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Stability conditions for fractional-order linear equations with delays

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Abstract. The problem of stability of the Grünwald-Letnikov-type linear fractional-order discrete-time systems with delays is discussed. For the stability analysis of the considered systems the $\mathscr{Z}$-transform is used. The sufficient conditions for the asymptotic stability of the considered systems are presented. Using conditions related to eigenvalues of the matrices defining the linear difference systems, one can determine the regions of location of eigenvalues of matrices associated to the systems in order to guarantee the asymptotic stability of the considered systems. Some of these regions are illustrated with relevant examples.

Key words: fractional calculus, discrete-time models with delays, stability.

1. Introduction

Fractional integrals, derivatives and differences of any order are the basic concepts in fractional calculus. The essentials of the corresponding mathematical theory were discovered over 300 years ago. Recently, fractional calculus in both continuous and discrete cases has played an important role in many scientific and engineering fields. For a comprehensive review of theory and applications of fractional calculus, we refer the reader to [1–5, 7–10, 16]. The importance of discrete case can be examined in applications, see [11–13].

One of the most important issues that should be solved for fractional order systems is stability analysis. It is difficult to find a valid tool to analyze the stability of fractional equations. In the case of linear fractional order difference systems, the $\mathscr{Z}$-transform can be used as an effective method for stability analysis, see for instance [5, 14, 15, 17–19]. In the literature one can find a couple of approaches to the notion of stability of difference fractional systems or equations. For example in [20] authors discuss qualitative properties of the two-term linear fractional difference equation; in [21] stability regions for linear fractional differential systems and their discretizations are presented.

In our paper we deal with the problem of working out a direct stability conditions for linear discrete-time fractional order systems with the delay. Observe that in [22] one can find the conditions for practical stability and for asymptotic stability of fractional discrete-time linear scalar systems with one constant delay, which is standard and positive. We consider matrix systems with a finite sequence of delays. However, the final condition is stated for systems with one delay and it is connected with eigenvalues of matrices of considered systems. Our results are illustrated by examples, where plots of stability regions are presented according to delays and orders.

We also stress that systems with step $h > 0$, order $\alpha \in (0, 1]$ and delays $k_0 > 0$ are studied. Note that taking step h tending to zero resembles the kind of approximations typical of continuous-time systems. However, a comparison with paper [23] that investigates stability and asymptotic properties of autonomous fractional differential systems with a time delay is not made.

The paper is organized as follows. In Sec. 2 we gather some results needed in the sequel. The equivalent descriptions of the considered linear fractional-order discrete-time systems with delays and their solutions are discussed in Sec. 3. Next, Sec. 4 contains the stability analysis of linear difference systems with positive fractional orders. Additionally, similarly as in [18, 19, 24] we prove the conditions connected with eigenvalues of the matrices that define the considered linear difference systems. Finally, we also include a simple example in order to illustrate the presented conditions and calculate trajectories for linear system with two variables.

2. Preliminaries

We begin the preliminaries by defining special coefficients, which is common in fractional discrete calculus. They play an important role in definition of the Grünwald-Letnikov-type fractional-order difference. The following sequence of coefficients is defined by:

$$a^{(\alpha)}(k) := \begin{cases} 1 & \text{for } \ k = 0 \\ (-1)^k \dfrac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} & \text{for } \ k \in \mathbb{N}, \end{cases}$$

where $\alpha \in \mathbb{R}$. Since $\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} = \binom{\alpha}{k}$, the sequence $\left(a^{(\alpha)}(k)\right)_{k\in\mathbb{N}_0}$ can be rewritten using the generalized binomial $\binom{\alpha}{k}$ as follows $a^{(\alpha)}(k) = (-1)^k\binom{\alpha}{k}$. It is much better to calculate the sequence using recurrence definition:

$$a^{(\alpha)}(0) := 1,$$
$$a^{(\alpha)}(k+1) := \left(1 - \tfrac{\alpha+1}{k+1}\right) a^{(\alpha)}(k), \quad k \in \mathbb{N} \cup \{0\} =: \mathbb{N}_0.$$

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It is easy to see that for $0 < \alpha \leq 1$ we have that $a^{(-\alpha)}(k) > 0$, $k \in \mathbb{N}_0$. While $a^{(\alpha)}(0) = 1$ and $a^{(\alpha)}(k) < 0$ for $k > 0$. For more properties of function $a^{(\alpha)} : \mathbb{N}_0 \to \mathbb{R}$, see [6,8].

Consider a discrete-variable bounded real-valued function $x(\cdot)$. Information and application of the Grünwald-Letnikov fractional-order backward difference with constant order can be found, for example, in [3,8,9].

Definition 1. The Grünwald-Letnikov fractional-order backward difference (GL-FOBD) with an order $\alpha \in \mathbb{R}$ and step $h > 0$ is defined as a sum

$$\left(\Delta_h^\alpha x\right)(kh) = \sum_{i=0}^{k} a^{(\alpha)}(i) x((k-i)h) h^{-\alpha} = \\ = \begin{bmatrix} 1 & a^{(\alpha)}(1) & \cdots & a^{(\alpha)}(k) \end{bmatrix} \begin{bmatrix} x(kh) \\ x((k-1)h) \\ \cdots \\ x(h) \\ x(0) \end{bmatrix} h^{-\alpha}. \tag{1}$$

The GL-FOBD may be expressed as a discrete convolution: $\left(\Delta_h^\alpha x\right)(kh) = h^{-\alpha}\left(\mathfrak{a} * \overline{x}\right)(k) = h^{-\alpha}\left(\overline{x} * \mathfrak{a}\right)(k)$, where $\mathfrak{a}(k) := a^{(\alpha)}(k)$, $\overline{x}(k) = x(kh)$. In the particular case of constant order function and $h = 1$, we have the following formulas: $\left(\Delta^0 x\right)(k) = x(k)$ and $\left(\Delta^q x\right)(k) = \sum_{i=0}^{k}(-1)^i \binom{q}{i} x(k-i)$ for $k > q-1$ and $q \in \mathbb{N}$.

Let us recall that one-sided $\mathscr{Z}$-transform of a sequence $(x(k))_{k\in\mathbb{N}_0}$ is a complex function given by

$$X(z) := \mathscr{Z}[x](z) = \sum_{k=0}^{\infty} \frac{x(k)}{z^k}, \tag{2}$$

where $z \in \mathbb{C}$ denotes a complex number for which the series (2) converges absolutely. It is a useful tool for solving difference equations with initial conditions. We assume that all discrete functions are zero for negative arguments. Note that since $a^{(\alpha)}(k) = (-1)^k \binom{\alpha}{k}$, then for $|z| > 1$ and $\alpha \in \mathbb{R}$ we have

$$\mathscr{Z}\left[a^{(\alpha)}\right](z) = \sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} z^{-k} = \left(1 - z^{-1}\right)^\alpha. \tag{3}$$

As the GL-FOBD for constant orders is discrete one-sided convolution, we obtain the following result.

Proposition 2. Let $\alpha \in \mathbb{R}$. Then

$$\mathscr{Z}\left[\Delta_h^\alpha x\right](z) = h^{-\alpha}\left(1 - z^{-1}\right)^\alpha X(z),$$

where $X(z) = \mathscr{Z}[\overline{x}](z)$ and $\overline{x}(k) = x(kh)$.

Proof. We need only to use formula (3) and $\mathscr{Z}$-transform of one-sided convolution. □

Proposition 3. Let $\alpha \in \mathbb{R}$, $k_0 \in \mathbb{N}_1$, and

$$y(k) := \left(\Delta_h^\alpha x\right)((k+k_0)h).$$

Then

$$\mathscr{Z}[y](z) \\ = z^{k_0}\left(h^{-\alpha}\left(1-z^{-1}\right)^\alpha X(z) - \sum_{p=0}^{k_0-1} z^{-p}\left(\Delta_h^\alpha x\right)(ph)\right), \tag{4}$$

where $X(z) = \mathscr{Z}[\overline{x}](z)$ and $\overline{x}(k) = x(kh)$.

Proof. It is the consequence of Proposition 2 and the properties of $\mathscr{Z}$-transform. □

For simple example let us observe that for $k_0 = 1$ we have that if $y(k) := \left(\Delta_h^\alpha x\right)((k+1)h)$, then $\mathscr{Z}[y](z) = zh^{-\alpha}\left(\left(1 - z^{-1}\right)^\alpha X(z) - x(0)\right)$, as $\left(\Delta_h^\alpha x\right)(0) = x(0)h^{-\alpha}$.

3. Linear fractional-order system descriptions

In this section we present the system's description that is used in the stability analysis. Let us consider the following system

$$\left(\Delta_h^\alpha x\right)(kh) = \sum_{i=0}^{k_0} A_i x((k-i)h), \qquad k \geq k_0 \tag{5}$$

with initial values $x(0), x(1), \ldots, x(k_0-1)$. Moreover, k_0 is number of delays: $k_0 \in \mathbb{N}_1$, $h > 0$, $\alpha \in (0,1] \cap \mathbb{Q}$, $x(k) \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$. For $k \geq 0$ we can rewrite system (5) in the following way:

$$\left(\Delta_h^\alpha x\right)((k+k_0)h) = \sum_{i=0}^{k_0} A_i x((k+k_0-i)h), \qquad k \geq 0 \tag{6}$$

with initial values $x(0), x(1), \ldots, x(k_0-1)$. Observe that if $k_0 = 1$ and $A_0 = 0$ in (5) (or equivalently (6)), then we get a system in the classical Grünwald-Letnikov forward difference form.

Taking into account the definition of the fractional-order operator and assuming that $I - h^\alpha A_0$ is invertible, the solution of (5) can be written in the following recurrence way, for $k \geq k_0$

$$x(kh) = \left(I - h^\alpha A_0\right)^{-1}\left(h^\alpha \sum_{i=1}^{k_0} A_i x((k-i)h) \right. \\ \left. - \sum_{i=1}^{k} a^{(\alpha)}(i) x((k-i)h)\right). \tag{7}$$

The case: $k_0 = 1$ and $A_0 = 0$ was considered in [5] and many other, and then the recurrence solution is as follows:

$$x(kh) = h^\alpha A_1 x((k-1)h) - \sum_{i=1}^{k} a^{(\alpha)}(i) x((k-i)h). \tag{8}$$

If $k_0 = 1$ and $A_0 \neq 0$, then we have

$$x(kh) = \left(I - h^\alpha A_0\right)^{-1}\left(h^\alpha A_1 x((k-1)h) \right. \\ \left. - \sum_{i=1}^{k} a^{(\alpha)}(i) x((k-i)h)\right). \tag{9}$$

Proposition 4. Let $l \in \mathbb{N}_1$, $h > 0$, $\alpha \in (0,1] \cap \mathbb{Q}$, $x(k) \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$. Then, system (5) with initial values $x(0), x(1), \ldots, x(l-1)$ has unique solution given by the following formula

$$x(kh) = \sum_{p=0}^{k_0-1} \Phi(k-p)\left(\Delta_h^\alpha x\right)(ph) - \sum_{i=0}^{k_0}\sum_{l=0}^{k_0-i-1} \Phi(k-l-i)A_i x(lh), \tag{10}$$

where

$$\Phi(k) = \mathscr{Z}^{-1}\left[h^\alpha\left(1-z^{-1}\right)^{-\alpha} M^{-1}\right](k)$$

with

$$M = I - \left(\frac{h}{1-z^{-1}}\right)^\alpha \sum_{i=0}^{k_0} A_i z^{-i}.$$

Proof. Taking $\mathscr{Z}$-transform of system (6) we get the following

$$z^{k_0}h^{-\alpha}(1-z^{-1})^\alpha X(z) - z^{k_0}\sum_{p=0}^{k_0-1} z^{-p}\left(\Delta_h^\alpha x\right)(ph)$$
$$= \sum_{i=0}^{k_0} A_i\left(z^{k_0-i}\left[X(z) - \sum_{l=0}^{k_0-i-1} z^{-l}x(lh)\right]\right),$$

where $X(z) = \mathscr{Z}[x](z)$. Then,

$$X(z) = h^\alpha(1-z^{-1})^{-\alpha}M^{-1}F(z), \tag{11}$$

where

$$M = I - \left(\frac{h}{1-z^{-1}}\right)^\alpha \sum_{i=0}^{k_0} A_i z^{-i}$$

and

$$F(z) = \sum_{p=0}^{k_0-1} z^{-p}\left(\Delta_h^\alpha x\right)(ph) - \sum_{i=0}^{k_0} A_i \sum_{l=0}^{k_0-i-1} z^{-l-i}x(lh).$$

Now to get the thesis we need to take inverse $\mathscr{Z}$-transform of (11). □

Formula (10) agrees with those proposed in [5] for the situation with $A_1 \neq 0$ and $A_i = 0$ for $i \neq 1$. Then,

$$M = I - \left(\frac{h}{1-z^{-1}}\right)^\alpha \frac{1}{z}A_1,$$

and

$$\Phi(k) = \mathscr{Z}^{-1}\left[h^\alpha(1-z^{-1})^{-\alpha}M^{-1}\right](k),$$

so $x(kh) = \Phi(k)x(0)h^{-\alpha}$, with simple form of $F(z) = x(0)h^{-\alpha}$.

4. Stability conditions and examples

We say that the constant vector $x^{eq} = (x_1^{eq}, \ldots, x_n^{eq})$ is an *equilibrium point* of fractional difference system (6) (or eqivalently (5)) if and only if

$$\left(\Delta_h^\alpha x_i^{eq}\right)((k+k_0)h) = \sum_{i=0}^{k_0} A_i x_j^{eq},$$

where $i = 1, \ldots, n$ and $k \in \mathbb{N}_0$. Note that the trivial solution $x \equiv 0$ is an equilibrium point of system (6) (or eqivalently (5)).

Let $\overline{x}(s) := (x_1(kh), x_2(kh), \ldots, x_n(kh))^{\mathrm{T}} \in \mathbb{R}^n$.

Definition 5. The equilibrium point $x^{eq} = 0$ of (6) (or eqivalently (5)) is said to be

(a) *stable* if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|\overline{x}(0)\| < \delta$ implies $\|\overline{x}(k)\| < \epsilon$, for all $k \in \mathbb{N}_0$.
(b) *asymptotically stable* if it is stable and $\lim_{k\to+\infty} \overline{x}(k) = 0$.

The fractional difference system (6) (or eqivalently (5)) is called *stable* (*asymptotically stable*), if its equilibrium point $x^{eq} = 0$ is stable (asymptotically stable).

Proposition 6. Let $\det(I - h^\alpha A_0) \neq 0$ and R be the set of all roots of the equation

$$\det\left(I - \left(\frac{h}{1-z^{-1}}\right)^\alpha \sum_{i=0}^{k_0} A_i z^{-i}\right) = 0\,. \tag{12}$$

Then the following items are satisfied:

(a) If all elements from R are strictly inside the unit circle, then system (6) (or equivalently (5)) is asymptotically stable.
(b) If there is $z \in R$ such that $|z| > 1$, then system (6) (or equivalently (5)) is not stable.

Proof. The proof is similar to those presented in [5]. Here we need to base the proof on the formula of $\mathscr{Z}$-transform of function $\Phi(\cdot)$ from Proposition 4. □

Now, let us present the example that illustrates the behaviour of trajectories of the considered systems in the scalar case, i.e. $A_i \in \mathbb{R}$, $i = 0, 1, \ldots, k_0$. For better visualization the points obtained as the values of systems's solutions are connected.

Example 7. Let us consider system (5) in the scalar case for $\alpha = 0.5$, $k_0 = 2$ and $A_0 = 0$, $A_1 = a_1$, $A_2 = a_2$:

$$\left(\Delta^{0.5}x\right)(k) = a_1 x(k-1) + a_2 x(k-2), \qquad k \geq 2 \tag{13}$$

with initial conditions $x(0) = x(1) = 1$. We change the common situation:

(a) $a_1 = -1.4142$, $a_2 = 0$,
(b) $a_1 = 0$, $a_2 = -1.1175$,
(c) $a_1 = -1.414$, $a_2 = -1.1175$.

Cases (a) and (b) are the limit of stability values, where we have only one delay, and in case (c) we analyze two parameters. We found the limit values by calculations in Maple package and we compare them with calculations of solution of region R given by (12). In the next Propositions we state the interval and region for eigenvalues of the matrices when we have only one delay. The lines for solutions with parameters from cases (a) and (b) are presented in Fig. 1.

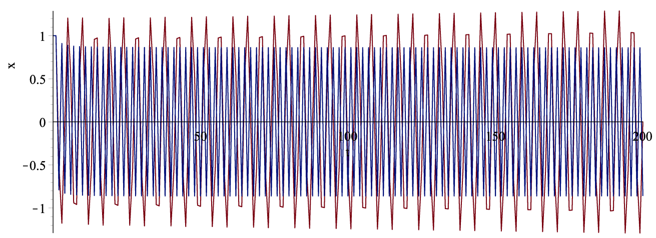

Fig. 1. Stable solutions (limit cases) of system (5) with $A_0 = 0$, $A_1 = a_1$, $A_2 = a_2$, $a_1, a_2 \in \mathbb{R}$, $T = 200$, $h = 1$, blue: (a) $a_1 = 1.4142$, $a_2 = 0$; red: (b) $a_1 = 0$, $a_2 = -1.1175$

In case (c): $\left(\Delta^{0.5}x\right)(k) = -1.4142x(k-1) - 1.1175x(k-2)$ we get unstable solution, see Fig. 2. For given a_1, a_2 the set R from condition (12) has elements from outside of the unit circle with $|z| = 1.020251430$.

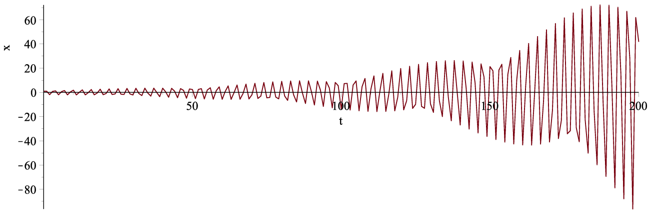

Fig. 2. Unstable solution of system (5): $\left(\Delta^{0.5}x\right)(k) = -1.4142x(k-1) - 1.1175x(k-2)$, $T = 200$, $h = 1$

However, for the case

(d) $a_0 = 0$, $a_1 = -2.4142$, $a_2 = -1$

we have $R = 0.9997756270$ at the limit of stability again, see Fig. 3.

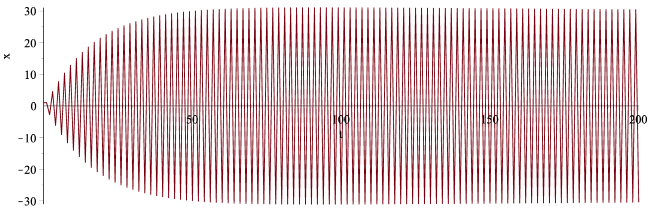

Fig. 3. Unstable solution of system (5): $\left(\Delta^{0.5}x\right)(k) = -2.4142x(k-1) - x(k-2)$, $T = 200$, $h = 1$

In the next proposition we state the condition for stability/instability only for the situation with one delay $k_0 \in \mathbb{N}_1$. Let us consider the system (5) with one delay $k_0 \in \mathbb{N}_1$. Therefore, we take $A_i = 0$ for $i \neq k_0$ and $A_{k_0} = A \neq 0$. Then, system (5) has the following form

$$\left(\Delta_h^\alpha x\right)(kh) = Ax((k-k_0)h), \qquad k \geq k_0 \tag{14}$$

with initial values $x(0), x(1), \ldots, x(k_0 - 1)$, $A \in \mathbb{R}^{n \times n}$, where $x(kh) \in \mathbb{R}^n$.

Now, we formulate the conditions for the stability/instability of (14) taking into account the eigenvalues of matrix A.

Proposition 8.

(a) If the following conditions are satisfied

(1) for all $i = 1, \ldots, n$

$$\arg \lambda_i \in \left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right], \tag{15}$$

(2) for all $i = 1, \ldots, n$

$$|\lambda_i| < |w_i|, \ i = 1, \ldots, n, \tag{16}$$

where $\arg \lambda_i$ and $|\lambda_i|$ are respectively the main argument and modulus of $\lambda_i \in Spec(A)$ and

$$|w_i| = \begin{cases} \left(\frac{2}{h}\left|\sin\frac{2\arg\lambda_i - \alpha\pi}{2(2k_0-\alpha)}\right|\right)^\alpha, & \arg\lambda_i \in [0,\pi], \\ \left(\frac{2}{h}\left|\sin\frac{2\arg\lambda_i - \alpha\pi + 4(k_0-1)\pi}{2(2k_0-\alpha)}\right|\right)^\alpha, & \arg\lambda_i \in (\pi, 2\pi), \end{cases} \tag{17}$$

then system (14) with $\alpha \in (0,1]$ is asymptotically stable.

(b) If there exists $\lambda_i \in Spec(A)$ such that $|\lambda_i| > |w_i|$, then system (14) with $\alpha \in (0,1]$ is not stable.

Proof. Let $z \in \mathbb{C}$ with $\varphi = \arg z \in [0, 2\pi]$ and $\alpha \in (0,1]$. Then

$$\log z^\alpha = \alpha \ln|z| + \alpha i\varphi + i2k\pi\,,$$

where $k \in \mathbb{N}_0$. We will show that conditions (15) and (16) are equivalent to the fact that all roots of the equation (12) are inside the unit circle. Let

$$w = \frac{z^{k_0}}{h^\alpha}\left(1 - z^{-1}\right)^\alpha$$

with $|z| = 1$. Then, $z = e^{i\varphi}$,

$$\xi = \arg\left(1 - z^{-1}\right) = \arctan\frac{\sin\varphi}{1-\cos\varphi} = \arctan\left(\tan\left(\frac{\pi}{2} - \frac{\varphi}{2}\right)\right) = \frac{\pi - \varphi}{2}$$

and $\left|1 - z^{-1}\right| = 2\sin\frac{\varphi}{2}$. Hence,

$$\log w = \ln\left(\frac{2\sin\frac{\varphi}{2}}{h}\right)^\alpha + i\left(k_0\varphi + \alpha\frac{\pi-\varphi}{2} + 2l\pi\right), \tag{18}$$

where $l \in \mathbb{Z}$.

Then we need to find the formula for the main argument of w, as corresponding eigenvalue λ satisfies $\arg\lambda = \arg w$, where $\varphi = \arg z \in [0, 2\pi]$.

From (18) we get there exists $l \in \mathbb{Z}$ such that $\arg w = k_0\varphi + \alpha\frac{\pi-\varphi}{2} + 2l\pi$. Hence one gets

$$\arg w = k_0\varphi + \alpha\frac{\pi-\varphi}{2} + 2l\pi = \left(k_0 - \frac{\alpha}{2}\right)\varphi + \alpha\frac{\pi}{2} + 2l\pi, \tag{19}$$

for some $l \in \mathbb{Z}$.

Observe that the conditions $0 \leq \varphi \leq 2\pi$ are equivalent to

$$\alpha\frac{\pi}{2} \leq \left(k_0 - \frac{\alpha}{2}\right)\varphi + \alpha\frac{\pi}{2} \leq 2k_0\pi - \alpha\frac{\pi}{2}. \tag{20}$$

Then

$$\alpha\frac{\pi}{2} \leq \arg w \leq 2\pi - \alpha\frac{\pi}{2}.$$

Therefore, condition (15) is satisfied if only z belongs to the unit circle. Now, we need to calculate from (19) the main argument φ and state that eigenvalues for which all roots of the equation (12) are inside the unit circle should have smaller modulus than corresponding points on the border, i.e. $|\lambda_i| < |w_i|$. However, from (18) $|w_i| = \left|\frac{2}{h}\sin\frac{\varphi}{2}\right|^\alpha$. From (19) if $\arg w \in [0,\pi]$ one gets

$$\frac{\varphi}{2} = \frac{2\arg w - \alpha\pi}{2(2k_0 - \alpha)}.$$

Observe that for $\arg w \in (\pi, 2\pi)$ by (19) there exists $l \in \mathbb{Z}$ such that

$$\frac{\varphi}{2} = \frac{2\arg w - \alpha\pi - 4l\pi}{2(2k_0 - \alpha)}.$$

Since $\varphi \leq 2\pi$, one gets $\arg w \leq 2(k_0 + l - 1)\pi + (1 - \alpha)\frac{\pi}{2}$. Therefore in order to guarantee $\arg w < 2\pi$ one has to take $l = 1 - k_0$. Hence (16) holds. Since conditions (15) and (16) are equivalent to the fact that all roots of the equation (12) are inside the unit circle, by Proposition 6 we get that system (14) is asymptotically stable and (a) holds. Observe that the existence of $\lambda_i \in Spec(A)$ such that $|\lambda_i| > |w_i|$ is equivalent to the fact that there is a root of (12) that is outside the unit circle. Then by Proposition 6 we get that system (14) is not stable and (b) holds. □

Corollary 9. For the scalar system (14), i.e. with $A = \lambda < 0$ and $\alpha \in (0, 1]$ we have that:

(a) If $|\lambda| < \left(\frac{2}{h} \sin \frac{2-\alpha}{2k_0 - \alpha} \frac{\pi}{2}\right)^{\alpha}$ (particularly for $k_0 = 1$: $|\lambda| < \left(\frac{2}{h}\right)^{\alpha}$), then system (14) is asymptotically stable.

(b) If $|\lambda| > \left(\frac{2}{h} \sin \frac{2-\alpha}{2k_0 - \alpha} \frac{\pi}{2}\right)^{\alpha}$ (particularly for $k_0 = 1$: $|\lambda| > \left(\frac{2}{h}\right)^{\alpha}$), then system (14) is not stable.

Proof. For the case $k_0 = 1$. Since $\lambda < 0$, $\arg \lambda = \pi$. Then

$$|w| = \left(\frac{2}{h}\left|\sin \frac{\pi - \alpha\frac{\pi}{2}}{2 - \alpha}\right|\right)^{\alpha} = \left(\frac{2}{h}\left|\sin \frac{\pi}{2}\right|\right)^{\alpha} = \left(\frac{2}{h}\right)^{\alpha}.$$

□

From simple calculations for scalar case of (14) we have also the following result:

Corollary 10. For the scalar system (14), i.e. with $A = \lambda < 0$, $k_0 \geq 2$ and $\alpha = \frac{2}{k_0}$ we have that:

(a) If $|\lambda| < \left(\frac{2}{h} \sin \frac{\pi}{k_0+1}\right)^{\alpha}$, then system (14) is asymptotically stable.

(b) If $|\lambda| > \left(\frac{2}{h} \sin \frac{\pi}{k_0+1}\right)^{\alpha}$, then system (14) is not stable.

Example 11. Conditions presented in Proposition 8 can be illustrated by areas of the location of eigenvalues of matrix $A_{k_0} = A$ for which the considered systems are stable, see Figs. 4–7.

a) $\alpha \in \{0.5, 0.7, 0.9\}$, $k_0 = 2$ b) $\alpha \in \{0.5, 0.7, 0.9\}$, $k_0 = 100$

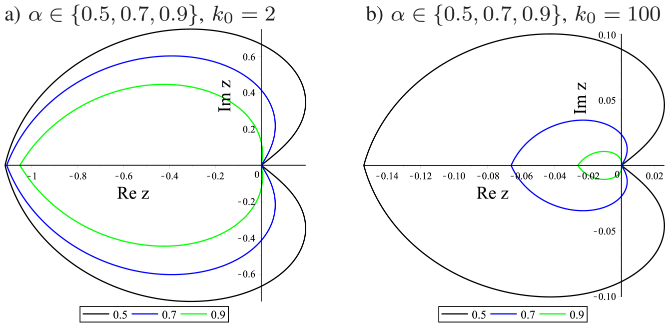

Fig. 4. Borders of areas of eigenvalues of matrix A for stability of system (14) with different orders $\alpha \in \{0.5, 0.7, 0.9\}$ and in: a) delay $k_0 = 2$, b) delay $k_0 = 100$

a) $\alpha \in \{0.5, 0.4, 0.2\}$, $k_0 = 2$ b) $\alpha \in \{0.5, 0.4, 0.2\}$, $k_0 = 100$

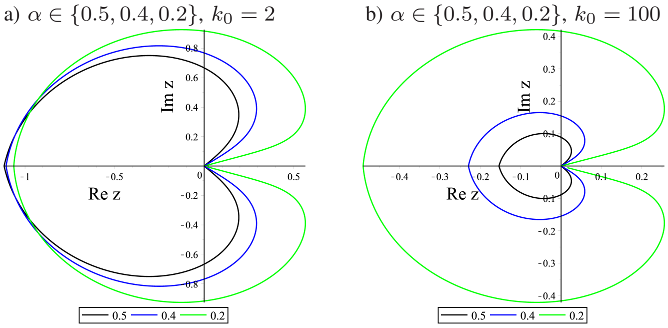

Fig. 5. Borders of areas of eigenvalues of matrix A for stability of system (14) with different orders $\alpha \in \{0.5, 0.4, 0.2\}$ and in: a) delay $k_0 = 2$, b) delay $k_0 = 100$

a) $\alpha \in \{0, 0.05, 0.1\}$, $k_0 = 2$ b) $\alpha \in \{0, 0.05, 0.1\}$, $k_0 = 10$

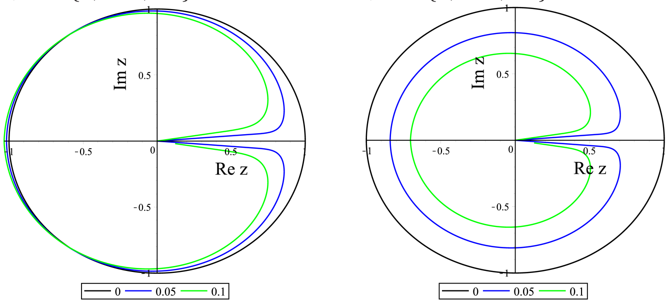

Fig. 6. Borders of areas of eigenvalues of matrix A for stability of system (14) with different orders $\alpha \in \{0, 0.05, 0.1\}$ and in: a) delay $k_0 = 2$, b) delay $k_0 = 10$

a) $\alpha = 0.5$, $k_0 \in \{1, 2, 100\}$ b) $\alpha = 0.9$, $k_0 \in \{1, 2, 10\}$

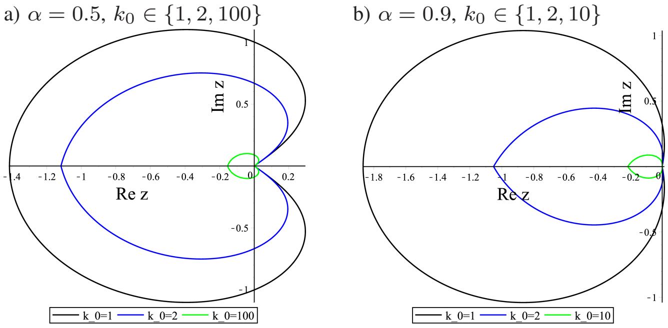

Fig. 7. Borders of areas of eigenvalues of matrix A for stability of system (14) with different delays and orders in: a) $\alpha = 0.5, k_0 \in \{1, 2, 100\}$, b) $\alpha = 0.9, k_0 \in \{1, 2, 10\}$

Example 12. Let us consider the system of the form (14) with $\alpha \in (0, 1]$, and $A = \begin{bmatrix} -1 & 0.5 \\ -0.1 & -1 \end{bmatrix}$ with eigenvalues $\lambda = -1 \pm 0.316227766016838 \cdot i$, $|\lambda| = 1.04880884817015$. For $k_0 = 2$ we have stability for $\alpha \leq 0.5$, plots of values x_1, x_2 are presented in Fig. 8a and in plane $x_1 x_2$ in Fig. 8b. If we change k_0 to $k_0 = 3$ then we need to have lower values of order to preserve stability.

5. Conclusions

The problem of stability of the Grünwald-Letnikov-type linear fractional-order discrete-time systems with delays was studied by using $\mathscr{Z}$-transform method. We presented the sufficient

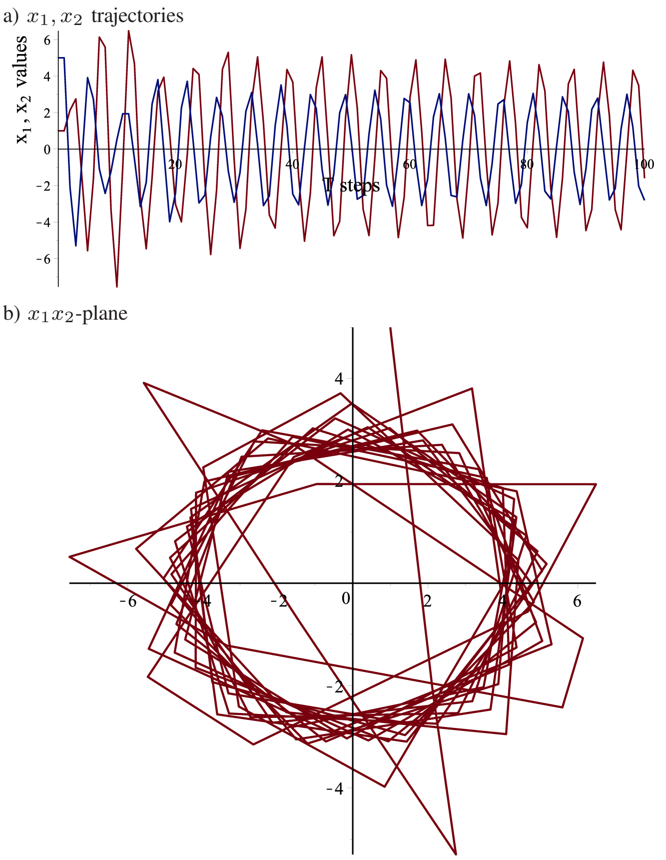

Fig. 8. Stable solutions of system (14), $x_1 x_2$-plane, $\alpha = 0.5$, $h = 1$, $k_0 = 2$

conditions for the asymptotic stability of the fractional-order discrete systems with delays. Using these conditions we determined the regions of location of eigenvalues of matrices associated to the systems in order to guarantee the asymptotic stability of the considered systems.

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