form of the conditional sampling density. We also show that MSF-MGARCH models based on the n -variate Gaussian white noise keep the ellipsoidal form of this density (like, e.g., t -MGARCH models), while GMSF-MGARCH models with $k > 2$ may lead to non-ellipsoidal density contours.

In this paper we focus not only on the distributional form of the conditional sampling density, but also on the specification of the conditional mean. Thus, the traditional VAR(1) form for n logarithmic return rates is extended to competing VAR(2) forms (for n logged prices) with the reduced rank long-run multiplier matrix. This enables considering relations among prices themselves, and not only between returns on different markets.

The paper is organised as follows. The next section (Section 2) is devoted to the presentation and general discussion of MSV-MGARCH models and the construction of

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the GMSF-MGARCH specifications. In Section 3 we consider the Bayesian VAR(2)-GMSF-SBEKK model framework as well as the general form of the posterior density function and its conditionals that enable to construct appropriate Gibbs samplers with Metropolis-Hastings steps. In Section 4 we present the empirical example, where six time series from three groups of markets are modelled using different MSF-SBEKK and GMSF-SBEKK specifications; the usefulness of the latter models is clearly illustrated. Concluding remarks are grouped in Section 5.

2 MSV-MGARCH models: a general form and the GMSF-MGARCH case

In this paper we focus on a new class of models for a n -variate time series of asset prices. Assume that s_t is an $(n \times 1)$ vector of (multiplied by 100) natural logarithms of n asset prices, observed at time t , and $r_t = s_t - s_{t-1}$ is the corresponding vector of logarithmic return rates in percentage points. Let $x_t = s_t - s_0$; now n prices are measured relatively to the period $t = 0$, so they are of similar orders of magnitude, but the return rates remain unchanged: $r_t = x_t - x_{t-1}$. Consider the following VAR(2) model for x_t :

$$r_t = \lambda + \Lambda r_{t-1} + \Pi x_{t-1} + \varepsilon_t \quad (t = 1, 2, \dots, T, \dots, T + h) \quad (1)$$

where $\Pi = 0$ corresponds to the usual (standard) VAR(1) specification for r_t , but non-zero reduced rank Π matrices are also considered here. The $(n \times 1)$ error term ε_t has the hybrid MSV-MGARCH form iff

$$\varepsilon_t = G_t^{\frac{1}{2}} H_t^{\frac{1}{2}} \xi_t, \quad (2)$$

where

1. $\{\xi_t\}$ is a strict n -variate white noise with unit covariance matrix, i.e. $\xi_t \sim iid^{(n)}(0, I_n)$;
2. G_t and H_t are square matrices of order n , symmetric and positive definite for all t in the observation period ($t = 1, 2, \dots, T$) as well as in the forecasting period ($t = T + 1, \dots, T + h$);
3. $H_t = f_{MGARCH}(\varepsilon_{t-1}, \varepsilon_{t-2}, \dots)$ is a non-constant function of the past of ε_t , corresponding to the conditional covariance matrix of some MGARCH specification;
4. $G_t = f_{MSV}(g_t)$ is a non-constant function of a k -variate non-trivial (i.e., assuming dependence over time) unobserved stochastic process $\{g_t\}$, independent of $\{\xi_t\}$.

Under these assumptions, the conditional distribution of r_t – given the past of x_t , denoted by ψ_{t-1} , and the k -dimensional current random variable g_t – is determined by the particular form of $D^{(n)}(0, I_n)$, and has mean vector $\mu_t = \lambda + \Lambda r_{t-1} + \Pi x_{t-1}$ and covariance matrix $\Omega_t = G_t^{\frac{1}{2}} H_t G_t^{\frac{1}{2}}$. The conditional covariance matrix Ω_t depends on current latent variables g_{t1}, \dots, g_{tk} (through $G_t^{\frac{1}{2}}$) as well as on the past of observations, through $H_t = f_{MGARCH}(r_{t-1} - \mu_{t-1}, r_{t-2} - \mu_{t-2}, \dots)$. Thus Ω_t has a hybrid form, joining basic features of pure MSV and pure MGARCH specifications.

In order to explain the specific conditions imposed through assumptions 3 and 4, let us consider their violation and its consequences.

- (i) Constant H_t and G_t ($H_t = H$, $G_t = G$) lead to ε_t that are *iid* with mean vector zero and covariance matrix $\Omega = G^{\frac{1}{2}} H G^{\frac{1}{2}}$; neither ARCH nor SV structure is present; the VAR(2) specification with *iid* errors cannot model (e.g.) volatility clustering, even if heavy-tailed error distributions (like Student t or α -stable) are employed.
- (ii) Constant H_t ($H_t = H$) together with time-varying G_t , a function of latent variables g_t that are independent over time, lead to essentially the same case as in (i); neither ARCH nor SV structure is present. Independence of both ξ_t and g_t over time results in independence of ε_t over time – only its distribution may be time-varying and belongs to a different class than the distribution of ξ_t . As an example assume that $\xi_t \sim iidN^{(n)}(0, I_n)$, $k = 1$, $G_t = g_t I_n$ and g_t^{-1} are independent gamma variables with mean 1 and variance $\frac{2}{\nu}$; then $\Omega_t = g_t H$ and ε_t are independent n -variate Student t variables with ν degrees of freedom, zero location vector and precision matrix H^{-1} . Remind that each n -variate Student t variable is a continuous scale mixture of n -variate normal variables. Let us mention another member of this scale mixture family, obtained under the assumption that $\ln g_t \sim iidN(0, \sigma^2)$; the log-normal scale mixture of n -variate normal distributions is interesting as it naturally appears in the context of MSV models.
- (iii) Non-constant $H_t = f_{MGARCH}(\varepsilon_{t-1}, \varepsilon_{t-2}, \dots)$ and constant G_t ($G_t = G$) lead to an MGARCH specification with the conditional distribution from the family determined by ξ_t and the conditional covariance matrix $G^{\frac{1}{2}} H_t G^{\frac{1}{2}}$.
- (iv) Non-constant H_t and G_t , with G_t depending on g_t , but g_t independent over time, lead to the MGARCH model with the conditional distribution modified by the presence of latent g_t ; there is no SV structure as there is no dependence of latent variables over time (the latent process $\{g_t\}$ is trivial). For example, if $\xi_t \sim iidN^{(n)}(0, I_n)$, $k = 1$, $G_t = g_t I_n$ and g_t^{-1} are independent gamma variables with mean 1 and variance $\frac{2}{\nu}$, then $\varepsilon_t = H_t^{\frac{1}{2}} \xi_t^*$ and $\xi_t^* = \sqrt{g_t} \xi_t$ has the Student t n -variate distribution with ν degrees of freedom, zero location vector and unit precision matrix. That is, in this example we obtain the pure MGARCH model with the Student t conditional distribution with covariance matrix $\frac{\nu}{\nu-2} H_t$.

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- (v) Constant H_t ($H_t = H$) and $G_t = f_{MSV}(g_t)$, with g_t dependent over time, lead to pure MSV models. The simplest specification, the so-called MSF (Multiplicative Stochastic Factor) model, amounts to assuming: $k = 1$, $G_t = g_t I_n$ and $\ln g_t = \varphi \ln g_{t-1} + \tau^{-1} \eta_t$, with $\tau \in \mathbb{R}_+$, $\varphi \in (-1, 1) \setminus \{0\}$, $\eta_t \sim iid N(0, 1)$ and $\eta_t \perp \xi_s$ for all (t, s) . If $\xi_t \sim iid N^{(n)}(0, I_n)$, the conditional distribution of ε_t (given its past and g_t) is n -variate normal with mean vector zero and covariance matrix $g_t H$. Since $\{\ln g_t\}$ is a Gaussian AR(1) stationary and causal process, the marginal distribution of each $\ln g_t$ is (assuming infinite past) $N(0, [\tau(1 - \varphi^2)]^{-1})$ and g_t is a log-normal variable. So we have $\varepsilon_t = H^{\frac{1}{2}} \tilde{\xi}_t$ with $\tilde{\xi}_t = \sqrt{g_t} \xi_t$, where ε_t are dependent over time due to the AR(1) structure of $\ln g_t$, but the marginal n -variate distribution of ε_t is always the same scale mixture of $N(0, g_t H)$ distributions with the appropriate log-normal distribution as the mixing one. The crucial difference between this particular MSV (namely MSF) specification and the case considered at the end of point (ii) lies in stochastic dependence of ε_t over time; independence appearing in (ii) corresponds to $\varphi = 0$, which is excluded in the proper MSV framework.

These considerations explain that we have formulated assumptions 3 and 4 in order to create truly hybrid MSV-MGARCH models, which would not reduce to either pure MSV or pure MGARCH or just independent error terms ε_t . The main goal of such hybrid modelling amounts to exploiting advantages of both model classes, while keeping H_t and G_t as simple as possible. A particularly simple MSV-MGARCH specification, the MSF-MGARCH structure corresponds to $G_t = g_t I_n$ ($k = 1$) and $\varepsilon_t = H_t^{\frac{1}{2}} \sqrt{g_t} \xi_t$ with $\{\ln g_t\}$ – a Gaussian AR(1) process as defined above in (v). Two simple MGARCH forms of H_t has been considered in this framework, the DCC structure of Engle (2002) and the simple scalar BEKK (SBEKK) structure, see Osiewalski and Pajor (2007, 2009), Osiewalski (2009), Pajor (2010). In particular, the MSF-SBEKK specification with conditional normality can be effectively applied in modelling large portfolios (of, e.g., $n = 50$ assets).

The MSF-MGARCH specification can be used to clarify the similarities and differences between MSV-MGARCH and pure MGARCH models. As we have already stressed in (iv), if the latent process $\{g_t\}$ is trivial – i.e., if $g_t \sim iid$ – then it only serves to make the tails of $p(r_t | \psi_{t-1}; \theta)$ heavier, but we stay within the class of pure MGARCH models. If, however, the latent process $\{g_t\}$ is non-trivial – i.e., if latent variables g_t are stochastically dependent – then we obtain the same effect of heavier tails of $p(r_t | \psi_{t-1}; \theta)$, but we also get an additional source of dependence within the observed time series and some extra parameters describing dependence in $\{g_t\}$, like φ in the MSF-MGARCH example in (v). This is the explanation of greater flexibility of hybrid models in comparison to MGARCH models.

Note that in order to define the exact hybrid extension of any MGARCH model with Student t conditional distribution, we should consider different latent processes than in the MSF specification. A latent process that suits the purpose has the form

$\ln g_t = \varphi \ln g_{t-1} - \ln \gamma_t$, where γ_t are independent gamma variables with mean 1 and variance $\frac{2}{\nu}$; if $\varphi = 0$ we are back in (iv) with conditionally t MGARCH model, but if $\varphi \neq 0$ we have its true hybrid MSV-MGARCH generalization. Although such a model can be an interesting competitor to the MSF-MGARCH structure, in this paper we do not go in this direction, leaving it to future research. Instead, we focus on hybrids with more latent processes.

The initial idea behind the MSF-MGARCH specification lies in using only one latent AR(1) process in order to add flexibility into volatility modelling with simple MGARCH structures. Note that the conditional covariance matrix in MSF-MGARCH models takes the form $g_t H_t$, where the latent process affects all the variances and covariances in the same way. Intuitively, it should be enough when the analysed portfolio consists of assets that are homogeneous as regards main sources of their volatility – e.g., they represent the same market or the same sector of an economy. If, however, the portfolio is built of k separate groups, each with n_i assets ($i = 1, \dots, k; n_1 + \dots + n_k = n$), then it seems reasonable to use k different latent processes $\{g_{t,i}\}$ ($i = 1, \dots, k$). Thus, the GMSF-MGARCH (Generalised MSF-MGARCH) hybrid model structure is defined by a block-diagonal form of G_t with k scalar blocks:

$$G_t = \begin{bmatrix} g_{t,1} I_{n_1} & & \\ & \ddots & \\ & & g_{t,k} I_{n_k} \end{bmatrix}, \quad \sum_{j=1}^k n_j = n, \quad (3)$$

where

$$\ln g_{t,i} = \phi_i \ln g_{t-1,i} + \sqrt{\tau_i^{-1}} \eta_{t,i}, \quad g_{0,i} = 0, \quad \phi_i \in (0, 1), \quad \tau_i > 0 \quad (4)$$

$$\eta_{t,i} \perp \eta_{s,j}, \quad t, s \in \{1, \dots, T\}, \quad i, j \in \{1, 2, \dots, k\}. \quad (5)$$

Independence holds, of course, except the case when $s = t$ and $i = j$ jointly.

The conditional covariance matrix of any GMSF-MGARCH specification takes the form

$$\begin{aligned} \Omega_t &= G_t^{\frac{1}{2}} H_t G_t^{\frac{1}{2}} = \\ &= \begin{bmatrix} g_{t,1} (H_t)_{1:n_1, 1:n_1} & \cdots & \sqrt{g_{t,1} g_{t,k}} (H_t)_{1:n_1, \left(\sum_{i=1}^{k-1} n_i + 1\right):n} \\ \cdots & \cdots & \cdots \\ \sqrt{g_{t,k} g_{t,1}} (H_t)_{\left(\sum_{i=1}^{k-1} n_i + 1\right):n, 1:n_1} & \cdots & g_{t,k} (H_t)_{\left(\sum_{i=1}^{k-1} n_i + 1\right):n, \left(\sum_{i=1}^{k-1} n_i + 1\right):n} \end{bmatrix}, \end{aligned} \quad (6)$$

where $M_{a:b,c:d}$ is a block of matrix M formed by its rows from a to b and columns c to d .

Imposing restrictions on the latent processes $g_{t,1} \equiv g_{t,2} \equiv \dots \equiv g_{t,k} \equiv g_t$ will reduce the model to the MSF-MGARCH one. Although the conditional covariances depend

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on the latent variables $g_{t,i}$, it is not the case for the conditional correlations, as they collapse to the form given in the H_t part of the specification:

$$\rho_{t,kl} = \frac{h_{t,kl}}{\sqrt{h_{t,kk}h_{t,ll}}}, \quad k, l = 1, \dots, n. \quad (7)$$

As regards distributional assumptions, we stay with conditional normality. That is, we assume that the $(n+k)$ -variate process $\{\zeta_t\}$, where $\zeta_t = [\xi_t' \eta_t']' = [\xi_t' \eta_{t,1} \dots \eta_{t,k}]'$ is a Gaussian white noise with unit covariance matrix: $\zeta_t \sim iid N^{(n+k)}(0, I_{n+k})$. Thus, the conditional distribution of r_t , given ψ_{t-1} and g_t , is $N^{(n)}(\mu_t, \Omega_t)$ with Ω_t in (6) and the density function

$$p(r_t | \psi_{t-1}, g_{t,1}, \dots, g_{t,k}; \theta) = f_N^n(r_t | \mu_t, \Omega_t) = (2\pi)^{-\frac{n}{2}} (\det \Omega_t)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \varepsilon_t' \Omega_t^{-1} \varepsilon_t\right), \quad (8)$$

where θ is a sufficient parametrization and $\varepsilon_t = r_t - \mu_t$. The density function of the conditional distribution of r_t given its own past alone is the following k -dimensional integral

$$p(r_t | \psi_{t-1}; \theta) = \int_{\mathbb{R}_+^k} f_N^n(\varepsilon_t | 0, \Omega_t) \prod_{i=1}^k f_{LN}(g_{t,i} | 0, [\tau_i (1 - \varphi_i^2)]^{-1}) dg_{t,1} \dots dg_{t,k}, \quad (9)$$

where $f_{LN}(\cdot | a, b)$ denotes the log-normal density function corresponding to the $N(a, b)$ distribution. That is, if $z \sim N(a, b)$ then the density function of the log-normal variable e^z is denoted as $f_{LN}(\cdot | a, b)$. In the special case of just one latent variable $g_{t,1}$, i.e. in the MSF-MGARCH model, $\Omega_t = g_{t,1} H_t$ and

$$p(r_t | \psi_{t-1}; \theta) = \int_0^\infty f_N^n(\varepsilon_t | 0, g_{t,1} H_t) f_{LN}(g_{t,1} | 0, [\tau_1 (1 - \varphi_1^2)]^{-1}) dg_{t,1} \quad (10)$$

Note that $p(r_t | \psi_{t-1}; \theta)$ is the density of a continuous scale mixture of n -variate normal distributions with the log-normal marginal distribution of the variance factor $g_{t,1}$ as the mixing distribution. Because $\ln g_{t,i}$ and $-\ln g_{t,i}$ have the same $N(0, [\tau_i (1 - \varphi_i^2)]^{-1})$ marginal distribution, the random precision factor $g_{t,1}^{-1}$ of $N^{(n)}(0, g_{t,1} H_t)$ has the same log-normal distribution as the random variance factor $g_{t,1}$. Thus we can compare (10) to the Student t class, which corresponds to gamma mixing distributions for the precision factor. Since scale mixtures of normals form a subclass of ellipsoidal distributions, the conditional distribution of r_t given ψ_{t-1} alone is ellipsoidal. By extending any conditionally normal MGARCH specification to the MSF-MGARCH model, we keep the original ellipsoidal structure of that MGARCH model, but make the tails of the conditional distribution heavier. However, when we assume some GMSF-MGARCH model with at least two latent processes ($k \geq 2$), the representation $\Omega_t = g_{t,1} \Omega_t^*$ leaves $k-1$ random variables $(\frac{g_{t,2}}{g_{t,1}}, \dots, \frac{g_{t,k}}{g_{t,1}})$ in Ω_t^* and

$p(r_t|\psi_{t-1};\theta)$ cannot be represented as a scale mixture of n -variable normal densities. Using GMSF-MGARCH specifications with $k \geq 2$ may significantly increase flexibility of the shape of the contours (isodensity lines) of $p(r_t|\psi_{t-1};\theta)$.

In spite of all these considerations, the conditional normality of our general GMSF-MGARCH framework may be seen as a restrictive assumption. Remind, however, that we condition not only on the past observations, but also on current latent variables, which – as it was explained above – lead to non-normal distributions with heavier tails on the level of $p(r_t|\psi_{t-1};\theta)$. By using simple hybrid structures with conditional normality we are already on a similar level of tail modelling as in e.g. MGARCH models with the conditional Student t distribution; the source of this property (and the price we pay for it in computations) lies in employing non-trivial latent processes. Obviously, we can go further and assume (within our hybrid structure) non-Gaussian ξ_t with heavy tails. In fact, Osiewalski (2015) considers the MSF-SBEKK model with n -variate Student t strict white noise $\{\xi_t\}$ and finds it a useful, but not crucial extension in his application. In this work we assume that $\{\xi_t\} \sim iidN(0, I_n)$.

3 Bayesian VAR(2)-GMSF-SBEKK models

In Bayesian statistical analysis of GMSF-MGARCH models we focus on the simplest possible choice for the MGARCH structure of H_t , namely the SBEKK one. So we assume

$$H_t = (1 - \beta - \gamma)A + \beta\varepsilon_{t-1}\varepsilon'_{t-1} + \gamma H_{t-1}, \quad (11)$$

where A is a free symmetric positive definite square matrix of order n , $\beta > 0$, $0 < \gamma < 1$, $\beta + \gamma < 1$ and $H_0 = h_0 I_n$, with $h_0 > 0$ either assumed arbitrarily or treated as one more free parameter of the SBEKK structure in (11). Our sampling model for T observations and $T \cdot k$ latent variables has a hierarchical structure described by the following density functions

$$p\left(r, g^{(1)}, \dots, g^{(k)}|\psi_0, \theta\right) = \prod_{t=1}^T \left[p(r_t|\psi_{t-1}, g_{t,1}, \dots, g_{t,k}, \theta) \prod_{i=1}^k p(g_{t,i}|g_{t-1,i}, \theta) \right], \quad (12)$$

where $r = [r_1 \dots r_T]$, $g^{(i)} = [g_{1,i} \dots g_{T,i}]'$, ψ_0 consists of initial observations on prices (x_{-2}, x_{-1}, x_0) ,

$$p(r_t|\psi_{t-1}, g_{t,1}, \dots, g_{t,k}, \theta) = f_N^n(r_t|\mu_t, \Omega_t), \quad (13)$$

$$p(g_{t,i}|g_{t-1,i}, \theta) = g_{t,i}^{-1} f_N^1(\ln g_{t,i}|\varphi_i \ln g_{t-1,i}, \tau_i^{-1}), \quad (14)$$

$g_{0,i} = 1$ (arbitrarily) and θ groups all the parameters in μ_t and Ω_t together with k pairs (φ_i, τ_i) .

Note that in the conditional mean of r_t , $\mu_t = \lambda + \Lambda r_{t-1} + \Pi x_{t-1}$ appearing in (1) and (13), we can assume $\Pi = 0$ (as usual) or introduce an error correction term (ECT) through a non-zero reduced rank matrix Π . It seems natural to treat such

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reduced rank Π as representing one or more co-integration relationships in levels of x_t . However, any white noise conditions for the process $\{\varepsilon_t\}$ in (1) and (2) have not been derived yet, so formally we cannot say that the variables $x_{t,1}, \dots, x_{t,n}$ are individually integrated and jointly co-integrated. Nevertheless, using reduced rank Π (and ECT) in the conditional mean μ_t enlarges the set of specifications that may appear useful in empirical modelling.

In this section, in order to cover Bayesian models with different assumptions on Π , we divide θ into $\theta^{(1)}$ and $\theta^{(2)}$:

$$\theta^{(1)} = \left(\lambda', (\text{vec}\Lambda) ', (\text{vech}A^{-1}) ', \beta, \gamma, h_0, \phi_1, \dots, \phi_k, \tau_1, \dots, \tau_k \right)',$$

which groups all the parameters except Π (i.e., all that appear in the standard case with $\Pi = 0$) and $\theta^{(2)}$, which denotes an appropriate sufficient parametrization of reduced rank Π . We also assume that $\Pi = 0$ can be obtained using a particular value of $\theta^{(2)}$.

The Bayesian VAR-GMSF-SBEKK model is represented by the product of the sampling density function $p(r, g^{(1)}, \dots, g^{(k)} | \psi_0, \theta)$ in (12), which keeps the same general structure irrespective of the particular form of Π , and the prior density function

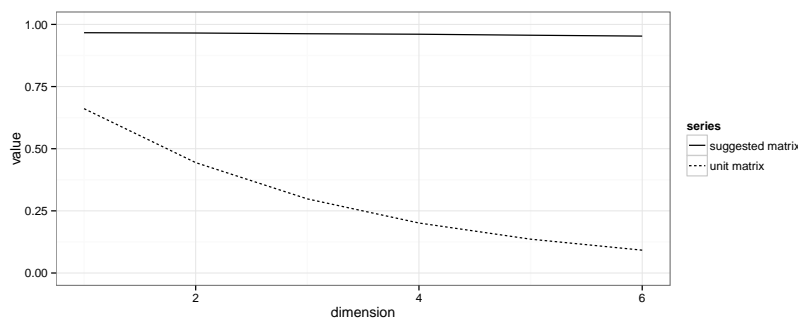
$$p(\theta) = p(\theta^{(1)})p(\theta^{(2)}), \quad (15)$$

where

$$p(\theta^{(1)}) = p(\lambda)p(\text{vec}\Lambda)p(A^{-1})p(\beta, \gamma)p(h_0) \prod_{i=1}^k [p(\phi_i)p(\tau_i)], \quad (16)$$

and the form of $p(\theta^{(2)})$ will be presented at the end of this section. In (16) we carefully choose rather diffuse prior densities for particular groups of parameters:

- i) $p(\lambda) = f_N^n(\lambda | 0, \frac{1}{9}I_n)$ – n -dimensional normal distribution with zero mean and $\frac{1}{9}I_n$ as the covariance matrix. Thanks to such construct, we can assure $P(|\lambda_i| < 1)$ is high enough (0.9973). This is a reflection of a prior belief that the unconditional expectation ($E(r_t) = (I_n - \Lambda)^{-1}\lambda$ when $\Pi = 0$) of each portfolio component should exceed (in absolute value) one percentage point with a low probability. Given the prior distribution of Λ as below, we can simulate the prior probability $P(E(r_t) \in [-1, 1]^n)$, which is shown in Figure 1. As one can see, the probability decreases monotonically in both cases, however, very slowly for the covariance matrix $\frac{1}{9}I_n$. In the case of $n = 6$ and the unit covariance matrix, it drops to as low as 0.091 – which is not reasonable.
- ii) $p(\text{vec}\Lambda) \propto f_N^{n^2}(\text{vec}\Lambda | 0, \frac{1}{9n}I_{n^2}) \mathbf{1}_{\{M: \rho(M) < 1\}}(\Lambda)$ – multivariate normal distribution truncated by the VAR stability restriction when $\Pi = 0$ (i.e. all eigenvalues of the matrix Λ lie within a unit ball); $\rho(M)$ denotes the spectral radius of matrix M and $\mathbf{1}_S(x)$ stands for the indicator function of the set S : $\mathbf{1}_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$.

Figure 1: Probability *a priori* of the condition $E(r_t) \in [-1, 1]^n$ 

Thus, in this prior distribution we assume that the covariances of the vector $\text{vec}\Lambda$ decrease when the portfolio dimension increases. As a result, the stability restriction does not effectively truncate the prior distribution. In the case of the unit covariance structure, probability *a priori* of the restriction $\rho(\Lambda) < 1$ is numerically close to zero when $n \geq 5$, see Figure 2.

- iii) $p(A^{-1}) = f_{Wishart}(A^{-1}|I_n, n+2)$ – Wishart distribution with a shape parameter I_n and $n+2$ degrees of freedom,
- iv) $p(\beta, \gamma) \propto \mathbf{1}_{\{(x,y) \in [0,1]^2: x+y < 1\}}(\beta, \gamma)$ – uniform distribution over the unit simplex,
- v) $p(h_0) = f_{Exp}(h_0|1)$ – exponential distribution with mean (and standard deviation) 1,
- vi) $p(\phi_i) \propto f_N(\phi_i|0, 100)\mathbf{1}_{\{|x| < 1\}}(\phi_i)$,
- vii) $p(\tau_i) = f_{Exp}(\tau_i|200)$ – exponential distribution with mean (and standard deviation) 200.

The joint posterior distribution of all parameters and latent variables has the density function

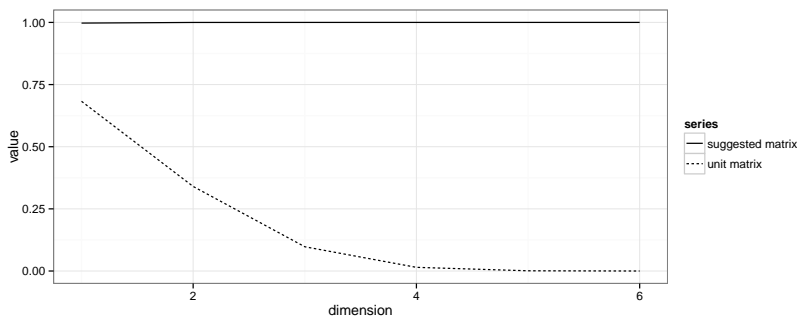
$$\begin{aligned}
 p(\theta, g^{(1)}, \dots, g^{(k)} | \psi_0, r) &= \frac{p(r, g^{(1)}, \dots, g^{(k)} | \psi_0, \theta) p(\theta)}{p(r | \psi_0)} \\
 &\propto p(\theta^{(1)}) p(\theta^{(2)}) \prod_{t=1}^T \left[f_N^n(x_t | x_{t-1} + \mu_t, \Omega_t) \prod_{i=1}^k g_{t,i}^{-1} f_N^1(\ln g_{t,i} | \varphi_i \ln g_{t-1,i}, \tau_i^{-1}) \right],
 \end{aligned} \tag{17}$$

which is highly multivariate and non-standard. The joint posterior will be analysed through MCMC simulation, using Gibbs sampling with Metropolis-Hasting steps.

For the sake of clarity, let us use the following notation: $g = (g^{(1)} \dots g^{(k)})$ groups all latent variables (in both spatial and temporal dimensions), $\phi = (\phi_1 \dots \phi_k)$ and

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Figure 2: Probability (*a priori*) of the stability restriction $\rho(\Lambda) < 1$



$\tau = (\tau_1 \dots \tau_k)$ group all the AR(1) and precision parameters of latent processes, and $\mu_t^* = x_{t-1} + \mu_t$ is the conditional mean of x_t .

- i) Parameters of the VAR(1) structure (λ and Λ) have following conditional posterior distributions:

$$p(\lambda|x, g, \Lambda, A, \beta, \gamma, h_0, \phi, \tau, \theta^{(2)}) \propto p(\lambda) \prod_{t=1}^T f_N^n(x_t|\mu_t^*, \Omega_t), \quad (18)$$

$$p(\text{vec}\Lambda|x, g, \lambda, A, \beta, \gamma, h_0, \phi, \tau, \theta^{(2)}) \propto p(\text{vec}\Lambda) \prod_{t=1}^T f_N^n(x_t|\mu_t^*, \Omega_t), \quad (19)$$

where $x = (x_1, \dots, x_T)$. Due to the fact that the MGARCH structure ties together all of the points in time, it is not possible to directly sample from the conditional distributions presented in (18) and (19). To obtain a (pseudo)random sample from these conditionals, we suggest using a random walk Metropolis-Hastings algorithm. The proposal distribution, used for obtaining candidate draws, can be multivariate normal parametrised by the previous state as the mean and an arbitrary (but dispersed enough) covariance matrix, tuned on the initial cycles of the algorithm to achieve the acceptance rate between 3-7%.

- ii) Parametrs of the SBEEK structure, i.e. A , (β, γ) and h_0 have the following conditional distributions:

$$p(A|x, g, \lambda, \Lambda, \beta, \gamma, h_0, \phi, \tau, \theta^{(2)}) \propto p(A) \prod_{t=1}^T f_N^n(x_t|\mu_t^*, \Omega_t), \quad (20)$$

$$p(\beta, \gamma, h_0|x, g, \lambda, \Lambda, A, \phi, \tau, \theta^{(2)}) \propto p(\beta, \gamma)p(h_0) \prod_{t=1}^T f_N^n(x_t|\mu_t^*, \Omega_t). \quad (21)$$

Again, we suggest using a random walk Metropolis-Hastings algorithm. For the matrix A^{-1} , the candidates are drawn from the Wishart distribution with m degrees of freedom and the parameter of scale being set to the previous state multiplied by $\frac{1}{m}$, so that the previous state is the expected value of the candidate one. As in the previous case, the value of m is tuned to achieve the acceptance rate around 5%. We use the similar method to draw samples from the conditional distribution of (β, γ) . In this case, the proposal distribution is normal, truncated by the restrictions: $\beta + \gamma < 1$, $\beta > 0$ and $\gamma > 0$. For the parameter h_0 , describing the initial condition for the MGARCH part of the conditional covariance matrix, we use the same procedure with a normal proposal distribution, truncated to \mathbb{R}_+ .

- iii) In the case of ϕ_1, \dots, ϕ_k and τ_1, \dots, τ_k , the parameters ruling the latent processes, direct sampling from the conditional posteriors is possible - see Pajor (2003):

$$\begin{aligned}
 & p(\phi_i | x, g, \lambda, \Lambda, A, \beta, \gamma, h_0, \phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_k, \tau, \theta^{(2)}) \\
 & \propto p(\phi_i) \prod_{t=1}^T f_N(\ln g_{t,i} | \phi_i \ln g_{t-1,i}, \tau_i^{-1}) \\
 & \propto f_N(\phi_i | \phi_i^*, s_i^{2*}) \mathbf{1}_{(-1,1)}(\phi_i),
 \end{aligned} \tag{22}$$

$$\begin{aligned}
 & p(\tau_i | x, g, \lambda, \Lambda, A, \beta, \gamma, h_0, \phi, \tau_1, \dots, \tau_{i-1}, \tau_{i+1}, \dots, \tau_k, \theta^{(2)}) \\
 & \propto p(\tau_i) \prod_{t=1}^T f_N(\ln g_{t,i} | \phi_i \ln g_{t-1,i}, \tau_i^{-1}) \\
 & \propto f_G(\tau_i | \frac{T}{2} + 1, \beta_i^*),
 \end{aligned} \tag{23}$$

where

$$s_i^{2*} = \left[0.01 + \tau_i \sum_{t=1}^T (\ln g_{t-1,i})^2 \right]^{-1}, \tag{24}$$

$$\phi_i^* = s_i^{2*} \tau_i \sum_{t=1}^T \ln g_{t,i} \ln g_{t-1,i}, \tag{25}$$

$$\beta_i^* = \left[0.005 + \frac{1}{2} \sum_{t=1}^T (\ln g_{t,i} - \phi_i \ln g_{t-1,i})^2 \right]^{-1}, \tag{26}$$

and $f_G(\cdot | a, b)$ stands for the density function of the gamma distribution with mean $\frac{a}{b}$ and variance $\frac{a}{b^2}$.

- iv) For every $t = 1, \dots, T-1$ and $i = 1, \dots, k$, $g_{t,i}$ has the following conditional posterior density (where $g_{(-ti)}^*$ denotes the set of all variables in g , except $g_{t,i}$):

$$\begin{aligned}
 & p(g_{t,i} | x, g_{(-ti)}^*, \lambda, \Lambda, A, \beta, \gamma, h_0, \phi, \tau, \theta^{(2)}) \\
 & \propto f_N(\ln g_{t,i} | \phi_i \ln g_{t-1,i}, \tau_i^{-1}) f_N(\ln g_{t+1,i} | \phi_i \ln g_{t,i}, \tau_i^{-1}) \cdot g_{t,i}^{-1} \cdot f_N^n(x_t | \mu_t^*, \Omega_t),
 \end{aligned} \tag{27}$$

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and for $t = T$:

$$p(g_{T,i}|x, g_{(-T,i)}^*, \lambda, \Lambda, A, \beta, \gamma, h_0, \phi, \tau, \theta^{(2)}) \propto f_N(\ln g_{T,i} | \phi_i \ln g_{T-1,i}, \tau_i^{-1}) \cdot g_{T,i}^{-1} \cdot f_N^*(x_T | \mu_T^*, \Omega_T). \quad (28)$$

Drawing random samples from the conditional posteriors defined by (27) and (28) is relatively straightforward in the case of $k = 1$, although Metropolis-Hastings steps are required. If $k = 1$, we can decompose the factor $(x_t - \mu_t^*)' \Omega_t (x_t - \mu_t^*)$ to a product of g_t and $(x_t - \mu_t^*)' H_t (x_t - \mu_t^*)$, and thus use a gamma proposal density function for g_t^{-1} based on the normal density function of $(x_t - \mu_t^*)$. In the general case, the proposal density we introduce is inspired by this simpler case. If we treat the appropriate blocks of Ω_t as the ones containing most of the information about $g_{t,i}$, then we arrive at the following proposal density:

$$p_c(g_{t,i}^{-1}) = f_G(g_{t,i}^{-1} | \varphi_{t,i}, \eta_{t,i}) \quad (29)$$

where

$$\varphi_{t,i} = (\exp(\sigma_i^2) - 1)^{-1} + \frac{n_i}{2}, \quad (30)$$

$$\eta_{t,i} = \left(\varphi_{t,i} - \frac{n_i}{2} \right) \exp\left(-s_{t,i} - \frac{\sigma_i^2}{2}\right) + \frac{1}{2} (x_t - \mu_t^*)'_{a:b,1:1} (H_t^{-1})_{a:b,a:b} (x_t - \mu_t^*)_{a:b,1:1}, \quad (31)$$

$a = \sum_{j=1}^{i-1} n_j + 1$, $b = \sum_{j=1}^i n_j$. We then use only the information about assets a to b of the portfolio – those associated with the latent process i .

All the conditional posterior densities presented above condition on $\theta^{(2)}$, so they appear in all Bayesian models considered in this section, irrespective of the assumptions on Π . In the standard case of $\Pi = 0$ these conditional densities are all what is needed to sample from the joint posterior distribution.

Now we present the forms of the priors and conditional posteriors for the cases of reduced rank Π that will be considered in our empirical example, where $n = 6$ assets are divided into three pairs (2 stock indexes, gold and silver, oil and natural gas). Assume that the rank of Π is m , where $0 < m < n$; then

$$\Pi = ab', \quad (32)$$

with both a and b of dimension $n \times m$ and full rank m . Since the representation in (32) is not unique, we use the identifying restriction

$$b'b = I_m, \quad (33)$$

which means that b is an element of the Stiefel manifold - see Strachan (2003), Koop, León-Gonzalez and Strachan (2009), and Wróblewska (2010). Note that b can be

parametrized in terms of its angular polar coordinates.

In the empirical part of this paper we consider only two obvious cases of rank reduced Π , namely

i) three possible relationships (one for each pair of assets):

$$\Pi = \begin{bmatrix} \Pi^{(1)} & 0_{[2 \times 2]} & 0_{[2 \times 2]} \\ 0_{[2 \times 2]} & \Pi^{(2)} & 0_{[2 \times 2]} \\ 0_{[2 \times 2]} & 0_{[2 \times 2]} & \Pi^{(3)} \end{bmatrix}, \quad (34)$$

where for $i \in \{1, 2, 3\}$ we assume:

$$\Pi^{(i)} = \begin{bmatrix} a_{2i-1} \\ a_{2i} \end{bmatrix} \begin{bmatrix} \cos(\kappa_i) & \sin(\kappa_i) \end{bmatrix}, \quad \kappa_i \in [0, \pi], \quad a_{2i-1}, a_{2i} \in \mathbb{R}; \quad (35)$$

ii) one global relationship linking all asset prices:

$$\Pi = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \begin{bmatrix} \cos(\kappa_1) \cos(\kappa_2) \cos(\kappa_3) \cos(\kappa_4) \cos(\kappa_5) \\ \sin(\kappa_1) \cos(\kappa_2) \cos(\kappa_3) \cos(\kappa_4) \cos(\kappa_5) \\ \sin(\kappa_2) \cos(\kappa_3) \cos(\kappa_4) \cos(\kappa_5) \\ \sin(\kappa_3) \cos(\kappa_4) \cos(\kappa_5) \\ \sin(\kappa_4) \cos(\kappa_5) \\ \sin(\kappa_5) \end{bmatrix}' \quad (36)$$

where, for $j \in \{1, 2, 3, 4, 5\}$ and $i \in \{1, 2, 3, 4, 5, 6\}$,

$$\kappa_j \in [0, \pi], \quad a_i \in \mathbb{R}. \quad (37)$$

That is, in each case $\theta^{(2)}$ is the vector grouping $n = 6$ parameters a_i and all the angles κ_j (three in the first case, which is denoted VEC(3) in the empirical part, five in the second case - denoted VEC(5)). Taking $a = 0$ would lead to $\Pi = 0$ (not considered now). In both cases we assume that each individual angle is (*a priori*) independent of other parameters and uniformly distributed over $[0, \pi]$ and the vector a of dimension $n = 6$ is *a priori* normally distributed with mean vector 0 and covariance matrix $\frac{1}{9}I_n$. It is important to stress that, for the bi-variate portfolio ($n = 2$), the uniform distribution over $[0, \pi]$ (assumed for the only angular coordinate of b) corresponds to the non-informative prior on the Stiefel manifold.

The full conditional posterior density of each individual angle takes the form

$$\begin{aligned} p(\kappa_j | x, g, \Lambda, A, \beta, \gamma, h_0, \phi, \tau, a, \kappa_1, \dots, \kappa_{j-1}, \kappa_{j+1}, \dots, \kappa_J) &\propto \\ &\propto \mathbf{1}_{[0, \pi]}(\kappa_j) \prod_{t=1}^T f_N^n(x_t | \mu_t^*, \Omega_t), \end{aligned} \quad (38)$$

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while for the vector a we have

$$p(a|x, g, \Lambda, A, \beta, \gamma, h_0, \phi, \tau, \kappa_1, \dots, \kappa_J) \propto f_N^n \left(a | 0, \frac{1}{g} I_n \right) \prod_{t=1}^T f_N^n(x_t | \mu_t^*, \Omega_t), \quad (39)$$

where $J = 3$ or $J = 5$ - depending on the case we consider. These conditional posteriors require Metropolis-Hastings steps in order to sample from; for details see Osiewalski and Osiewalski (2013), and Osiewalski (2015).

4 An empirical example

4.1 The data

As an empirical evidence of the usefulness of the GMSF-SBEKK specification, we analyse a 6-dimensional portfolio ($n = 6$) composed of assets belonging to three different types of markets:

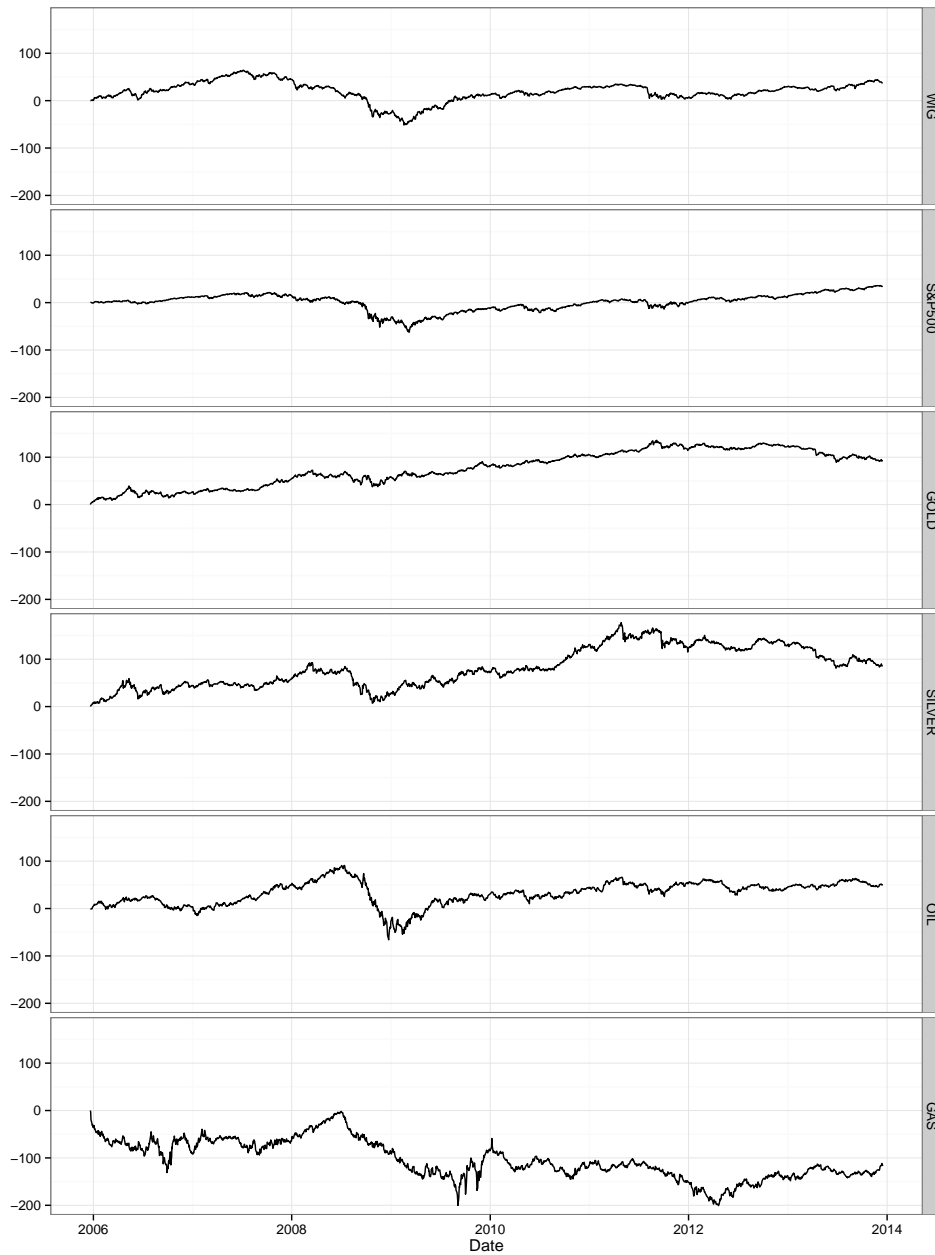
- stock markets: represented by the Warsaw Stock Exchange index (WIG) and the American S&P500;
- precious metals markets: represented by gold and silver in London fixing (PM) prices in USD/Ounce (international troy ounce ≈ 31.1 g);
- energy commodities markets: represented by oil (West Texas Intermediate – WTI, priced in USD per barrel; 1 bbl = 42 gallons ≈ 159 liters) and natural gas (priced as the Henry Hub spots (NYMEX) in USD per MMBTU, i.e. million British thermal units; 1 BTU ≈ 1055 J, which is the amount of energy required to raise the temperature of one pound (454 g) of water by one degree Fahrenheit ($\frac{5}{9}$ °C)).

We analyse daily data from December 21, 2005 till December 16, 2013. This results in $T = 2066$ days when at least one asset was valued. For all the time series, the missing observations have been filled using linear interpolation – as recommended by Osiewalski and Osiewalski (2012b). Such approach prevents from: losing available

Table 1: Descriptive statistics of the logarithmic return rates

	WIG	SP500	GOLD	SILVER	OIL	GAS
min	-8.289	-9.47	-9.596	-18.693	-12.827	-25.529
max	6.084	10.957	6.841	18.279	16.414	28.391
average	0.019	0.017	0.043	0.04	0.026	-0.057
standard deviation	1.362	1.373	1.327	2.526	2.346	3.804
skewness	-0.433	-0.286	-0.523	-0.599	0.066	0.379
kurtosis	3.243	10.002	4.412	7.841	5.918	6.801

Figure 3: Modelled multivariate time series x_t



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information, modifying time axis, creating artificial outliers and thus changing volatility estimates within multivariate framework.

The descriptive statistics of the logarithmic returns can be found in Table 1, while in Table 2 we show the above diagonal part of the empirical correlation matrix. Original time series of logs of prices (or price indexes) are plotted in Figure 3.

Table 2: Correlation coefficients between the logarithmic return rates

	WIG	SP500	GOLD	SILVER	OIL	GAS
WIG		0.418	0.167	0.183	0.342	0.064
SP500			0.002	0.007	0.355	0.018
GOLD				0.600	0.210	0.092
SILVER					0.150	0.136
OIL						0.072
GAS						

4.2 Model comparison

Let us present the formal model comparison first, and then - in the next subsection - the posterior results obtained in each individual model, as they explain the ranking of our models.

Bayesian model comparison is based, for any pair of competing models, on the Bayes factor, which is the ratio of the marginal data density (MDD) values for both models. Each MDD value has been computed (within the MCMC simulation from the posterior distribution) using exactly the same approach as in Osiewalski and Osiewalski (2013). Namely, we have used the harmonic mean estimator (HME, Newton and Raftery, 1994), corrected in the spirit of Lenk (2009). Such approach, which amounts to applying a particularly adjusted version of HME, is formally justified by Pajor and Osiewalski (2013-14), although it does not have so good properties as the corrected arithmetic mean estimator (CAME) proposed by Pajor (2016). However, the use of CAME in dynamic models with latent processes is not easy at all, due to very high dimension of the Monte Carlo simulation space (exceeding 6200 for our GMSF-SBEKK models).

In Table 3 one can find the Newton-Raftery HME values, the Lenk corrections and the final, adjusted HME values of the marginal data density. It is worth noting, that for all of the 6-dimensional models the prior sizes of appropriate cubes resulting from the Lenk corrections are almost identical for most of the common parameters (Λ , A , (β, γ) , h_0) and in the $\Pi = 0$ and $\Pi \neq 0$ cases for λ . Naturally, the correction varies for the MSV part due to the dimension difference; not surprisingly, the correction for g is higher for models with three latent processes, similarly for the τ and ϕ parameters. Sum of all the corrections reduces the advantage of the GMSF-SBEKK model over the MSF-SBEKK one by about 13 orders of magnitude, yet it is still clear that the extended model explains the observations much better.

The comparison is not so clear in the case of the models with the ECM part versus the

GMSF-SBEKK with $\Pi = 0$. Although the original HME indicates a slight advantage of the models with the ECM part, yet this advantage is wholly taken by the Lenk correction (on the parameter a). It is worth mentioning that this also means high fragility of the final result. As an example, if we reduce the prior standard deviation of the parameter a from $\frac{1}{3}$ to $\frac{1}{10}$, the correction size would be smaller by 3.1 orders of magnitude and the final marginal data densities would be equal for models with $\Pi = 0$ and $rank(\Pi) = 1$. This means that the hypothesis of one global long-run relation, linking prices on all six markets, cannot be strongly rejected (especially when it is supported by prior beliefs). On the other hand, such global relation does not help in explaining our data; the short-run effects and dependencies are crucial.

Table 3: MDD estimates and Bayes Factors for the competing models

	MSF-SBEKK $\Pi = 0$	GMSF-SBEKK			3x 2-dimensional MSF-SBEKK
		VEC(3)	VEC(5)	$\Pi = 0$	
	$\log_{10} Pr(A_k M_k)$				
λ	-3.288	-1.626	-0.764	-3.095	-3.317
Λ	-17.546	-17.767	-18.118	-18.266	-6.217
A	-34.592	-31.247	-30.522	-30.706	-2.886
(β, γ)	-3.612	-3.763	-3.788	-3.783	-8.852
h_0	-0.042	-0.150	-0.187	-0.163	-0.251
φ	-0.643	-1.846	-1.822	-1.793	-1.257
τ	-2.947	-8.427	-8.440	-8.389	-8.127
g	-19.237	-29.137	-28.131	-28.517	-18.972
a	-	-9.492	-8.576	-	-
κ	-	0.000	-0.010	-	-
	sum of Lenk's correction				
$\log_{10} Pr(A_k M_k)$	-81.907	-103.455	-100.358	-94.712	-49.878
	Newton-Raftery estimator				
$\log_{10} p_{NR}(x M_k)$	-9583.6	-9484.7	-9481.6	-9484.1	-9681.1
	final estimates of the marginal data density				
$\log_{10} p(x M_k)$	-9665.5	-9588.2	-9582	-9578.8	-9731.3
	$\log_{10}(BF_{GMSF-SBEKK:MSF-SBEKK}) \approx 86.7$ $\log_{10}(BF_{GMSF-SBEKK:3x\ 2-dim\ MSF-SBEKK}) \approx 152.5$ $\log_{10}(BF_{VEC(5)-GMSF-SBEKK:GMSF-SBEKK}) \approx -3.2$ $\log_{10}(BF_{VEC(3)-GMSF-SBEKK:VEC(5)-GMSF-SBEKK}) \approx -6.2$				

Our competing models reflect two main aspects of modelling short-run dependencies between - and volatility of - different groups of markets. Firstly, we model each group separately, assuming complete independence and using three unrelated bivariate MSF-SBEKK models. Secondly, we apply a joint MSF-SBEKK specification, which allows for parsimonious modelling of dependencies between different types of markets, but it does not distinguish their volatility patterns strongly enough. The third option, that is the joint GMSF-SBEKK specification, is based on separate latent processes and thus it allows for deep differences in volatility patterns. The values of Bayes factors clearly indicate that joint modelling of prices on different markets leads to much,

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much better statistical explanation of the multidimensional data. The case of three separate bivariate MSF-SBEKK models is very strongly rejected. Using one or three latent processes seems a secondary issue, when we look from the perspective of joint or independent modelling of different prices. However, when we adopt the strategy of joint modelling, then using the GMSF-SBEKK structure appears inevitable, although it is numerically more demanding than the MSF-SBEKK specification.

4.3 Posterior results within the main models

In this subsection we present the marginal posterior distributions of the model parameters, focusing on the differences between the analysed models, namely the basic MSF-SBEKK specification and the proposed GMSF-SBEKK extensions (with and without the ECM terms). Since, looking at the marginal data density values (see subsection 4.2), the case of rank one matrix Π , representing one global relation among price levels, seems more relevant than the case of three relations (one per each type of markets), we will only present the results for the case of $rank(\Pi) = 1$, denoted VEC(5). In the following tables, the posterior expectation value was marked with bold whenever zero was not in the 95% highest posterior density (HPD) interval. Let us discuss the conditional mean parameters first. After introducing more latent processes, the posterior of λ in the GMSF-SBEKK model with $\Pi = 0$ is very similar to its counterpart in the MSF-SBEKK model. The situation changes in the case of the GMSF-SBEKK models with non-zero Π . This is expected, as the interpretation of λ changes due to the presence of the ECM term.

MSF-SBEKK model:						
	WIG	S&P500	GOLD	SILVER	OIL	GAS
$E(\lambda' x)$ $(D(\lambda' x))$	0.057 (0.022)	0.088 (0.018)	0.055 (0.023)	0.060 (0.041)	0.101 (0.036)	-0.059 (0.059)
GMSF-SBEKK model:						
	WIG	S&P500	GOLD	SILVER	OIL	GAS
$E(\lambda' x)$ $(D(\lambda' x))$	0.055 (0.021)	0.087 (0.018)	0.080 (0.022)	0.079 (0.037)	0.080 (0.036)	-0.043 (0.061)
VEC(5)-GMSF-SBEKK model:						
	WIG	S&P500	GOLD	SILVER	OIL	GAS
$E(\lambda' x)$ $(D(\lambda' x))$	0.141 (0.097)	0.115 (0.057)	0.039 (0.081)	0.144 (0.104)	0.293 (0.137)	-0.318 (0.185)

As expected, the matrix Λ has very similar posteriors in each case. In some cases there exist minor differences: an interesting example is the case of Λ_{66} , where due to slightly increased dispersion of the posterior in the GMSF-SBEKK model with $\Pi = 0$, zero

Hybrid MSV-MGARCH Models ...

crossed the border of the HPD interval, causing the parameter to be “insignificant”. One important conclusion we can make is that there exist significant parameters outside block-diagonal market division, e.g. lagged gold return significantly influences the current S&P500 return. This is another evidence in favour of modelling these assets jointly.

MSF-SBEKK model:

	WIG ₋₁	S&P500 ₋₁	GOLD ₋₁	SILVER ₋₁	OIL ₋₁	GAS ₋₁	
$E(\Lambda x)$ $(D(\Lambda x))$	-0.019 (0.024)	0.209 (0.026)	-0.034 (0.024)	0.012 (0.012)	0.012 (0.013)	-0.014 (0.007)	WIG
	0.033 (0.019)	-0.064 (0.023)	-0.052 (0.020)	0.015 (0.010)	-0.005 (0.011)	0.003 (0.005)	S&P500
	-0.039 (0.022)	0.043 (0.025)	-0.041 (0.025)	-0.005 (0.012)	0.073 (0.013)	0.005 (0.006)	GOLD
	0.002 (0.037)	0.144 (0.040)	0.587 (0.039)	-0.313 (0.022)	0.192 (0.022)	0.013 (0.011)	SILVER
	-0.024 (0.035)	0.067 (0.040)	0.018 (0.038)	-0.004 (0.020)	-0.025 (0.024)	0.002 (0.011)	OIL
	-0.091 (0.050)	0.075 (0.056)	-0.073 (0.056)	-0.009 (0.031)	0.147 (0.034)	0.045 (0.022)	GAS

GMSF-SBEKK model:

	WIG ₋₁	S&P500 ₋₁	GOLD ₋₁	SILVER ₋₁	OIL ₋₁	GAS ₋₁	
$E(\Lambda x)$ $(D(\Lambda x))$	-0.021 (0.024)	0.215 (0.026)	-0.030 (0.023)	0.015 (0.011)	0.011 (0.013)	-0.015 (0.006)	WIG
	0.030 (0.019)	-0.059 (0.024)	-0.053 (0.019)	0.015 (0.009)	-0.012 (0.011)	0.004 (0.005)	S&P500
	-0.035 (0.021)	0.041 (0.025)	-0.037 (0.025)	-0.004 (0.012)	0.068 (0.012)	-0.002 (0.006)	GOLD
	0.007 (0.035)	0.159 (0.040)	0.593 (0.040)	-0.310 (0.023)	0.174 (0.020)	0.005 (0.010)	SILVER
	-0.037 (0.034)	0.097 (0.040)	0.024 (0.036)	0.003 (0.020)	-0.033 (0.024)	0.005 (0.011)	OIL
	-0.094 (0.052)	0.079 (0.056)	-0.043 (0.055)	-0.022 (0.031)	0.143 (0.034)	0.045 (0.023)	GAS

VEC(5)-GMSF-SBEKK model:

	WIG ₋₁	S&P500 ₋₁	GOLD ₋₁	SILVER ₋₁	OIL ₋₁	GAS ₋₁	
$E(\Lambda x)$ $(D(\Lambda x))$	-0.022 (0.024)	0.215 (0.027)	-0.032 (0.023)	0.015 (0.011)	0.011 (0.013)	-0.015 (0.006)	WIG
	0.030 (0.019)	-0.058 (0.024)	-0.054 (0.019)	0.016 (0.010)	-0.012 (0.011)	0.004 (0.005)	S&P500
	-0.033 (0.021)	0.036 (0.025)	-0.041 (0.025)	-0.006 (0.012)	0.066 (0.012)	0.000 (0.006)	GOLD
	0.007 (0.035)	0.157 (0.038)	0.588 (0.041)	-0.311 (0.023)	0.175 (0.020)	0.005 (0.010)	SILVER
	-0.037 (0.035)	0.100 (0.040)	0.023 (0.036)	0.003 (0.020)	-0.033 (0.023)	0.004 (0.011)	OIL
	-0.094 (0.051)	0.084 (0.053)	-0.041 (0.055)	-0.023 (0.029)	0.136 (0.036)	0.046 (0.023)	GAS

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MSF-SBEKK model:

	WIG	S&P500	GOLD	SILVER	OIL	GAS	
$E(A x) =$ $(D(A x))$	1.009 (0.104)	0.277 (0.054)	0.183 (0.062)	0.320 (0.108)	0.305 (0.101)	0.000 (0.168)	WIG
		0.638 (0.068)	0.041 (0.049)	0.067 (0.087)	0.396 (0.086)	-0.086 (0.139)	S&P500
			1.070 (0.111)	1.086 (0.146)	0.322 (0.102)	0.158 (0.172)	GOLD
				3.252 (0.343)	0.303 (0.172)	0.436 (0.307)	SILVER
					2.622 (0.271)	0.208 (0.276)	OIL
						8.292 (0.868)	GAS
	$E(\beta x) = \mathbf{0.026}$ $(D(\beta x))$		$E(\gamma x) = \mathbf{0.956}$ $(D(\gamma x))$		$E(h_0 x) = \mathbf{0.980}$ $(D(h_0 x))$		

GMSF-SBEKK model:

	WIG	S&P500	GOLD	SILVER	OIL	GAS	
$E(A x) =$ $(D(A x))$	1.079 (0.156)	0.320 (0.065)	0.204 (0.059)	0.353 (0.100)	0.370 (0.096)	0.029 (0.153)	WIG
		0.704 (0.104)	0.057 (0.046)	0.090 (0.079)	0.470 (0.085)	-0.089 (0.125)	S&P500
			0.848 (0.095)	0.823 (0.127)	0.321 (0.091)	0.188 (0.150)	GOLD
				2.383 (0.285)	0.368 (0.153)	0.535 (0.264)	SILVER
					2.413 (0.286)	0.191 (0.245)	OIL
						6.988 (0.839)	GAS
	$E(\beta x) = \mathbf{0.020}$, $(D(\beta x))$		$E(\gamma x) = \mathbf{0.964}$, $(D(\gamma x))$		$E(h_0 x) = \mathbf{1.411}$ $(D(h_0 x))$		

VEC(5)-GMSF-SBEKK model:

	WIG	S&P500	GOLD	SILVER	OIL	GAS	
$E(A x) =$ $(D(A x))$	1.077 (0.159)	0.317 (0.066)	0.198 (0.060)	0.342 (0.102)	0.362 (0.097)	0.033 (0.156)	WIG
		0.702 (0.105)	0.054 (0.046)	0.083 (0.080)	0.465 (0.086)	-0.086 (0.129)	S&P500
			0.816 (0.096)	0.783 (0.129)	0.315 (0.093)	0.195 (0.154)	GOLD
				2.289 (0.288)	0.353 (0.155)	0.531 (0.268)	SILVER
					2.397 (0.283)	0.192 (0.250)	OIL
						6.949 (0.827)	GAS
	$E(\beta x) = \mathbf{0.020}$, $(D(\beta x))$		$E(\gamma x) = \mathbf{0.964}$, $(D(\gamma x))$		$E(h_0 x) = \mathbf{1.401}$ $(D(h_0 x))$		

Now let us move to the MGARCH parameters. Differences in the posterior expectations of the parameters of matrix A can be seen in the cases of A_{33} , A_{66} , A_{45} and A_{46} , and only between the models with one versus three latent processes (practically the cases of GMSF-SBEKK and VEC(5)-GMSF-SBEKK are identical). The diagonal elements of A , which were mentioned before, are both lower in the case when more latent processes are present. This suggests that newly introduced stochastic components may partially explain specific volatility patterns in gold and natural gas markets. It is also interesting that the non-diagonal elements of A are higher (in absolute value) when the number of latent processes increases. Finally, let us focus on the MSV part of the GMSF-SBEKK model. The basic characteristics of the parameters ϕ and τ can be found below. Posterior densities are plotted in Figure 4. It is worth analysing Figures 4 and 5 jointly: we can observe a significantly different behavior of the latent process associated with the precious metals market (spikes, low persistence), which is reflected in a different location and dispersion of the parameters associated with it. The highest auto-correlation is present in the latent process describing the stock markets – it is driven by the posterior of ϕ_1 close to 1 (expected value of 0.920, comparing to 0.370 and 0.753 for precious metals and energy commodities, respectively). Also, for the latent process describing the stock markets, the posterior mean of the unconditional variance, $E((\tau_1^{-1}/(1 - \phi_1^2)) | x)$, is somewhat smaller than in other cases: 0.293, comparing to 0.370 for precious metals and 0.299 for energy commodities.

An interesting conclusion can be drawn when we compare the latent processes that appear in different models. The single process in the MSF-SBEKK case looks like an average of the three very different processes in the GMS-SBEKK model. This is another evidence that the three latent processes (corresponding to three types of markets) should not be collapsed to one, “global” or “average”, latent process. As one might anticipate, the ECM term does not have any visible impact on the GMSF-SBEKK latent processes’ structures.

MSF-SBEKK model:					
$E(\phi x) = \mathbf{0.606}$		$E(\tau^{-1} x) = \mathbf{0.195}$			
$(D(\phi x))$		$(D(\tau^{-1} x))$			
(0.044)		(0.022)			
GMSF-SBEKK model:					
$E(\phi_1 x) = \mathbf{0.920}$	$E(\phi_2 x) = \mathbf{0.370}$	$E(\phi_3 x) = \mathbf{0.753}$			
$D(\phi_1 x)$	$D(\phi_2 x)$	$D(\phi_3 x)$			
(0.023)	(0.084)	(0.060)			
$E(\tau_1^{-1} x) = \mathbf{0.044}$	$E(\tau_2^{-1} x) = \mathbf{0.440}$	$E(\tau_3^{-1} x) = \mathbf{0.130}$			
$D(\tau_1^{-1} x)$	$D(\tau_2^{-1} x)$	$D(\tau_3^{-1} x)$			
(0.014)	(0.057)	(0.035)			
VEC(5)-GMSF-SBEKK model:					
$E(\phi_1 x) = \mathbf{0.919}$	$E(\phi_2 x) = \mathbf{0.305}$	$E(\phi_3 x) = \mathbf{0.747}$			
$D(\phi_1 x)$	$D(\phi_2 x)$	$D(\phi_3 x)$			
(0.024)	(0.080)	(0.062)			
$E(\tau_1^{-1} x) = \mathbf{0.045}$	$E(\tau_2^{-1} x) = \mathbf{0.504}$	$E(\tau_3^{-1} x) = \mathbf{0.132}$			
$D(\tau_1^{-1} x)$	$D(\tau_2^{-1} x)$	$D(\tau_3^{-1} x)$			
(0.015)	(0.057)	(0.037)			

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Figure 4: Histograms of the posterior distributions of the parameters ruling the latent processes in the GMSF-SBEKK and MSF-SBEKK models; prior distributions are marked with dashed lines

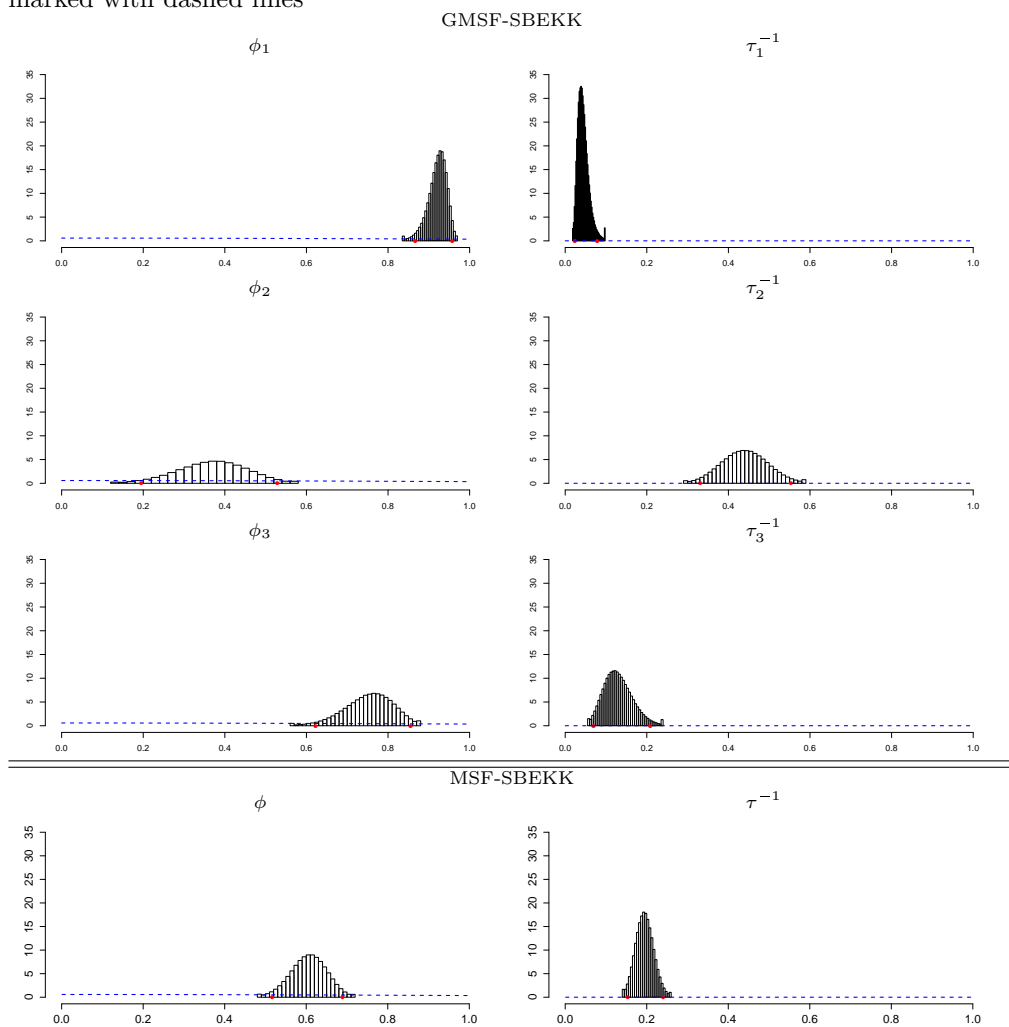
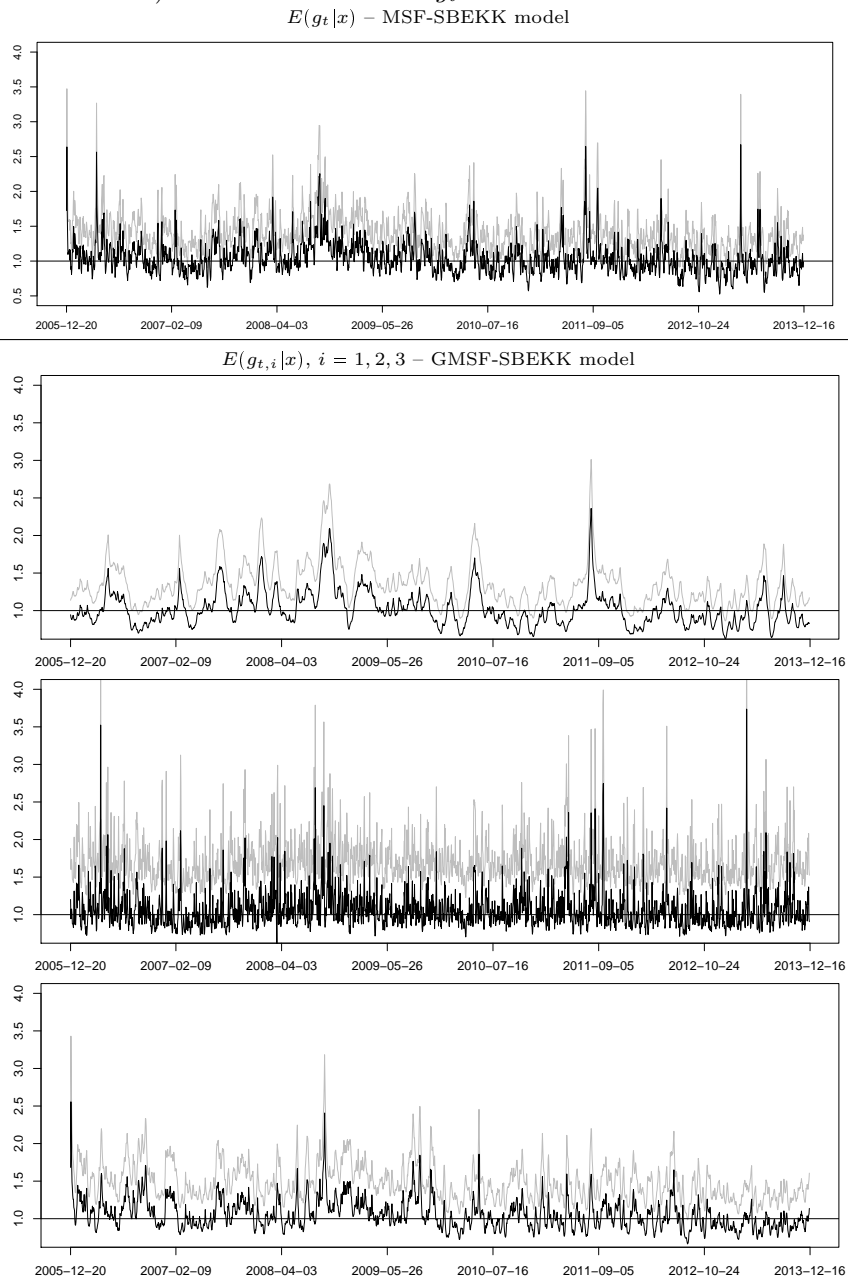


Figure 5: Posterior expectations (and posterior expectations plus one posterior standard deviation) of the latent variables g_t



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Figure 6: Posterior expectations of the conditional correlation coefficient (\pm two posterior standard deviations) in the GMSF-SBEKK model

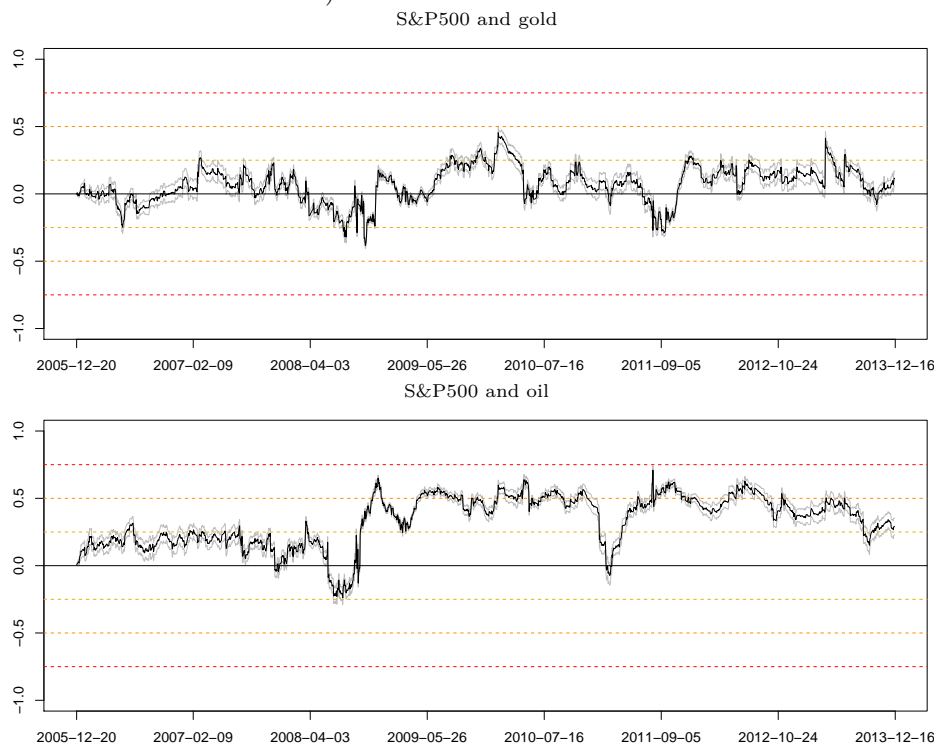


Table 4: Averages over time of posterior means (and standard deviations) of the conditional correlation coefficients $\rho_{t,ij}$ in MSF-SBEKK (above diagonal) and GMSF-SBEKK (below diagonal) models

	WIG	S&P500	GOLD	SILVER	OIL	GAS
WIG		0.405 (0.12)	0.183 (0.148)	0.189 (0.115)	0.254 (0.148)	0.025 (0.089)
S&P500	0.409 (0.111)		0.06 (0.142)	0.074 (0.124)	0.315 (0.202)	-0.006 (0.086)
GOLD	0.194 (0.139)	0.068 (0.131)		0.609 (0.079)	0.217 (0.118)	0.068 (0.093)
SILVER	0.203 (0.11)	0.082 (0.116)	0.607 (0.083)		0.151 (0.108)	0.105 (0.102)
OIL	0.268 (0.136)	0.335 (0.187)	0.227 (0.108)	0.168 (0.1)		0.066 (0.089)
GAS	0.029 (0.078)	-0.006 (0.074)	0.076 (0.084)	0.121 (0.095)	0.066 (0.08)	

Table 5: Averages over time of posterior means (and standard deviations) of conditional standard deviations $\sigma_{t,i}$ in MSF-SBEKK and GMSF-SBEKK models

	WIG	S&P500	GOLD	SILVER	OIL	GAS
MSF-SBEKK	1.225 (0.492)	1.121 (0.721)	1.232 (0.475)	2.142 (0.848)	2.049 (1.051)	3.401 (1.339)
GMSF-SBEKK	1.222 (0.504)	1.127 (0.737)	1.195 (0.434)	2.055 (0.781)	2.04 (0.994)	3.372 (1.379)
correlation coefficient	0.980	0.989	0.976	0.979	0.979	0.979

In order to conclude the discussion of our empirical example, we may say that both the model comparison (see subsection 4.2) and the posterior results within individual models prove that joint analysis of different types of markets is clearly preferred. Their analysis should not be reduced to three independent 2-dimensional models – due to e.g. significant off block-diagonal elements in the matrices A and Λ . This conclusion is also supported by the conditional correlations between different markets. In Figure 6 we can see that, e.g., conditional correlation between the returns of gold or oil and S&P500 remains strongly different from zero for long periods. Positive correlation between these assets since second half of 2009 suggests that there were limited risk diversification opportunities within such constructed portfolio. This means, again, that these markets did not operate independently in the analysed time frame.

The model choice (and thus the number of latent processes) does not impact the conclusions above: one can see in Table 4 that the estimates (posterior expectations) of the conditional correlation coefficients are very similar in both cases (one or three latent processes). The same happens with the conditional standard deviations (see Table 5). Apart from similar levels, the co-movements of conditional standard deviations do not significantly vary between the models (see bottom row of Table 5) – which is also true for conditional correlations, where the lowest correlation coefficient between posterior expectations of conditional correlation coefficients was 0.981.

Again, introducing the ECM part to the conditional mean of the GMSF-SBEKK model did not change the volatility estimates. Thus, for the sake of clarity we skip the VEC(5)-GMSF-SBEKK model in these final considerations, figures and tables.

5 Concluding remarks

The hybrid MSV-MGARCH structures (for VAR error terms) enable using only very simple MGARCH and MSV specifications to jointly describe volatilities of many assets (or markets) as well as their relationships. The simplest versions of MSV-MGARCH structures, namely the MSF-MGARCH models, amount to multiplying the MGARCH conditional covariance matrix by a non-trivial scalar latent process. When compared to a pure MGARCH model, its MSF-MGARCH extension leads to heavier tails of the conditional distribution (given the past of the observed process) and introduces extra parameters as well as an additional source of dependence over time.

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In this paper we focus mainly on the GMSF-MGARCH models that use as many latent processes as there are groups of assets (or markets); they lead to even more flexibility and enable to capture differences in volatility of assets from different groups – at a relatively low computational cost. We also show that extending standard volatility models in order to capture long-run relationships among price levels is straightforward within the proposed VAR(2)-GMSF-SBEKK framework. It is stressed that our Bayesian VAR(2)-GMSF-SBEKK models require relatively careful prior elicitation, but simple MCMC tools are sufficient for simulating posterior distributions; Gibbs chains with easy Metropolis-Hastings steps are constructed and programmed.

For Bayesian model comparison through posterior model probabilities, Newton and Raftery's harmonic mean estimator (of the marginal data density value, MDD) needs the important correction in the spirit of Lenk (2009). This, however, is not a final solution and further research is required to find other – feasible but more reliable – numerical tools for MDD evaluations in models with latent processes.

While MSF-SBEKK and GMSF-SBEKK specifications successfully compete with even more complicated pure MGARCH or MSV structures, their empirical comparison (using formal Bayesian approach) to models based on dynamic copulas is required. The main question is whether good fit and relative simplicity of our hybrids can justify their use instead of copulas (at least for large portfolios). Also, theoretical work on hybrid stochastic processes is needed in order to find conditions of their covariance stationarity and establish other properties.

Acknowledgements

The first author would like to thank Anna Pajor for her cooperation on construction and Bayesian analysis of the first MSV-MGARCH models. He also acknowledges support from the funds granted to the Faculty of Management at Cracow University of Economics, within the framework of the subsidy for the maintenance of research potential.

The second author is grateful to Anna Pajor, the supervisor of his doctoral thesis, for her invaluable help and many useful discussions, and to Ryszard Doman and Marek Męczarski, who refereed the thesis, for their useful comments that helped in writing the present paper. Both authors express their thanks to the referee of the paper.

References

- [1] Almeida C. and Czado C. (2012), Efficient Bayesian inference for stochastic time-varying copula models, *Computational Statistics and Data Analysis* 56, 1511–1527.
- [2] Bauwens L., Laurent S. and Rombouts J.V.K. (2006), Multivariate GARCH models: A survey, *Journal of Applied Econometrics* 21, 79–109.

- [3] Doman R. (2011), *Zastosowania kopuli w modelowaniu dynamiki zależności na rynkach finansowych*, Wydawnictwo Uniwersytetu Ekonomicznego w Poznaniu, Poznań.
- [4] Engle R.F. (2002), Dynamic Conditional Correlation - A simple class of multivariate GARCH models, *Journal of Business and Economic Statistics* 20, 339–350.
- [5] Hafner C.M. and Manner H. (2012), Dynamic stochastic copula models: Estimation, inference and applications, *Journal of Applied Econometrics* 27, 269–295.
- [6] Jondeau E. and Rockinger M. (2006), The Copula-GARCH model of conditional dependencies: An international stock market application, *Journal of International Money and Finance* 25, 827–853.
- [7] Koop G., León-Gonzalez R. and Strachan R. (2009), Efficient posterior simulation for cointegrated models with priors on the cointegration space, *Econometric Reviews* 29, 224–242.
- [8] Lenk P. (2009), Simulation pseudo-bias correction to the harmonic mean estimator of integrated likelihoods, *Journal of Computational and Graphical Statistics* 18, 941–960.
- [9] Marimoutou V., Raggad B. and Trabelsi A. (2009), Extreme Value Theory and Value at Risk: Application to oil market, *Energy Economics* 31, 519–530.
- [10] Newton M. and Raftery A. (1994), Approximate Bayesian inference by Weighted Likelihood Bootstrap (with discussion), *Journal of the Royal Statistical Society, series B* 56, 3–48.
- [11] Osiewalski J. (2009), New hybrid models of multivariate volatility (a Bayesian perspective), *Przegląd Statystyczny* 56, 15–22.
- [12] Osiewalski J. and Osiewalski K. (2011), Modele hybrydowe MSV-MGARCH z trzema procesami ukrytymi w badaniu zmienności cen na różnych rynkach, *Folia Oeconomica Cracoviensia* 52, 71–85.
- [13] Osiewalski J. and Osiewalski K. (2012a), Modele hybrydowe MSV-MGARCH z dwoma procesami ukrytymi, *Zeszyty Naukowe Uniwersytetu Ekonomicznego w Krakowie, seria Finanse* nr 895, 5–18.
- [14] Osiewalski J. and Pajor A. (2007), Flexibility and parsimony in multivariate financial modelling: A hybrid bivariate DCC-SV model [in:] *Financial Markets: Principles of Modelling, Forecasting and Decision-Making*, FindEcon Monograph Series No 3 [eds.] Milo W. and Wdowiński P., Łódź University Press.

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- [15] Osiewalski J. and Pajor A. (2009), Bayesian analysis for hybrid MSF-SBEKK models of multivariate volatility, *Central European Journal of Economic Modelling and Econometrics* 1, 179–202.
- [16] Osiewalski J. and Pajor A. (2010), Bayesian Value-at-Risk for a portfolio: multi- and univariate approaches using MSF-SBEKK models, *Central European Journal of Economic Modelling and Econometrics* 2, 253–277.
- [17] Osiewalski K. (2015), *Hybrydowe modele MSV-MGARCH w łącznej analizie zmienności cen na wybranych rynkach*, unpublished doctoral dissertation, Warsaw School of Economics.
- [18] Osiewalski K. and Osiewalski J. (2012b), Missing observations in daily returns – Bayesian inference within the MSF-SBEKK model, *Central European Journal of Economic Modelling and Econometrics* 4, 169–197.
- [19] Osiewalski K. and Osiewalski J. (2013), A long-run relationship between daily prices on two markets: the Bayesian VAR(2)-MSF-SBEKK model, *Central European Journal of Economic Modelling and Econometrics* 5, 65–83.
- [20] Pajor A. (2003), *Procesy zmienności stochastycznej w bayesowskiej analizie finansowych szeregów czasowych*, Wydawnictwo Akademii Ekonomicznej w Krakowie, Kraków.
- [21] Pajor A. (2010), *Wielowymiarowe procesy wariancji stochastycznej w ekonometrii finansowej. Ujęcie bayesowskie*, Wydawnictwo Uniwersytetu Ekonomicznego w Krakowie, Kraków.
- [22] Pajor A. (2016), Estimating the marginal likelihood using the arithmetic mean identity, *Bayesian Analysis* (forthcoming), doi: 10.1214/16-BA1001, <http://projecteuclid.org/euclid.ba/1459772735>
- [23] Pajor A. and Osiewalski J. (2012), Bayesian Value-at-Risk and Expected Shortfall for a large portfolio (multi- and univariate approaches), *Acta Physica Polonica A* vol 121 issue 2B, 101–109.
- [24] Pajor A. and Osiewalski J. (2013-14), A note on Lenk’s correction of the harmonic mean estimator, *Central European Journal of Economic Modelling and Econometrics* 5 (2013), 271–275; correction: vol. 6 (2014), 69.
- [25] Patton A.J. (2006), Modelling asymmetric exchange rate dependence, *International Economic Review* 47, 527–556.
- [26] Patton A.J. (2012), A review of copula models for economic time series, *Journal of Multivariate Analysis* 100, 4–18.

- [27] Strachan R. (2003), Valid Bayesian estimation of the cointegrating error correction model, *Journal of Business and Economic Statistics* 21, 185–195.
- [28] Tsay R. (2005), *Analysis of Financial Time Series, 2nd ed.*, Wiley.
- [29] Vo M. (2011), Oil and stock market volatility: A multivariate stochastic volatility perspective, *Energy Economics* 33, 956–966.
- [30] Wróblewska J. (2010), *Modele i metody bayesowskiej analizy kointegracji*, Wydawnictwo Uniwersytetu Ekonomicznego w Krakowie, Kraków.