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The analysis of non-linear free vibration of FGM nano-beams based on the conformable fractional non-local model

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Abstract. Continuum models generalized by fractional calculus are used in different mechanical problems. In this paper, by using the conformable fractional derivative (CFD) definition, a general form of Eringen non-local theory as a fractional non-local model (FNM) is formulated. It is then used to study the non-linear free vibration of a functional graded material (FGM) nano-beam in the presence of von-Kármán non-linearity. A numerical solution is obtained via Galerkin and multiple scale methods and effects of the integer and non-integer (fractional) order of stress gradient (in the non-local stress-strain relation) on the ratio of the non-local non-linear natural frequency to classical non-linear natural frequency of simply-supported (S-S) and clamped-free (C-F) FGM nano-beams are presented.

Key words: FGM nano beam, non-linear free vibration, fractional calculus, integer and non-integer stress gradient's order, fractional non-local model.

1. Introduction

Nanotechnology is primarily concerned with the fabrication of FGM and engineering structures at a nanoscale, which provides a new category of materials with revolutionary properties and devices with enhanced functionality. One of these structures is the nano-beam, which is widely used in many systems and devices such as nano-wires, nano-probes, atomic force microscope, nano-actuators and nano-sensors. To avoid the peak resonance, the designer must know the natural frequency so the analysis of free vibration of structural elements must be performed during the designing process. This problem becomes more pronounced for nanostructures like oscillators, clocks and sensor devices and is still not solved in the mechanical community.

There are two different methods for mechanical analysis of nanostructures [1]: the molecular dynamic model and continuum models. Continuum models provide a simpler description of the nanostructures. Due to the absence of the length scale parameter (although the techniques of the traditional micro to macro averaging theory – implicit non-locality – are of a broad applicability also cf. [37]), the classical continuum theories fail to accurately predict the mechanical behavior of nanostructures [2], therefore non-conventional continuum theories which contain an additional material length scale parameter have been proposed, such as strain-gradient theories [3, 4], micro-polar theories [5–8] or theories of material surfaces [9]. Among these models, the non-local theory is one of the most likely used. The well-known constitutive relation for non-local elasticity involves an integral over the whole body. Since solving the integral constitutive equation is difficult, Eringen has approximated it with a simplified equation of a differential form (see [10]) and showed that approximated dispersion curves based on his model are in good agreement with experimental evidence. Forasmuch, as other types of approximations are possible, Challamel et al. [11] presented a non-local elasticity model in terms of fractional calculus (based on the Caputo definition) that generalizes Eringen non-local theory (ENT) and showed that the fractional derivative model gives a perfect matching with the dispersive wave properties of the Born–Kármán model of lattice dynamics and its results are in better agreement with experimental data than ENT (cf. also [22]).

Fractional calculus is a branch of mathematical analysis that studies differential operators of an arbitrary (real or complex) order and is a different approach to non-local mechanics. Fractional differential operators introduce non-locality to the description in a natural way [12] and they make the description more realistic. They also are useful in describing the occurrence of vibrations in engineering practice [13] which was pointed out by many authors [14], therefore attention concentrates on this field.

In solid mechanics, the fractional calculus approach has been introduced especially in describing viscous behavior of materials [15–17]. The idea to include a fractional term in the governing equation of the elastic problem has been proposed by Lazopoulos, who showed in [18] that the strain energy density depends both on the local value and on fractional order of strain. Demir et al. [13] studied the application of fractional calculus in the dynamic analysis of beam and concluded that both the order and the coefficient of the fractional derivative have a significant effect on the natural frequency and the amplitude of vibrations. Palfalvi presented an efficient solution of a vibration equation involving fractional derivatives [19]. Atanackovic et al. [20] studied the motion of a one-dimensional continuum whose deformation was described by a strain measure of non-local type.

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Sumelka [12, 21] generalized the Kirchhoff–Love plate's theory by using fractional calculus, and studied the new concept of space-fractional continuum body utilizing the Riesz– Caputo derivative. He also studied a free axial vibration of a nano-rod based on this model [22], and presented a fractional non-local Euler-Bernoulli beam theory [23] for better approximation of experimental Young modulus values. More recently, Rahimi et al. [33, 36] presented a generalized form of non-local elasticity theory using the conformable fractional derivatives definition (CFDD) and showed its application in analysis of free vibration of homogenous nano-beams and a study of critical point instability of micro- and nano-beams.

As mentioned above, Challamel et al. [11] presented a general form of ENT based on the Caputo definition. However, the integral form of fractional derivatives definition like Riemann–Liouville, Caputo, and Grunwald-Letnikov makes the numerical solution of the governing equations difficult. Due to this limitation, this paper presents a general form of ENT based on the CFD definition, without an integral form. The presented model is used to study the non-linear vibration of a FGM nano-beam. To the best of the authors' knowledge, the application of a conformable non-local model in non-linear vibration and problems related to FGM has not been studied elsewhere.

1.1. Conformable fractional derivative. There are different types of definition of fractional derivatives but herein the CFD has been used instead of the most popular ones like Riemann–Liouville, Caputo or Grunwald-Letnikov [14]. The reason is that RL, C, GL definitions, as well as many others that are not mentioned here, have some drawbacks (see Khalil et al. [24]). Furthermore, the solution of fractional differential equations that is achieved by these definitions is difficult due to the necessity of formulation complex numerical approximations [25–29]. This difficulty can limit the application of fractional calculus in real life complex problems. The CFD definition [24,38,39] has been used because it has no integral form and makes it possible to solve the obtained equations in the usual way. It also eliminates some of the defects of other definitions [14, 30, 31].

The CFD definition is (cf. Appendix A):

$$\begin{aligned} D_x^{\alpha}(f)(x,y) &= x^{(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}f(x,y)}{dx^{\lceil\alpha\rceil}}, \\ D_x^{\alpha} &= \frac{\partial^{\alpha}}{\partial x^{\alpha}}, \\ D_y^{\alpha}(f)(x,y) &= y^{(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}f(x,y)}{dy^{\lceil\alpha\rceil}}, \\ D_y^{\alpha} &= \frac{\partial^{\alpha}}{\partial y^{\alpha}}, \end{aligned} \tag{1}$$

where α is a fractional parameter, $\alpha \in (n-1, n]$ and is the smallest integer greater than or equal to α.

Here we consider $1 < \alpha \leq 2$ because we want to compare our fractional model with the classical non-local theory of the Eringen type. It is clear that when α is an integer Eq. (1a) reduces to the following (classical/local) form:

$$\begin{aligned} D_x^{n}(f)(x,y) &= \frac{d^{n}f(x,y)}{dx^{n}}, \\ D_y^{n}(f)(x,y) &= \frac{d^{n}f(x,y)}{dy^{n}}. \end{aligned} \tag{2}$$

2. Mathematical modeling of FGM nano-beam

Consider the functionally graded S-S and C-F nano-beams of length L, width b, and thickness h. Material properties of the beam, i.e., Young and shear modulus and mass density vary continuously along the beam thickness as functions of the z coordinate according to the power law distribution

$$p(z) = (p_l - p_u)\left(\frac{2z+h}{2h}\right)^k + p_u, \tag{3}$$

where subscripts u and l refer to material properties of the upper and lower surfaces, respectively, and k is a non-negative number that dictates the material variation profile through the thickness of the beam.

The displacement field, based on the Euler-Bernoulli beam theory, is restricted to:

$$\begin{aligned} u(x,z,t) &= u_0(x,t) - z\frac{\partial w_0(x,t)}{\partial x}, \\ w(x,z,t) &= w_0(x,t), \end{aligned} \tag{4}$$

where u_0 and w_0 are the axial and the transverse displacements, respectively, of any point on the axis, and t denotes time. In accordance, the von Kármán type non-linear strain–displacement relationship is

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \frac{\partial u_0}{\partial x} - z\left(\frac{\partial^2 w_0}{\partial x^2}\right) + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2. \tag{5}$$

Using the Hamilton principle, the equations of motion are derived as below:

$$\frac{\partial N}{\partial x} = D_1\frac{\partial^2 u_0}{\partial t^2}, \tag{6}$$

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial}{\partial x}\left(N\frac{\partial w_0}{\partial x}\right) = D_1\frac{\partial^2 w_0}{\partial t^2}, \tag{7}$$

where N and M are the stress resultants:

$$\mathrm{N} = \int_A \sigma_{\mathrm{xx}}\mathrm{dA} = bA_1\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right) - bB_1\left(\frac{\partial^2 w_0}{\partial x^2}\right), \tag{8}$$

$$\begin{aligned} \mathrm{M} &= \int_A \sigma_{\mathrm{xx}}\mathrm{zdA} = \\ &= bB_1\left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right) - bC_1\left(\frac{\partial^2 w_0}{\partial x^2}\right), \end{aligned} \tag{9}$$

where the parameters A_1, B_1, C_1 are:

$$\{A_1, B_1, C_1\} = \int_{-h/2}^{+h/2} E(z) = \{1, z, z^2\} dz, \qquad D_1 = \int_{-h/2}^{+h/2} \rho(z) dz. \tag{10}$$

Below we present a fractional non-local elasticity model. This model can be understood as a possible generalization of Eringen non-local elastic model, with a non-integer order stress gradient in the stress–strain equation. This model contains two additional material parameters compared to the classical local formulation: the characteristic length scale and the stress gradient order.

The fractional non-local model that is a generalization of the Eringen model is defined as below [11]:

$$\sigma_{xx} - \mu \frac{d^\alpha \sigma_{xx}}{dx^\alpha} = E \varepsilon_{xx}, \tag{11}$$

where σ and ε are, respectively, the uniaxial stress and strain, E is the Young modulus and $\mu = (e_0 a)\alpha$, e_0 is a material constant to be determined experimentally, a is the internal characteristic length (e.g. lattice parameter, granular size, distance between C-C bonds) and α is an order of stress gradient. The classical form of Eringen non-local model is obtained when $\alpha = 2$ and a local form of strain-stress is achieved when $\mu = 0$.

For the Euler-Bernoulli beam the constitutive relations are given by:

$$\mathrm{N} - \mu \frac{d^\alpha \mathrm{N}}{dx^\alpha} = bA_1 \hat{\varepsilon} + bB_1 \hat{k}, \tag{12}$$

$$\mathrm{M} - \mu \frac{d^\alpha \mathrm{M}}{dx^\alpha} = bB_1 \hat{\varepsilon} + bC_1 \hat{k}. \tag{13}$$

Using the CFD definition for $1 < \alpha \leq 2$, Eqs (12–13) gives:

$$\mathrm{N} - \mu x^{2-\alpha} \frac{d^2 \mathrm{N}}{dx^2} = bA_1 \hat{\varepsilon} + bB_1 \hat{k}, \tag{14}$$

$$\mathrm{M} - \mu x^{2-\alpha} \frac{d^2 \mathrm{M}}{dx^2} = bB_1 \hat{\varepsilon} + bC_1 \hat{k}. \tag{15}$$

If the axial inertia is neglected, Eq. (6) gives N = N0 = cte, so Eq. (12) can be simplified to

$$\mathrm{N} = bA_1 \hat{\varepsilon} + bB_1 \hat{k}. \tag{16}$$

For a nano-beam with an immovable ends (i.e. u_0 and $w_0 = 0$, at x = 0 and L) and by assuming N = N0 = cte, integration of Eq. (16) with respect to x leads to [34, 35]

$$N = N_0 = \frac{bA_1}{2L} \int_0^L \left(\frac{\partial w_0}{\partial x}\right)^2 dx - \frac{bB_1}{L} \int_0^L \frac{\partial^2 w_0}{\partial x^2} dx. \tag{17}$$

In order to express the bending moment in terms of deflection, Eq. (8) can be rewritten as follows

$$bB_1 \left(\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right) = \frac{B_1}{A_1}\left(N + bB_1\left(\frac{\partial^2 w_0}{\partial x^2}\right)\right). \tag{18}$$

Taking the second derivative of Eq. (10b) and substituting Eq. (13), Eq. (17) and Eq. (18) into it, we obtain the fractional non-linear equation of the Euler-Bernoulli functionally graded nano-beam

$$\begin{aligned} &\left(bC_1 - b\frac{B_1^2}{A_1} - \mu N_0 x^{2-\alpha}\right)\frac{\partial^4 w_0}{\partial x^4} + \\ &+ \left(2\mu(1-\alpha)N_0 x^{1-\alpha}\right)\frac{\partial^3 w_0}{\partial x^3} + \\ &+ \left((2-\alpha)(1-\alpha)x^{-\alpha} - 1\right)N_0 \frac{\partial^2 w_0}{\partial x^2} - \\ &- \left(\mu x^{2-\alpha} D_1\right)\frac{\partial^4 w_0}{\partial x^2 \partial t^2} - \\ &- \left(\mu\left(2(2-\alpha)x^{1-\alpha} D_1\right)\right)\frac{\partial^3 w_0}{\partial x \partial t^2} + \\ &+ \left(D_1 - D_1 \mu(2-\alpha)(1-\alpha)x^{-\alpha}\right)\frac{\partial^2 w_0}{\partial t^2} = 0. \end{aligned} \tag{19}$$

It should be noticed, as mentioned, that the Eringen non-local model is recovered when $\alpha = 2$ and a classical local form is achieved when $\mu = 0$. For convenience, the following non-dimensional variables are used:

$$\begin{aligned} &\hat{w} = \frac{w}{L},\ \hat{x} = \frac{w}{L},\ \hat{t} = \frac{t}{T},\ T = \sqrt{\frac{D_1 L^4}{bC_1}},\ \hat{z} = \frac{z}{h},\ \hat{N}_0 = \frac{bA_1 h^2}{2L^2}\int_0^1 \left(\frac{\partial \hat{w}_0}{\partial x}\right)^2 d\hat{x} - \frac{bB_1 h}{L^2}\int_0^1 \frac{\partial^2 \hat{w}_0}{\partial \hat{x}^2} d\hat{x} = \hat{N}_{01} + \hat{N}_{02}, \\ &\left(bC_1 - b\frac{B_1^2}{A_1} + \mu^\alpha \hat{N}_0 (L\hat{x})^{2-\alpha}\right)\left(\frac{h}{L^4}\right)\frac{\partial^4 \hat{w}_0}{\partial \hat{x}^4} + \left(2\mu^\alpha(2-\alpha)\hat{N}_0 (L\hat{x})^{1-\alpha}\right)\left(\frac{h}{L^3}\right)\frac{\partial^3 \hat{w}_0}{\partial \hat{x}^3} + \\ &+ \left(\mu^\alpha(2-\alpha)(1-\alpha)(L\hat{x})^{-\alpha} - 1\right)\hat{N}_0\left(\frac{h}{L^2}\right)\frac{\partial^2 \hat{w}_0}{\partial \hat{x}^2} - \left(\mu^\alpha (L\hat{x})^{2-\alpha}\hat{D}_1\right)\hat{N}_0\left(\frac{h}{L^2 T^2}\right)\frac{\partial^4 \hat{w}_0}{\partial \hat{x}^2 \partial \hat{t}^2} - \\ &- \left(\mu^\alpha\left(2(2-\alpha)(L\hat{x})^{1-\alpha}\hat{D}_1\right)\right)\left(\frac{h}{LT^2}\right)\frac{\partial^3 \hat{w}_0}{\partial \hat{x} \partial \hat{t}^2} + \left(\hat{D}_1 - \hat{D}_1 \mu^\alpha (2-\alpha)(1-\alpha)(L\hat{x})^{-\alpha}\right)\left(\frac{h}{T^2}\right)\frac{\partial^2 \hat{w}_0}{\partial \hat{t}^2}. \end{aligned} \tag{20}$$

3. Non-linear vibration analysis of FGM nano-beams

For non-linear free vibration analysis, the Galerkin method is used to convert Eq. (20) into an ordinary differential equation. The solution for the n-th mode is approximated as

$$w(\hat{x},\hat{t}) = \varphi(\hat{x}).q(\hat{t}), \tag{21}$$

where $\varphi(x)$ is a suitable mode shape (cf. [32]) and q (t) is a time dependent function to be determined.

Next, substituting Eq. (21) into Eq. (20), multiplying it by the mode shape and integrating the outcome from 0 to 1, we obtain

$$b_1\ddot{q}(\hat{t}) + b_2 q(\hat{t}) + b_3 q^2(\hat{t}) + b_4 q^3(\hat{t}) = 0, \tag{22}$$

where:

$$b_1 = D_1\int_0^1 \begin{bmatrix} \left[-\mu^\alpha (L\hat{x})^{2-\alpha}\left(\frac{h_1}{L^2T^2}\right)\varphi''\varphi\right] - \\ \left[2\mu^\alpha(2-\alpha)(L\hat{x})^{1-\alpha}\left(\frac{h}{LT^2}\right)\varphi'\varphi\right] + \\ \left[\left\{1-\mu^\alpha(2-\alpha)(1-\alpha)(L\hat{x})^{-\alpha}\right\}\left(\frac{h}{T^2}\right)\right] \end{bmatrix} d\hat{x},$$

$$b_2 = \left[b\hat{C}_1 - b\frac{\hat{B}_1^2}{\hat{A}_1}\right]\left(\frac{h}{L^4}\right)\int_0^1 \varphi(x)\varphi(x)'''' \, d\hat{x},$$

$$b_3 = \int_0^1 \hat{N}_{02} \begin{bmatrix} \left[\mu^\alpha (L\hat{x})^{2-\alpha}\left(\frac{h}{L^4}\right)\varphi''''\varphi\right] + \\ \left[2\mu^\alpha(2-\alpha)(L\hat{x})^{1-\alpha}\left(\frac{h}{L^3}\right)\varphi'''\varphi\right] + \\ \left[\left\{\mu^\alpha(2-\alpha)(1-\alpha)(L\hat{x})^{-\alpha}-1\right\}\left(\frac{h}{L^2}\right)\varphi''\varphi\right] \end{bmatrix} d\hat{x},$$

$$b_4 = \int_0^1 \hat{N}_{01} \begin{bmatrix} \left[\mu^\alpha (L\hat{x})^{2-\alpha}\left(\frac{h}{L^4}\right)\varphi''''\varphi\right] + \\ \left[2\mu^\alpha(2-\alpha)(L\hat{x})^{1-\alpha}\left(\frac{h}{L^3}\right)\varphi'''\varphi\right] + \\ \left[\left\{\mu^\alpha(2-\alpha)(1-\alpha)(L\hat{x})^{-\alpha}-1\right\}\left(\frac{h}{L^2}\right)\varphi''\varphi\right] \end{bmatrix} d\hat{x}.$$

Finally, Eq. (22) can be rewritten as below:

$$\begin{aligned} &\ddot{q}(\hat{t}) + \beta_{11}q(\hat{t}) + \beta_{12}q^2(\hat{t}) + \beta_{13}q^3(\hat{t}) = 0, \\ &\beta_{11} = \frac{b_2}{b_1},\ \beta_{12} = \frac{b_3}{b_1},\ \beta_{13} = \frac{b_4}{b_1}. \end{aligned} \tag{23}$$

3.1. Numerical solution. To derive the solution of Eq. (23), a set of the first-order approximations is applied

$$q(t) = \sum_{i=1}^{3} \varepsilon^i q_i(T_0, T_1, T_2) + \ldots, \tag{24}$$

where $T_0 =$ t is the fast time scale characterizing the motions corresponding to the unperturbed linear system, and $T_1 = \varepsilon t$ is the slow time scale characterizing the modulation of the amplitudes and phases due to non-linearity. The derivatives with respect to t become expansions in terms of partial derivatives with respect to T_n according to:

$$\frac{d}{dt} = \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} + \ldots = D_0 + \varepsilon D_1 + \ldots, \tag{25}$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \ldots. \tag{26}$$

Substituting Eqs (24) and (25, 26) into Eq. (23) and equating the coefficients of ε^1, ε^2, ε^3 … to zero, we obtain:

$$\varepsilon^1 : D_0^2 q_1 + q_1 = 0, \tag{27}$$

$$\varepsilon^2 : D_0^2 q_2 + q_2 + 2D_0D_1q_1 + \beta_1 q_1^2 = 0, \tag{28}$$

$$\begin{aligned} \varepsilon^3 : &D_0^2 q_3 + \beta_2 q_3 + \left(D_1^2 + 2D_0D_1\right)q_1 + \\ &+ 2D_0D_1q_1 + 2\beta_1 q_1 q_2 = 0, \end{aligned} \tag{29}$$

where $\beta_1 = \frac{\beta_{12}q_{\max}}{\beta_{11}}$, $\beta_2 = \frac{\beta_{13}q_{\max}^2}{\beta_{11}}$, $q(0) = q_{\max}$, $\dot{q}(0) = 0$.

The general solution of Eq. (27) is

$$q_1 = A_2(T_1, T_2)\exp(jT_0) + \bar{A}_2\exp(-jT_0), \tag{30}$$

where A_2 is an unknown complex function and $\bar{A}_2$ is a complex conjugate of A_2. Substituting Eq. (30) into Eq. (28) gives:

$$\begin{aligned} D_0^2 q_2 + q_2 = &-2jD_1A_2(T_1,T_2)\exp(jT_0) - \\ &- \beta_1\left(\bar{A}_2^2\exp(2jT_0)\right) + A_2\bar{A}_2 + cc, \end{aligned} \tag{31}$$

where cc denotes the complex conjugate of the preceding term. In order to avoid secular behavior in q_2:

$$D_1A_2 = 0,$$

therefore A_2 must be independent of T_1 or equivalently $A_2 = A_2(T_2)$.

With $D_1A_2 = 0$, the solution of Eq. (31) is:

$$q_2 = \frac{\beta_2 A^2}{3}\exp(2jT_0) - \beta_2 A_2\bar{A}_2 + cc, \tag{32}$$

where the solution of the homogeneous equation is not needed.

Substituting Eq. (32) and Eq. (30) into Eq. (29) leads to:

$$\begin{aligned} D_0^2 q_3 + q_3 = &\left(-2jD_2A_2 + \left(10\beta_2A_2^2\bar{A}_2/3\right) - \right. \\ &\left. - 3\beta_2A_2^2\bar{A}_2)\right)\exp(jT_0) - \\ &- \left(\left(2\beta_1^2A_2^3/3\right) + \beta_1A_2^3\right)\exp(3jT_0) + cc. \end{aligned} \tag{33}$$

To eliminate the secular term from q_3 we must have:

$$-2jD_2A_2 + \left(10\beta_2 A_2^2 \bar{A}_2/3\right) - 3\beta_2 A_2^2 \bar{A}_2 = 0. \quad (34)$$

Obtaining A_2 from Eq. (34) and stating it in a polar form leads to:

$$A_2 = \frac{1}{2}\varsigma \exp\left(j\lambda\right), \quad (35)$$

where η and λ are real functions of T_2. Next, by substituting Eq. (35) into Eq. (33) and separating the result into imaginary and real parts, we obtain:

$$\frac{d\varsigma}{dT^2} = 0 \rightarrow \textit{here it is assumed } \varsigma = \frac{1}{\varepsilon} \quad (36)$$

$$-\varsigma\frac{d\lambda}{dT_2} - \frac{3}{8}\varsigma^3\beta_2 + \varsigma^3\left(\frac{5\beta_1^2}{12}\right) = 0 \rightarrow$$
$$\rightarrow \lambda = \frac{3}{8}\varsigma^2\beta_2 - \varsigma^2\left(\frac{5\beta_1^2}{12}\right)T_2 + c_0, \quad (37)$$

where c_0 is constant. Substituting ς and λ from Eq. (28) into Eq. (35) and $T_2 = \varepsilon^2 t$ leads to:

$$A_2 = \frac{1}{2}\varsigma \exp\left(j\left(\frac{3}{8}\varsigma^2\beta_2 - \frac{5\beta_1^2}{12}\varsigma^2\right)\varepsilon^2 t + c_0\right). \quad (38)$$

Finally, by substituting q_1 and q_2 from Eqs (30) and (32) into Eq. (24) and using Eq. (38), we get:

$$q = \varepsilon q_1 + \varepsilon^2 q_2 = \varepsilon\eta\cos\left(\omega t + c_0\right) +$$
$$+ \varepsilon^2\left(\left(\beta_1\varsigma^2/6\right)\cos\left(\omega t + c_0\right) - \left(\beta_1\varsigma^2/2\right)\right) + \quad (39)$$
$$+ b\cos\left(t + \varpi\right),\ q(0) = q_{max},\ \dot{q}(0) = 0,$$

where the fractional non-local non-linear natural frequency based on Eq. (36) can be obtained as follows:

$$\omega_{nl}^{Fnl} = \left(\sqrt{\beta_{11}}\right)\left(1 + \frac{3}{8}\left(\frac{9\beta_{13}\beta_{11} - 10\beta_{12}^2}{9\beta_{11}^2}\right)q_{max}^2\right) + \quad (40)$$
$$+ O\left(\varepsilon^3\right).$$

3.2. Results. In this section the natural frequency of FGM nano-beams is studied based on the FNM. The FGM beams are composed of aluminum and silicon. Their properties are varying along the thickness, based on the power law. Their bottom and upper surfaces are pure silicon and pure aluminum, respectively. The material properties of silicon and aluminum are shown in Table 1.

Table 1
Material properties of FGM nano-beams

material	E	ρ
Silicon	210 GPa	2370 kg/m^3
Aluminum	70 GPa	2700 kg/m^3

To validate our results we compared the non-linear frequency ratio (NFR) which we obtained with our formulation with the Nezamnezhad results [32]. The NFR is defined as the ratio of non-dimensional non-linear non-local frequency to the non-dimensional non-linear classic frequency. Since the model presented in this paper is new, the results are validated with those in [32] for α = 2 only – other results are an original extension of this work and show an unusual flexibility.

Table 2
A comparison of non-linear frequency ratio $\left(r = \sqrt{(I/A)}\right)$

L = 10 nm	q_{max}/r	μ^2 (nm^2)	k	Nazemnezhad [32] (Eringen non-local theory)	Present (fractional non-local theory α = 2) or Eringen non-local theory)
	1	2	0	0.9293	0.9294
			1	0.9247	0.9250

To study the effect of different orders (integer or non-integer) of the stress gradient in the constitutive equation on non-linear frequency, the NFR has been given for the S-S and the C-F nano-beams in Table 3 and Table 4, respectively, where L = 10 nm, h = b = 1 nm and $\mu = \sqrt{2}$ nm. The fractional parameter (α) that controls the order of the stress gradient as mentioned above is between 1 and 2 (when α is equal to 2, the FNM reduces to ENT). As it can be seen from Table 3 and Fig. 1a, the decrease of the stress gradient order from 2 to 1.2 decreases the NFR where the gradient index is constant. Also for the C-F beam, it can be seen from Table 4 and Fig. 1b that decreasing the order of the stress gradient causes an increase

Table 3
The non-linear frequency ratio (NFR) for a S-S beam based on the FNM

L (nm)	$\frac{q_{max}}{r}$	k	α = 2 (ENT)	$\mu = \sqrt{2}$ nm				
				α = 1.84	α = 1.76	α = 1.6	α = 1.36	α = 1.2
10 nm	1	0	0.9294	0.9065	0.8956	0.8748	0.8462	0.8295
		1	0.9250	0.9015	0.8899	0.8667	0.8298	0.8018
		2	0.9224	0.8986	0.8867	0.8625	0.8226	0.7904
		3	0.9223	0.8985	0.8867	0.8627	0.8237	0.7928

Table 4
The non-linear frequency ratio (NFR) for a C-F beam based on the FNM

L (nm)	$\frac{q_{max}}{r}$	k	α = 2 (ENT)	$\mu = \sqrt{2}$ nm				
				α = 1.84	α = 1.76	α = 1.6	α = 1.36	α = 1.2
10 nm	1	0	1.0228	1.0238	1.0245	1.0272	1.0364	1.0497
		1	1.0229	1.0238	1.0245	1.0270	1.0361	1.0494
		2	1.0219	1.0227	1.0234	1.0257	1.0346	1.0479
		3	1.0214	1.0221	1.0227	1.0251	1.0339	1.0472

in the NFR. It is visible that the effect of changing order of the stress gradient is more pronounced for the S-S beam than for the C-F beam.

The influence of the order of the stress gradient on the NFR for S-S and C-F nano-beams is shown in Fig. 1 for different values of the gradient index $(k = 0, 1, 2, 3)$ where $L = 10$ nm and $q_{max}/r = 1$. As it is visible for the S-S beam, the influence of α on the NFR becomes more pronounced for a larger gradient index. In other words, the effect of the gradient index on the NFR becomes more intensive when α decreases. In Fig. 1b for the C-F beam the diagrams for different values of the gradient index are almost the same and for different values of α the increase of the gradient index decreases the NFR.

Figure 2a and Fig. 2b show the effect of a large range of the gradient index on the NFR for different values of the fractional parameter $(\alpha = 2, 1.76, 1.6, 1.36, 1.2)$. For both kinds of beams, for increasing gradient index the diagrams tend to be linear for different values of α. In Fig. 2a, as α decreases,

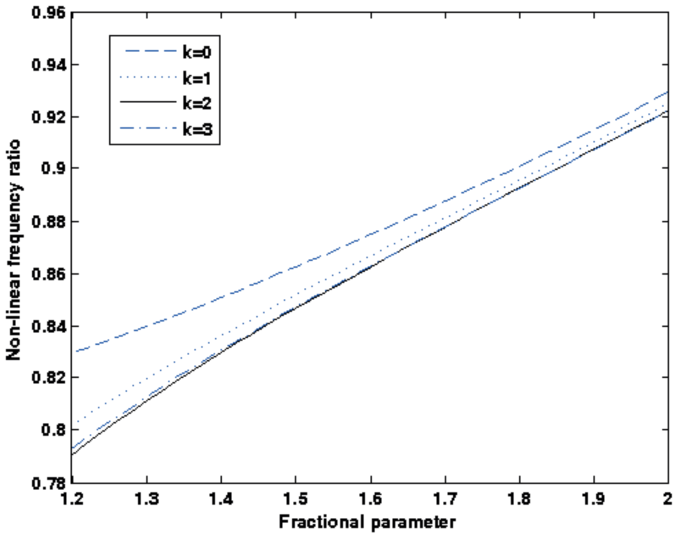

Fig. 1a. The NFR versus the order of the stress gradient for different values of the gradient index in an S-S nano-beam $(L = 10$ nm, $\mu = \sqrt{2}$ nm, $q_{max}/r = 1)$

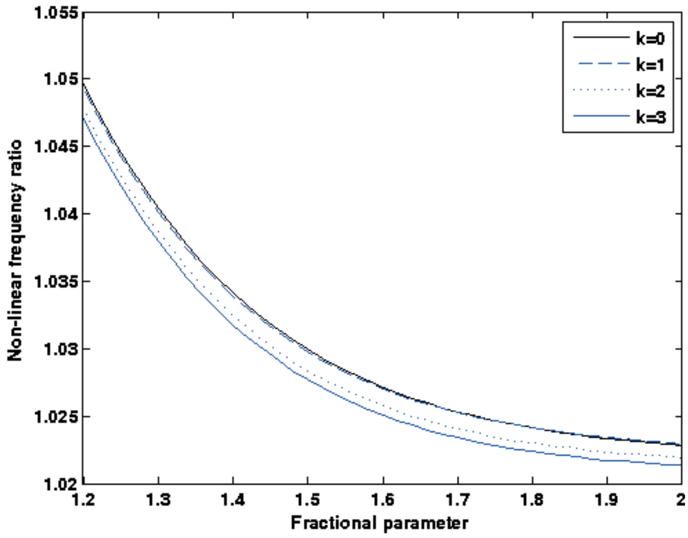

Fig. 1b. The NFR versus the order of the stress gradient for different values of the gradient index in a C-F nano-beam $(L = 10$ nm, $\mu = \sqrt{2}$ nm, $q_{max}/r = 1)$

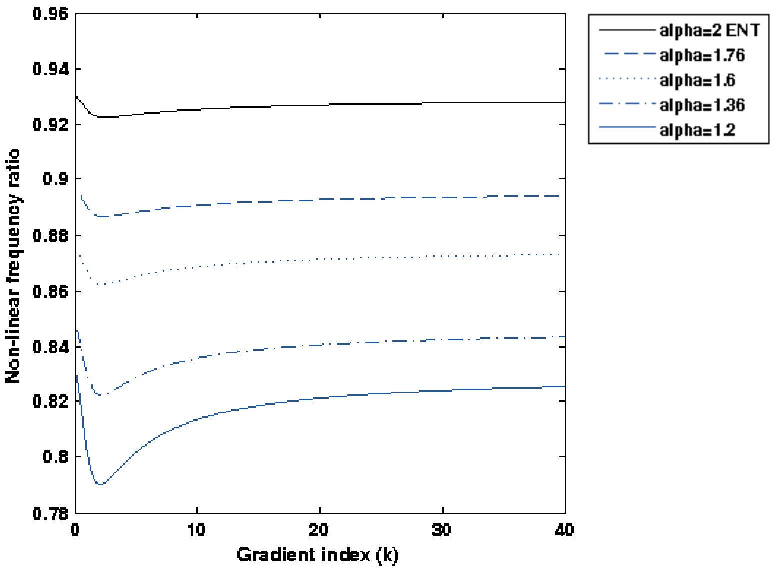

Fig. 2a. The NFR versus the gradient index based on the FNM in an S-S nano-beam, where $\alpha = 2$ the FNM reduced to ENT $(L = 10$ nm, $\mu = \sqrt{2}$ nm, $q_{max}/r = 1$, $k = 1)$

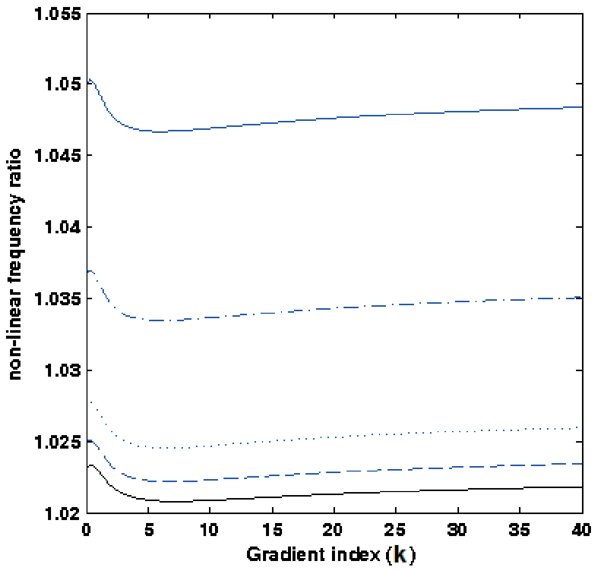

Fig. 2b. The NFR versus the gradient index based on the FNM in C-F nano-beam, where $\alpha = 2$ the FNM reduced to ENT $(L = 10$ nm, $\mu = \sqrt{2}$ nm, $q_{max}/r = 1$, $k = 1)$

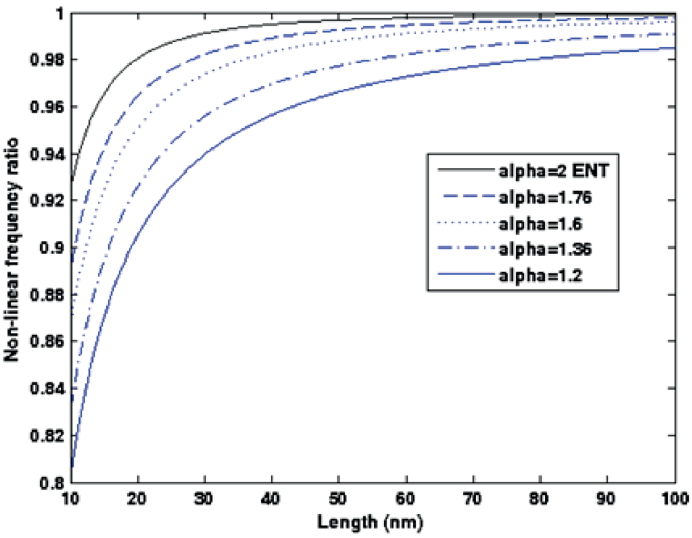

Fig. 3a. The NFR versus the length of an S-S nano-beam based on the FNM, where $\alpha = 2$ the FNM reduced to ENT $(L = 10$ nm, $\mu = \sqrt{2}$ nm, $q_{max}/r = 1, k = 1)$

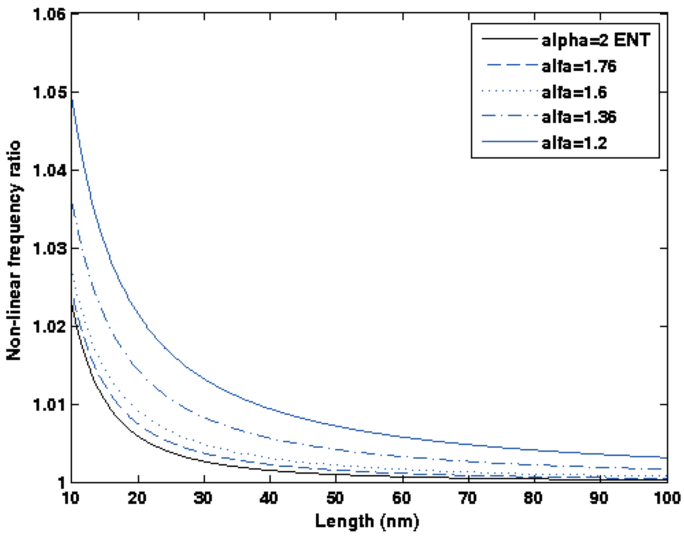

Fig. 3b. The NFR versus the length of a C-F nano-beam based on the FNM, where $\alpha = 2$ the FNM reduced to ENT $(L = 10$ nm, $\mu = \sqrt{2}$ nm, $q_{max}/r = 1, k = 1)$

the diagram is linear for the bigger value of the gradient index and the effect of gradient index values between 0–10 increases change of the NFR. But, Fig. 2b shows that the shape of the diagrams is the same for different values of α.

The NRT versus the length of the nano-beam is shown in Fig. 3a and Fig. 3b for both S-S and C-F beams, respectively, for different values of the order of the stress gradient $(\alpha = 2, 1.76, 1.6, 1.36, 1.2)$. For both beams, when the length increases, the diagram of the FNM becomes linear and when the length of the nano-beam decreases, the effect of the order of the stress gradient on the NRT increases. Note that the effects of length and the gradient index on the NFR based on ENT $(\alpha = 2)$ for the S-S nano-beam are the same as the results of Nezamnezhad et al. [31].

4. Conclusions

A non-local model which is the generalization of ENT by application of the CFD definition has been presented in the paper. The model has a simple numerical solution in contrast to former fractional derivatives models due to the absence of integral in the constitutive relation. In addition to a non-local parameter, the model has one extra free parameter that controls the order of the stress gradient in the constitutive non-local differential equation. This parameter makes the modeling more flexible and powerful since both integer and non-integer order of the stress gradient can be used in the modeling of physical phenomena. This new parameter is known as fractional parameter. In this paper its value is between 1 and 2 $(1 < \alpha \leq 2)$ and when it is equal to 2, the FNM reduces to ENT.

Finally, the non-linear free vibration of the S-S and C-F FGM nano-beams based on conformable FNM has been presented (the non-linearity is due to von-Kármán non-linearity and fractional derivatives). The non-linear governing equation was solved by the parameter expansion method and the following results were obtained:

- For the S-S nano-beam the influence of the gradient index on the NFR becomes more pronounced when the stress gradient order decreases, while for the C-F beam this effect is less visible.
- For both kinds of beams, increasing gradient index results in a linear diagram for different values of the stress gradient order.
- For the S-S nano-beam, as the stress gradient order decreases the diagram tends to be linear for larger values of the gradient index. The effect of the gradient index on the NFR increases for gradient values between 0–10.
- For the C-F nano-beam, the effect of the gradient index is smaller than for the S-S case for different values of the stress gradient order.
- In both kinds of beams, for different values of the stress gradient order by increasing the length of the nano-beam the diagram of the FNM becomes linear. The length of the nano-beam reduces the effect of the stress gradient order on the NRT due to the fact that beam dimensions become significantly higher than the length scale.

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Appendix A

Let us consider the CFD for a multi-variables function.
Assuming the function f(x, y), we have:

$$f_x(x,y)=\frac{df(x,y)}{dx}=lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h},$$
$$f_y(x,y)=\frac{df(x,y)}{dy}=lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}.$$

Next, based on the CFDD, we have [11]:

$$f_x^{\alpha}(x,y)=\frac{d^{\alpha}f(x,y)}{dx^{\alpha}}=lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{\lceil\alpha\rceil-\alpha},y)-f(x,y)}{\varepsilon},$$

$$f_y^{\alpha}(x,y)=\frac{d^{\alpha}f(x,y)}{dy^{\alpha}}=lim_{\varepsilon\to 0}\frac{f(x,y+\varepsilon y^{\lceil\alpha\rceil-\alpha})-f(x,y)}{\varepsilon}.$$

If $0<\alpha\leq 1$ one has $h=\varepsilon x^{\alpha-1}$, $h=\varepsilon y^{\alpha-1}$ and then from the above relations:

$$f_x^{\alpha}(x,y)=\frac{d^{\alpha}f(x,y)}{dx^{\alpha}}=lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{1-\alpha},y)-f(x,y)}{\varepsilon}=x^{1-\alpha}lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=x^{1-\alpha}\frac{df(x,y)}{dx},$$
$$f_y^{\alpha}(x,y)=\frac{d^{\alpha}f(x,y)}{dy^{\alpha}}=lim_{\varepsilon\to 0}\frac{f(x,y+\varepsilon y^{1-\alpha})-f(x,y)}{\varepsilon}=y^{1-\alpha}lim_{h\to 0}\frac{f(x,y)-f(x,y+h)}{h}=y^{1-\alpha}\frac{df(x,y)}{dy},$$

A smooth transition to the integer derivative is obtained for $\alpha=1$, namely:

$$f_x(x,y)=\frac{df(x,y)}{dx},$$
$$f_y(x,y)=\frac{df(x,y)}{dy}.$$