www.czasopisma.pan.pl PAN www.journals.pan.pl

GEODESY AND CARTOGRAPHY
Vol. 67, No. 2, 2018, pp. 271–294

DOI: https://doi.org/10.24425/gac.2018.125474

Original research paper

The method of construction of cylindrical and azimuthal equal-area map projections of a tri-axial ellipsoid

Paweł Pędzich

Warsaw Technical University, Faculty of Geodesy and Cartography
1 Plac Politechniki, 00–661 Warsaw, Poland
e-mail: ppedzich@wp.pl, ORCID: https://orcid.org/0000-0001-5292-4887

Received: Received: 8 October 2018 / Accepted: 10 November 2018

Abstract: The paper presents a method of construction of cylindrical and azimuthal equal-area map projections of a triaxial ellipsoid. Equations of a triaxial ellipsoid are a function of reduced coordinates and functions of projections are expressed with use of the normal elliptic integral of the second kind and Jacobian elliptic functions. This solution allows us to use standard methods of solving such integrals and functions. The article also presents functions for the calculation of distortion. The maps illustrate the basic properties of developed map projections. Distortion of areas and lengths are presented on isograms and by Tissot's indicatrixes with garticules of reduced coordinates. In this paper the author continues his considerations of the application of reduced coordinates to the construction of map projections for equidistant map projections. The developed method can be used in planetary cartography for mapping irregular objects, for which tri-axial ellipsoids have been accepted as reference surfaces. It can also be used to calculate the surface areas of regions located on these objects. The calculations were carried out for a tri-axial ellipsoid with semi-axes $a = 267.5$ m, $b = 147$ m, $c = 104.5$ m accepted as a reference ellipsoid for the Itokawa asteroid.

Keywords: map projection, distortion in map projection, equal-area map projection, tri-axial ellipsoid, reduced coordinates

1. Introduction

Nyrtsov et al. (2015) presented a method of construction of cylindrical and azimuthal equal-area map projections of a tri-axial ellipsoid with application of planetocentric coordinates. This paper presents an alternative method of calculating coordinates in such type of projections as well as a method of determination of map projection distortion. These methods depend on the application of reduced coordinates. Final equations are expressed by Jacobian elliptic functions and the normal elliptic integrals of the second kind. In this paper the author continues his considerations of the application of reduced coordinates to the construction of map projections as it was proposed in the paper (Pędzich, 2017) for equidistant map projections.

Cylindrical and azimuthal map projections of a tri-axial ellipsoid significantly differ from commonly known map projections of an oblate ellipsoid or a sphere. In cylindrical map projections of a triaxial ellipsoid, the meridians are projected as straight lines parallel to x axis and parallels are projected as curves. It is the most significant difference compared to map projections of an oblate ellipsoid wherein parallels are projected as straight lines. Azimuthal map projections of a tri-axial ellipsoid also differ from projections of an oblate ellipsoid or a sphere. The main difference is the fact that parallels are projected as ellipses not as circles like in projections of an oblate ellipsoid or a sphere.

Cylindrical map projections are used for mapping of an entire globe. However because of huge distortion in the polar regions, azimuthal map projections are recommended. Equal area map projections are used for mapping of entire objects and for calculating the surface areas of regions. Map projections of a tri-axial ellipsoid are used for mapping of extraterrestrial objects, especially when their shapes differ from a sphere or an oblate ellipsoid.

Equal-area map projections of a sphere have been known for ages. Ptolemy was one of the first who described this type of map projections. In 2nd century in his Geography he developed a map projection similar to equal-area. His second projection was developed by Rigobert Bonne in 1752 and then it was applied to the creation of topographic maps in many countries. In 1772 J.H. Lambert published a book on map projections, which described, among others, cylindrical and azimuthal equal-area map projections of a sphere. Hence such map projections are called Lambert's projections. Map projections of a tri-axial ellipsoid are rarely a subject of research. Nyrtsov et al. (2015) developed cylindrical and azimuthal map projections of a tri-axial ellipsoid with the application of planetocentric coordinates. He also presented maps of irregularly shaped extraterrestrial bodies. Their surfaces were approximated by a tri-axial ellipsoid. Another interesting equal-area mapping method for irregularly shaped bodies was presented by Bertohoud (2005). He applied the original method that consisted in dividing an object into small quadrangles and then projecting them on a plane preserving their areas. His maps show the approximate shape of object. Despite of maintaining area of object, an irregular shape of a graticule discouraged potential users. In literature, conformal (Bugayevsky, 1987, 1991; Snyder, 1985; Fleis et al., 2013; Nyrtsov, 2014) and equidistant (Bugaevsky, 1999; Nyrtsov et al., 2012) projections of a tri-axial ellipsoid are described more often than equal-area projections. Tri-axial ellipsoids are usually described with use of planetocentric and planetographic coordinates. In this paper, reduced coordinates are applied. They are a generalization of reduced coordinates of an oblate ellipsoid to a tri-axial ellipsoid. This enables the expression of map projection functions by means of Jacobian elliptic functions and the normal elliptic integrals of the second kind.

The main purpose of the article is to show the method of construction of equal-area map projections with application of reduced coordinates, normal elliptic integrals and Jacobian elliptic functions. Series of maps were created to present the graticules and map projections distortion. The calculations were carried out for a tri-axial ellipsoid with semi-axes $a = 267.5$ m, $b = 147$ m, $c = 104.5$ m accepted as a reference ellipsoid for the Itokawa asteroid (Nyrtsov et al., 2014).

2. Basic assumptions

One of the most interesting direction of research in cartography is mapping of small extraterrestrial objects, such as asteroids, comets or small satellites. Often their surfaces are approximated by tri-axial ellipsoids. Itokawa asteroid is an example of such object. It is a stony asteroid, classified as near-Earth object and potentially hazardous asteroid. It was the first asteroid to be the target of a sample return mission (the Japanese space probe Hayabusa) and the smallest asteroid photographed and visited by spacecraft (wikipedia). Therefore it is interesting object for mapping and is used in the paper as an example for calculation.

In the paper as a reference surface in map projections a tri-axial ellipsoid is assumed. Such a surface is often used as an approximation of small celestial bodies. In the paper equations of a tri-axial ellipsoid are expressed in reduced coordinates and have the following form:

$$\vec{r} = [X = a\cos u \cos v,\ Y = b\cos u \sin v,\ Z = c\sin u] \tag{1}$$

where a, b, c – semi-axis of ellipsoid, u, v – reduced coordinates $u \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$, $v \in \langle -\pi, \pi)$.

In the paper formulae of equal-area cylindrical and azimuthal projections are derived. These types of projections are used for mapping of entire extraterrestrial objects and because such maps preserve surface area they are especially useful for mapping the distribution of geological features on irregular objects (Berthoud 2005). They can also be used to calculate the surface areas of regions located on these objects.

General formulae for map projections of a tri-axial ellipsoid are as follows:

$$\vec{r}' = [x = x(u,v),\ y = x(u,v)]\,. \tag{2}$$

In equal-area map projections the following condition must be fulfilled:

$$H' = H, \tag{3}$$

where $H' = |\vec{r}'_u \times \vec{r}'_v|$ is a determinant of partial derivatives for projection functions and $H = |\vec{r}_u \times \vec{r}_v|$ is a determinant for a reference surface.

In the paper also formulae for distortion are derived. The equations for Tissot indicatrix and also scale of linear distortion are as follows (Balcerzak, Panasiuk 2005), (Pędzich, 2014):

$$\vec{\mu} = \vec{\mu}_1 \cos A + \vec{\mu}_2 \sin A, \tag{4}$$

where:

$$\vec{\mu}_1 = \frac{\vec{r}'_u}{\sqrt{E}}\,, \qquad \vec{\mu}_2 = \frac{E\vec{r}'_v - F\vec{r}'_u}{H\sqrt{E}}\,,$$

and $\vec{r}'_u$, $\vec{r}'_v$ are partial derivatives of map projection functions.

Based on scales $\vec{\mu}_1$, $\vec{\mu}_2$ extreme scales m, n can be calculated, according to the following equations:

$$\begin{aligned} \vec{m} &= \vec{\mu}_1 \cos A_e + \vec{\mu}_2 \sin A_e\,, \\ \vec{n} &= -\vec{\mu}_1 \sin A_e + \vec{\mu}_2 \cos A_e\,, \end{aligned} \tag{5}$$

where:

$$\tan A_e = \frac{2Q}{P-R}$$

and

$$P = \mu_1^2, \qquad Q = \vec{\mu}_1 \circ \vec{\mu}_2\,, \qquad R = \mu_2^2\,.$$

Angular distortion is calculating according to the formula:

$$\sin\frac{\omega}{2} = \frac{m-n}{m+n}\,. \tag{6}$$

For calculation of distortion partial derivatives $\vec{r}\,'_u$, $\vec{r}\,'_v$ are required:

$$\vec{r}\,'_u = \left[\frac{dx}{du}, \frac{dy}{du}\right], \qquad \vec{r}\,'_v = \left[\frac{dx}{dv}, \frac{dy}{dv}\right],$$

and coefficients of the Gaussian fundamental form.

3. Derivation of formulae in equal-area map projections of a tri-axial ellipsoid

This chapter shows the method of construction of equal-area map projections with application of reduced coordinates, normal elliptic integrals and Jacobian elliptic functions. There are presented equations of a tri-axial ellipsoid expressed by reduced coordinates and coefficients of the Gaussian fundamental form. Then functions of cylindrical and azimuthal equal-area map projections are derived along with their partial derivatives for map projection distortion.

3.1. Coefficients of the Gaussian fundamental form

Partial derivatives of function (1) have the form:

$$\vec{r}_u = \left[\frac{dX}{du} = -a\sin u\cos v,\ \frac{dY}{du} = -b\sin u\sin v,\ \frac{dZ}{du} = c\cos u\right],$$
$$\vec{r}_v = \left[\frac{dX}{dv} = -a\cos u\sin v,\ c\frac{dY}{dv} = b\cos u\cos v,\ c\frac{dZ}{dv} = 0\right]$$

and coefficients of the Gaussian fundamental form are as follows:

$$\begin{aligned}
&E = |\vec{r}_u|^2 = \left(\frac{dX}{du}\right)^2 + \left(\frac{dY}{du}\right)^2 + \left(\frac{dZ}{du}\right)^2,\\
&E = \sin^2 u\left(a^2\cos^2 v + b^2\sin^2 v\right) + c^2\cos^2 u,\\
&F = \vec{r}_u \circ \vec{r}_v = \frac{dX}{du}\frac{dX}{dv} + \frac{dY}{du}\frac{dY}{dv} + \frac{dZ}{du}\frac{dZ}{dv},\\
&F = a^2\sin u\cos u\sin v\cos v - b^2\sin u\cos u\sin v\cos v = \left(a^2-b^2\right)\sin u\cos u\sin v\cos v.
\end{aligned} \tag{7}$$

The determinant of derivatives has the following form:

$$H = |\vec{r}_u \times \vec{r}_v| =$$
$$= \sqrt{\left(\frac{dY}{du}\frac{dZ}{dv} - \frac{dY}{dv}\frac{dZ}{du}\right)^2 + \left(\frac{dX}{du}\frac{dZ}{dv} - \frac{dX}{dv}\frac{dZ}{du}\right)^2 + \left(\frac{dX}{du}\frac{dY}{dv} - \frac{dX}{dv}\frac{dY}{du}\right)^2} =$$
$$= abc\cos u\sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v + \frac{1}{b^2}\sin^2 v\right) + \frac{1}{c^2}\sin^2 u}\,. \tag{8}$$

3.2. Cylindrical equal-area map projections of a tri-axial ellipsoid

General formulas for cylindrical map projections of a tri-axial ellipsoid are as follows:

$$\vec{r}' = [x = x(u,v),\ \ y = Av]\,, \tag{9}$$

where A is a constant.

In that case the meridians are projected as straight lines and the distances between them are equal. For construction of equal-area map projection partial derivatives of function (9) are required and their determinant. For function (9) partial derivatives are as follows:

$$\vec{r}'_u = \left[\frac{dx}{du},\, 0\right], \qquad \vec{r}'_v = \left[\frac{dx}{dv},\, A\right] \tag{10}$$

and the determinant has the form:

$$H' = \left|\vec{r}'_u \times \vec{r}'_v\right| = A\frac{dx}{du}\,.$$

In equal-area map projections the condition (3) must be fulfilled. Hence for an equal-area map projection of a tri-axial ellipsoid the following condition is obtained:

$$\frac{dx}{du} = \frac{1}{A}abc\cos u\sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v + \frac{1}{b^2}\sin^2 v\right) + \frac{1}{c^2}\sin^2 u}\,. \tag{11}$$

Integration of (11) from $u = 0$ to $u = u_i$ leads to:

$$x = \frac{1}{A}abc\int_0^{u_i}\cos u\sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v + \frac{1}{b^2}\sin^2 v\right) + \frac{1}{c^2}\sin^2 u}\,du. \tag{12}$$

Equation (12) may be written in the following form:

$$x = \frac{1}{A}abc I_1\,, \tag{13}$$

where

$$I_1 = \int_0^{u_i} \cos u \sqrt{\cos^2 u \left(\frac{1}{a^2} \cos^2 v + \frac{1}{b^2} \sin^2 v \right) + \frac{1}{c^2} \sin^2 u}\, du. \tag{14}$$

Substituting

$$B^2 = \frac{1}{a^2} \cos^2 v + \frac{1}{b^2} \sin^2 v \quad \text{and} \quad C^2 = \frac{1}{c^2} \tag{15}$$

to equation (14) for the integral I_1 results in:

$$I_1 = \int_0^{u_i} \cos u \sqrt{B^2 \cos^2 u + C^2 \sin^2 u}\, du \tag{16}$$

hence

$$I_1 = B \int_0^{u_i} \cos u \sqrt{1 + \left(\frac{C^2}{B^2} - 1 \right) \sin^2 u}\, du = B I_2 , \tag{17}$$

where

$$I_2 = \int_0^{u_i} \cos u \sqrt{1 + n^2 \sin^2 u}\, du \quad \text{and} \quad n^2 = \frac{C^2}{B^2} - 1. \tag{18}$$

Transforming the integral I_2:

$$I_2 = \int_0^{u} \cos u \frac{1 + n^2 \sin^2 u}{\sqrt{1 + n^2 \sin^2 u}}\, du = \int_0^{u} \frac{\cos u}{\sqrt{1 + n^2 \sin^2 u}}\, du + n^2 \int_0^{u} \frac{\cos u \sin^2 u}{\sqrt{1 + n^2 \sin^2 u}}\, du =$$

$$= \int_0^{u} \frac{\cos u}{\sqrt{1 + n^2 \sin^2 u}}\, du + n^2 \int_0^{u} \frac{\cos u \left(1 - \cos^2 u\right)}{\sqrt{1 + n^2 \sin^2 u}}\, du =$$

$$= \int_0^{u} \frac{\cos u}{\sqrt{1 + n^2 \sin^2 u}}\, du + n^2 \int_0^{u} \frac{\cos u}{\sqrt{1 + n^2 \sin^2 u}}\, du - n^2 \int_0^{u} \frac{\cos^3 u}{\sqrt{1 + n^2 \sin^2 u}}\, du.$$

Finally basing on (Byrd Friedman 1954) the integral I_2 takes a form:

$$I_2 = G_1 \left(1 + n^2\right) - n^2 G_3 , \tag{19}$$

where:

$$G_1 = \int_0^{u_i} \frac{\cos u}{\sqrt{1 + n^2 \sin^2 u}}\, du = \frac{k'}{k} \ln \frac{1 + k \operatorname{sn} \omega}{\operatorname{dn} \omega} \Bigg|_0^{\omega_i} , \tag{20}$$

$$G_3 = \int_0^{u_i} \frac{\cos^3 u}{\sqrt{1 + n^2 \sin^2 u}}\, du = \frac{k'}{2k^3} \left[(1 + k^2) \ln \left(\frac{1 + k \operatorname{sn} \omega}{\operatorname{dn} \omega} \right) - k k'^2 \operatorname{sn} \omega \operatorname{nd}^2 \omega \right] \Bigg|_0^{\omega_i} \tag{21}$$

and:

$$\mathrm{sn}^2\omega=\frac{\left(1+n^2\right)\sin^2 u}{1+n^2\sin^2 u},\qquad k^2=\frac{n^2}{1+n^2},\qquad k'=\sqrt{1-k^2},$$

$$\mathrm{dn}\,\omega=\sqrt{1-k^2\mathrm{sn}^2\omega},\qquad \mathrm{nd}\,\omega=\frac{1}{\mathrm{dn}\,\omega},$$

snω, dnω,ndω – Jacobian elliptic functions.

Finally, the equations for cylindrical equal-area projections may be written as:

$$\begin{aligned} x&=\frac{1}{A}abc\,B\left(G_1\left(1+n^2\right)-n^2G_3\right),\\ y&=Av. \end{aligned}\tag{22}$$

One can assume that constant A in order to the length of the image of the equator on plane was equal to the length of the equator on a tri-axial ellipsoid. The equation for the length of a parallel (Pędzich, 2017) on a tri-axial ellipsoid has the form:

$$s_p=b\cos u\frac{1}{k_p'}\left[E\left(\psi_p,k_p\right)-k_p^2 sn\vartheta_p\; cd\vartheta_p\right],\tag{23}$$

where:

$E\left(\psi_p,\, k_p\right)$ is the normal elliptic integral of the second kind,

$$k_p'=\sqrt{1-k_p^2},\qquad k_p^2=\frac{n_p^2}{1+n_p^2},\qquad n_p^2=\frac{a^2}{b^2}-1,\qquad \mathrm{sn}\,\vartheta_p=\sqrt{\frac{\left(1+n_p^2\right)\sin^2 v}{1+n_p^2\sin^2 v}},$$

$$\mathrm{cd}\,\vartheta_p=\sqrt{\frac{1-\mathrm{sn}^2\vartheta_p}{1-k_p^2\mathrm{sn}^2\vartheta_p}},\qquad \psi_p=\arcsin\left(\sqrt{\frac{\left(1+n_p^2\right)\sin^2 v}{1+n_p^2\sin^2 v}}\right).$$

Substituting $u=0$ and $v=\dfrac{\pi}{2}$ the equation for the length of 1/4 of the equator is obtained:

$$s_p=\frac{b}{k_p'}E\left(\frac{\pi}{2},\,k_p\right)\tag{24}$$

and hence the constant A has a form:

$$A=\frac{2b}{\pi k_p'}E\left(\frac{\pi}{2},\,k_p\right).\tag{25}$$

If we assume a cylindrical map projection in the following form:

$$\vec{r}'=\left[x=x(u,v),\;\; y=y(v)\right],\tag{26}$$

then meridians are projected as straight lines but the distances between them are different depending on the function $y = y(v)$. Partial derivatives of functions (26) have the form:

$$\vec{r}'_u = \left[\frac{dx}{du},\ 0\right], \qquad \vec{r}'_v = \left[\frac{dx}{dv},\ \frac{dy}{dv}\right], \tag{27}$$

and the determinant of partial derivatives can be written as follows:

$$H' = \left|\vec{r}'_u \times \vec{r}'_v\right| = \frac{dx}{du}\frac{dy}{dv}. \tag{28}$$

If we assume that the distances between meridians are preserved, it also means that the equator is projected isometrically, and then basing on (Pędzich, 2017) it may be written as follows:

$$y = b\int_0^{v_i}\sqrt{1+n_p^2\sin^2 v}\,dv, \tag{29}$$

where: $n_p^2 = \dfrac{a^2}{b^2} - 1$.

Hence derivative

$$\frac{dy}{dv} = b\sqrt{1+n_p^2\sin^2 v}\,. \tag{30}$$

Then determinant H' has a form:

$$H' = \frac{dx}{du}b\sqrt{1+n_p^2\sin^2 v}\,.$$

Basing on equal-area condition $H' = H$ we may write:

$$\frac{dx}{du}b\sqrt{1+n_p^2\sin^2 v} = abc\cos u\sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v + \frac{1}{b^2}\sin^2 v\right) + \frac{1}{c^2}\sin^2 u}$$

hence

$$\frac{dx}{du} = \frac{ac\cos u}{\sqrt{1+n_p^2\sin^2 v}}\sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v + \frac{1}{b^2}\sin^2 v\right) + \frac{1}{c^2}\sin^2 u} \tag{31}$$

and then

$$x = \frac{ac}{\sqrt{1+n_p^2\sin^2 v}}\int_0^{u_i}\cos u\sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v + \frac{1}{b^2}\sin^2 v\right) + \frac{1}{c^2}\sin^2 u\,du}. \tag{32}$$

Equation (32) may be written in the form:

$$x = \frac{ac}{\sqrt{1+n_p^2\sin^2 v}}BI_2 \tag{33}$$

where B is given by (15) and I_2 is given by (18).

Finally projection functions in a cylindrical equal-area map projection with the equator projected without distortion are in the following form:

$$x = \frac{ac}{\sqrt{1+n_p^2\sin^2 v}} B\left(G_1\left(1+n^2\right)-n^2G_3\right),$$
$$y = b\frac{1}{k_p'}\left[E(\psi_p,k_p)-k_p^2\operatorname{sn}\vartheta_p\operatorname{cd}\vartheta_p\right] \tag{34}$$

and G_1, G_3, n and B are given by (15), (18), (20), (21), $E(\psi_p,k_p)$, k_p', k_p^2, n_p^2, $\operatorname{sn}\vartheta_p$, $\operatorname{cd}\vartheta_p$, ψ_p are calculated using (23).

3.3. Distortion in cylindrical equal-area map projections

In case of a map projection with equations given by (9) the partial derivatives have form (10). Derivatives $\frac{dy}{du}=0$, $\frac{dy}{dv}=A$. Derivative $\frac{dx}{du}$ is given by (11) and $\frac{dx}{dv}$ is obtained by differentiation of (12) with respect to v.

According to the Leibnitz rule we can write:

$$\frac{dx}{dv} = \frac{1}{A}abc\int_0^{u_i}\left[\cos u\frac{\cos^2 u\left(-\frac{2}{a^2}\cos v\sin v+\frac{2}{b^2}\sin v\cos v\right)}{2\sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v+\frac{1}{b^2}\sin^2 v\right)+\frac{1}{c^2}\sin^2 u}}\right]du =$$
$$= \frac{1}{A}abc\sin v\cos v\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\int_0^{u_i}\frac{\cos^3 u}{\sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v+\frac{1}{b^2}\sin^2 v\right)+\frac{1}{c^2}\sin^2 u}}du$$

hence:

$$\frac{dx}{dv} = \frac{1}{A}abc\sin v\cos v\left(\frac{1}{b^2}-\frac{1}{a^2}\right)I_3, \tag{35}$$

where:

$$I_3 = \int_0^{u_i}\frac{\cos^3 u}{\sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v+\frac{1}{b^2}\sin^2 v\right)+\frac{1}{c^2}\sin^2 u}}du.$$

Then substituting (15) to the integral I_3 the following is obtained:

$$I_3 = \int_0^{u_i}\frac{\cos^3 u}{\sqrt{B^2\cos^2 u+C^2\sin^2 u}}du = \frac{1}{B}\int_0^{u_i}\frac{\cos^3 u}{\sqrt{1+\left(\frac{C^2}{B^2}-1\right)\sin^2 u}}du. \tag{36}$$

Then the integral (36) can be written in the form:

$$I_3 = \frac{1}{B}\int_0^{u_i} \frac{\cos^3 u}{\sqrt{1+n^2\sin^2 u}}\,du = \frac{1}{B}G_3\,. \tag{37}$$

Finally the derivative $\frac{dx}{dv}$ has the form:

$$\frac{dx}{dv} = \frac{1}{A}\frac{1}{B}abc\sin v\cos v\left(\frac{1}{b^2}-\frac{1}{a^2}\right)G_3\,, \tag{38}$$

where G_3 is given by (21).

Substituting derived partial derivatives to formulae (4), (5) and (6) it is possible to calculate distortion.

In case of a cylindrical map projection with the equator projected without distortion (26) spatial derivatives are given by (27). The derivative $\frac{dx}{du}$ is given by (31), $\frac{dy}{dv}$ is given by (30) and derivative $\frac{dx}{dv}$ can be calculated by differentiation formula (32) with respect to v:

$$\begin{aligned}\frac{dx}{dv} &= \frac{ac}{\sqrt{1+np^2\sin^2 v}\,B}\sin v\cos v\left(\frac{1}{b^2}-\frac{1}{a^2}\right)G_3 + \\ &\quad -\frac{ac}{\left(1+np^2\sin^2 v\right)^{3/2}}np^2\sin v\cos vB\left(G_1(1+n^2)-n^2G_3\right) = \\ &= \frac{ac\sin v\cos v}{\sqrt{1+np^2\sin^2 v}}\left[\frac{1}{B}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)G_3 - \frac{np^2B}{1+np^2\sin^2 v}\left(G_1(1+n^2)-n^2G_3\right)\right].\end{aligned} \tag{39}$$

Substituting derived partial derivatives to formulae (4), (5) and (6) it is possible to calculate distortion.

3.4. Azimuthal equal-area map projections of a tri-axial ellipsoid

The general formulas for azimuthal map projections of a tri-axial ellipsoid are as follows:

$$\vec{r}\,' = \left[\rho(u,v)\cos v,\ \rho(u,v)\sin v\right]. \tag{40}$$

Partial derivatives of functions (40) have the following form:

$$\begin{aligned}\vec{r}\,'_u &= \left[\frac{d\rho}{du}\cos v,\ \frac{d\rho}{du}\sin v\right], \\ \vec{r}\,'_v &= \left[\frac{d\rho}{dv}\cos v-\rho\sin v,\ \frac{d\rho}{dv}\sin v+\rho\cos v\right]\end{aligned} \tag{41}$$

and the determinant of partial derivatives:

$$H' = \rho \frac{d\rho}{du}. \tag{42}$$

Because equal-area map projections meet the condition (3) hence for azimuthal projection (40) the following differential equation is obtained:

$$\rho \frac{d\rho}{du} = H \tag{43}$$

then:

$$\rho d\rho = -H\,du. \tag{44}$$

The minus sign in (44) appeared because function ρ decreases when u grows. After differentiation substituting (8) the following result is obtained:

$$\frac{\rho^2}{2} = -\int\limits_{\pi/2}^{u_i} H\,du = abc\left(I_1\big|_0^{\pi/2} - I_1\big|_0^{u_i}\right) \tag{45}$$

when I_1 is given by (17).

Then after some modification:

$$\rho = \sqrt{2abc\left(I_1\big|_0^{\pi/2} - I_1\big|_0^{u_i}\right)}. \tag{46}$$

Finally function (46) after substitution (19) takes the form:

$$\rho = \sqrt{2abcB\left[\left(G_1(1+n^2) - n^2G_3\right)\big|_0^{\pi/2} - \left(G_1(1+n^2) - n^2G_3\right)\big|_0^{u_i}\right]}, \tag{47}$$

where n, G_1, G_3, are given by (18), (20), (21).

The basic property of azimuthal map projections is that angles in the pole are projected without distortion. That condition is not fulfilled in a map projection (40). To fulfill this condition, a map projection must be modified. Instead of the reduced latitude v, the planetocentric latitude λ should be substituted. Then an equation for a map projection in the reduced coordinates will take the form:

$$x = \rho\cos(\lambda(v)), \qquad y = \rho\sin(\lambda(v)), \tag{48}$$

where:

$$\tan\lambda = \frac{b}{a}\tan v. \tag{49}$$

Hence, the partial derivatives of the function (48) take the form:

$$\begin{aligned} \vec{r}_u' &= \left[\frac{d\rho}{du}\cos\lambda,\ \frac{d\rho}{du}\sin\lambda\right], \\ \vec{r}_v' &= \left[\frac{d\rho}{dv}\cos\lambda - \rho\sin\lambda\frac{d\lambda}{dv},\ \frac{d\rho}{dv}\sin\lambda + \rho\cos\lambda\frac{d\lambda}{dv}\right]. \end{aligned} \tag{50}$$

Then the determinant of partial derivatives can be written:

$$H' = \rho \frac{d\rho}{du} \frac{d\lambda}{dv}. \tag{51}$$

On the basis of (51) the derivative is calculated:

$$\frac{d\lambda}{dv} = \frac{b}{a} \frac{\cos^2 \lambda}{\cos^2 v}. \tag{52}$$

By converting (51) we can write:

$$\frac{\sqrt{1-\cos^2 \lambda}}{\cos \lambda} = \frac{b}{a} \tan v,$$

hence:

$$\frac{1-\cos^2 \lambda}{\cos^2 \lambda} = \left(\frac{b}{a}\right)^2 \tan^2 v$$

and then:

$$\frac{1}{\cos^2 \lambda} = \left(\frac{b}{a}\right)^2 \tan^2 v + 1$$

and

$$\cos^2 \lambda = \frac{1}{\left(\frac{b}{a}\right)^2 \tan^2 v + 1}. \tag{53}$$

The result of substituting (53) to (52) and then transforming it is the derivative:

$$\frac{d\lambda}{dv} = \frac{b}{a} \frac{1}{\left(\frac{b}{a}\right)^2 \sin^2 v + \cos^2 v}. \tag{54}$$

Then from the condition $H = H'$ a differential equation is obtained:

$$\rho\, d\rho = -\frac{a}{b} \left(\left(\frac{b}{a}\right)^2 \sin^2 v + \cos^2 v \right) H\, du. \tag{55}$$

Integration results in the following formula:

$$\frac{\rho^2}{2} = -\frac{a}{b} \left(\left(\frac{b}{a}\right)^2 \sin^2 v + \cos^2 v \right) \int_{\pi/2}^{u_i} H\, du \tag{56}$$

hence:

$$\rho = \sqrt{\frac{2a}{b} \left(\left(\frac{b}{a}\right)^2 \sin^2 v + \cos^2 v \right) \left(\int_0^{\pi/2} H\, du - \int_0^{u_i} H\, du \right)} \tag{57}$$

and then:

$$\rho = \sqrt{2a^2c\left(\left(\frac{b}{a}\right)^2 \sin^2 v + \cos^2 v\right)\left(I_1\big|_0^{\pi/2} - I_1\big|_0^{u_i}\right)}, \tag{58}$$

where I_1 is given by (17).

Finally, after taking into account (14) the equation will take the form:

$$\rho = \sqrt{2a^2cB\left(\left(\frac{b}{a}\right)^2 \sin^2 v + \cos^2 v\right)\left(\left(G_1(1+n^2) - n^2G_3\right)\big|_0^{\pi/2} - \left(G_1(1+n^2) - n^2G_3\right)\big|_0^{u_i}\right)}, \tag{59}$$

where G_1, G_3, n and $B$are given by (15), (18), (20), (21).

3.5. Distortion in azimuthal equal-area map projections

The basis for determining the distortion are partial derivatives, which, in the case of a map projection in the form (40), are given by equation (41). Derivative $\frac{d\rho}{du} = -\frac{H}{\rho}$. The derivative $\frac{d\rho}{dv}$ is calculated by differentiating (46) with respect to the variable v.

Hence:

$$\frac{d\rho}{dv} = \frac{abc}{\rho}\left(\frac{dI_1\big|_0^{\pi/2}}{dv} - \frac{dI_1\big|_0^{u_i}}{dv}\right). \tag{60}$$

Then, on the basis of (14) the derivative $\frac{dI_1\big|_0^{u_i}}{dv}$ is calculated in the following way:

$$\begin{aligned}
\frac{dI_1}{dv} &= \frac{d\int\limits_0^{u_i} \cos u \sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v + \frac{1}{b^2}\sin^2 v\right) + \frac{1}{c^2}\sin^2 u}\, du}{dv} = \\
&= \int\limits_0^{u_i}\left[\cos u \frac{\cos^2 u\left(-\frac{2}{a^2}\cos v \sin v + \frac{2}{b^2}\sin v \cos v\right)}{2\sqrt{\cos^2 u\left(\frac{1}{a^2}\cos^2 v + \frac{1}{b^2}\sin^2 v\right) + \frac{1}{c^2}\sin^2 u}}\right] du = \\
&= \sin v \cos v\left(\frac{1}{b^2} - \frac{1}{a^2}\right) I_3,
\end{aligned} \tag{61}$$

where:

$$I_3 = \frac{1}{B}\int\limits_0^{u_i} \frac{\cos^3 u}{\sqrt{1+n^2\sin^2 u}}\, du = \frac{1}{B} G_3. \tag{62}$$

The derivative $\dfrac{dI_1|_0^{\pi/2}}{dv}$ is calculated in a similar way, changing only the limits of integration.

Finally, the derivative will take the form:

$$\frac{d\rho}{dv}=\frac{abc}{B\rho}\sin v\cos v\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left(G_3|_0^{\pi/2}-G_3|_0^{u_i}\right), \tag{63}$$

where G_3 is given by the equation (21).

Substituting derived partial derivatives to formulae (4), (5) and (6) it is possible to calculate distortion.

In the case of a map projection (48), the partial derivatives are in the form (50). In formulas for distortion except derivatives $\dfrac{d\rho}{du}$ and $\dfrac{d\rho}{dv}$ the derivative $\dfrac{d\lambda}{dv}$ appears, which is given by (54). The derivative $\dfrac{d\rho}{du}$ is calculated based on (55):

$$\frac{d\rho}{du}=-\frac{aH\left(\left(\dfrac{b}{a}\right)^2\sin^2 v+\cos^2 v\right)}{b\rho}.$$

The derivative $\dfrac{d\rho}{dv}$ is calculated by differentiation of the formula (58):

$$\begin{aligned}\frac{d\rho}{dv}=\frac{a^2c}{\rho}\sin v\cos v&\left[2\left(\left(\frac{b}{a}\right)^2-1\right)\left(I_1|_0^{\pi/2}-I_1|_0^{u_i}\right)+\right.\\&\left.+\left(\left(\frac{b}{a}\right)^2\sin^2 v+\cos^2 v\right)\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left(\frac{dI_1|_0^{\pi/2}}{dv}-\frac{dI_1|_0^{u_i}}{dv}\right)\right]\end{aligned}$$

hence

$$\begin{aligned}\frac{d\rho}{dv}=\frac{a^2c\sin v\cos v}{\rho}&\left[2B\left(\left(\frac{b}{a}\right)^2-1\right)\left(\left(G_1(1+n^2)-n^2G_3\right)|_0^{\pi/2}+\right.\right.\\&\left.\left.-\left(G_1(1+n^2)-n^2G_3\right)|_0^{u_i}\right)+\frac{1}{B}\left(\left(\frac{b}{a}\right)^2\sin^2 v+\cos^2 v\right)\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left(G_3|_0^{\pi/2}-G_3|_0^{u_i}\right)\right].\end{aligned}$$

where G_1, G_3, n and $B$are given by (15), (18), (20), (21).

Substituting derived partial derivatives to formulae (4), (5) and (6) it is possible to calculate distortion.

4. Analysis of results

Derived formulae allow to calculate coordinates and distortion measures in elaborated map projections. There are formulae for two types of projections: azimuthal and cylindrical. Using that formulae coordinates and distortion measures were calculated in map projections of tri-axial ellipsoid used as a reference surface for asteroid Itokawa. Then maps of distortion were created.

4.1. Analysis of results in cylindrical map projections

Coordinates of nodes were calculated using equations (22) and then graticule for entire ellipsoid were drawn (Figure 1).

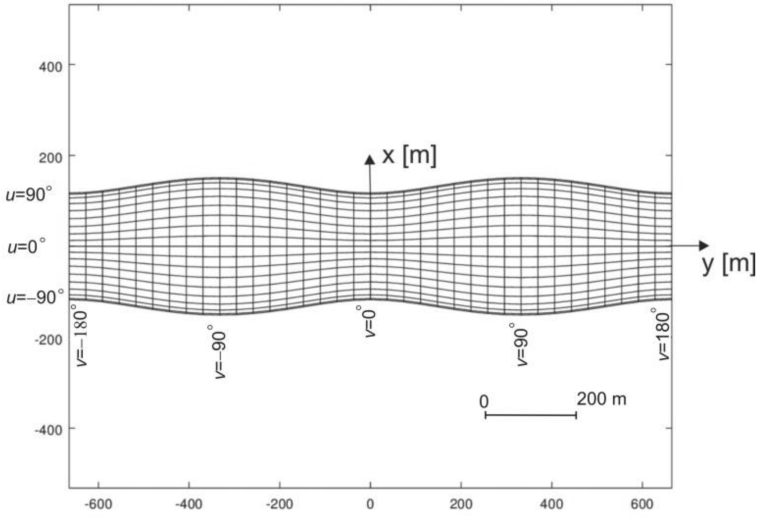

Fig. 1. A graticule of reduced coordinates in a cylindrical equal-area map projection of a tri-axial ellipsoid with equally spaced images of meridians $v = \text{const}$

Characteristic property of a graticule presented in Fig. 1 is that distances between images of meridians are equal and the length of an entire equator is preserved, but the projection is not equidistant. Meridians v =const on a tri-axial ellipsoid are distributed along the equator in different distances. Therefore conditions were introduced so that the distances between images of meridians were equal to the distances between meridians along the equator on a tri-axial ellipsoid.

Using developed projection (22), maps of distortion were created. On the first map (Figure 2) the Tissot's indicatrices are drawn. The second map (Figure 3) presents isograms for maximum angular deformation at 10° interval. The third one (Figure 4) shows isograms of extreme linear scales. Generally the interval is equal to 1 but in the range (1,2) the interval is equal to 0.1.

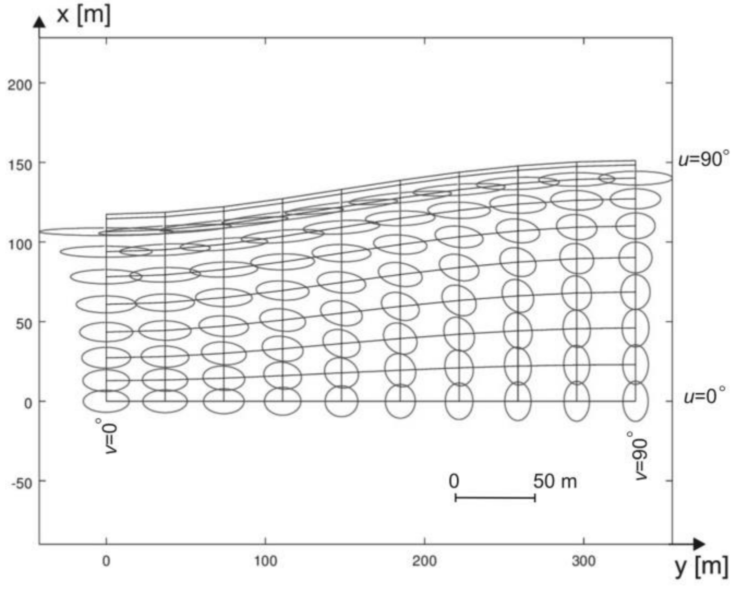

Fig. 2. Tissot's indicatrices in a cylindrical equal-area map projection of a tri-axial ellipsoid with equally spaced images of meridians $v = \text{const}$

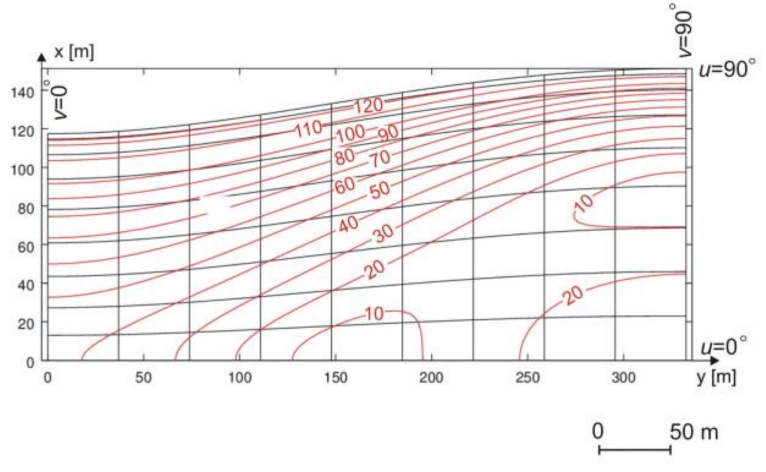

Fig. 3. Isograms for maximum angular deformations in a cylindrical equal-area map projection of a tri-axial ellipsoid with equally spaced images of meridians $v = \text{const}$

The result is a map projection with an interesting distribution of distortion. Generally, relatively small distortion on the equator grows with latitude. If we start from the point ($u = 0$, $v = 90°$) distortion slightly decreases at first and then starts growing.

Coordinates of nodes of a graticule were calculated basing on equations (34) and drawn in a cylindrical equal-area map projection of an entire tri-axial ellipsoid (Figure 5).

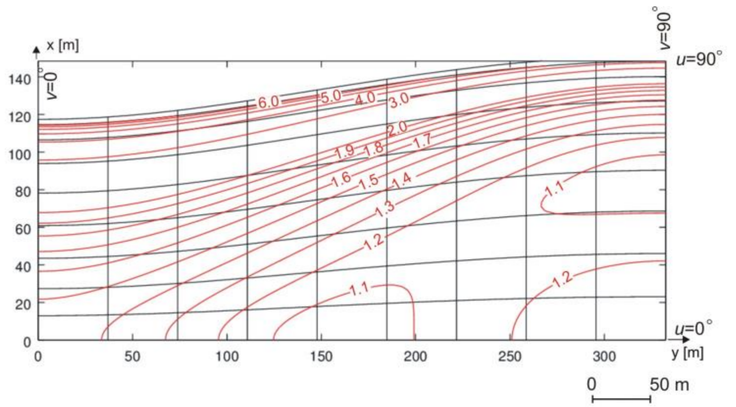

Fig. 4. Isograms for extreme linear scales in a cylindrical equal-area map4 projection of a tri-axial ellipsoid with equally spaced images of meridians v = const

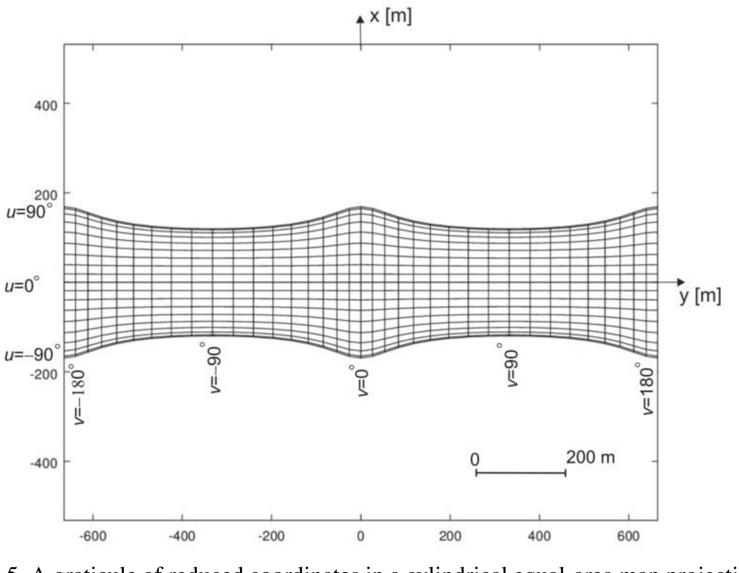

Fig. 5. A graticule of reduced coordinates in a cylindrical equal-area map projections of a tri-axial ellipsoid with the equator projected without distortion

After the comparison of graticules in Figures 1 and 5 the differences between them become easily noticeable. Especially the differences between lengths of meridians $v = 0$ and $v = \pm 90^\circ$ are interesting. In the graticule in Figure 1 the image of the central meridian $v = 0$ (overlap x-axis) is shorter than the images of meridians $v = \pm 90^\circ$ and in the graticule on Figure 5 there is inverse situation, i.e. the image of the central meridian $v = 0$ is longer than the images of meridians $v = \pm 90^\circ$. It is logical because in the graticule in Fig. 1 the image is shrunk near the central meridian in a direction of x-axis and in the graticule in Figure 5 the image is stretch near central meridian in a direction of x-axis.

The developed projection was used to create maps of distortion. On the first map (Figure 6) the Tissot's indicatrices are drawn. The second map (Figure 7) presents isograms for maximum angular deformation at 10° interval. The third one (Figure 8) shows isograms of extreme linear scales. Generally interval is equal to 1 but in the range (1, 2) interval is equal to 0.1.

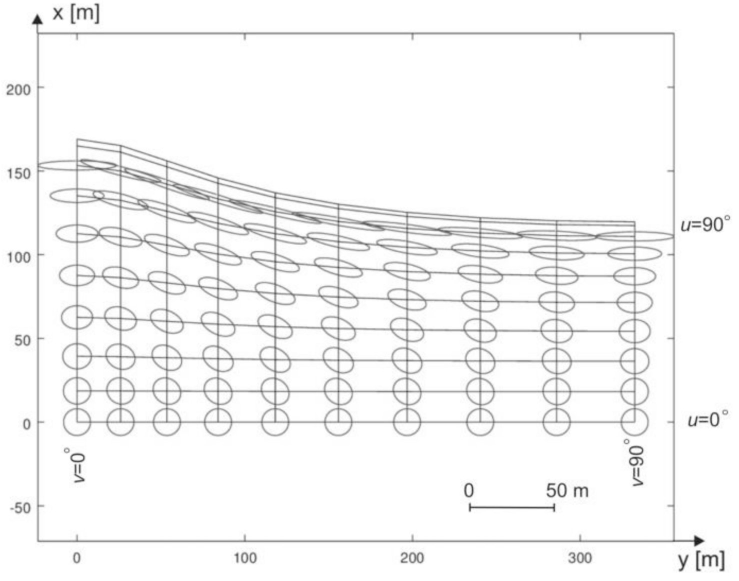

Fig. 6. Tissot's indicatrices in a cylindrical equal-area map projection of a tri-axial ellipsoid with the equator projected without distortion

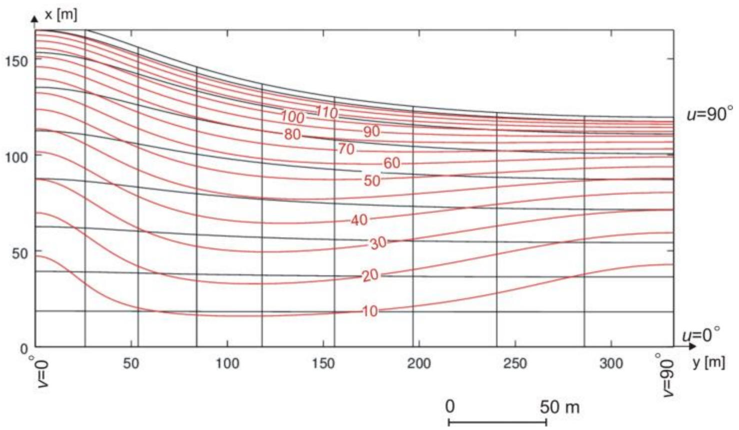

Fig. 7. Isograms for maximum angular deformations in a cylindrical equal-area map projection of a tri-axial ellipsoid with the equator projected without distortion

In a developed map projection the equator is projected without distortion and distortion grows with latitude. The shapes of isolines are interesting as they are curves that resemble waves.

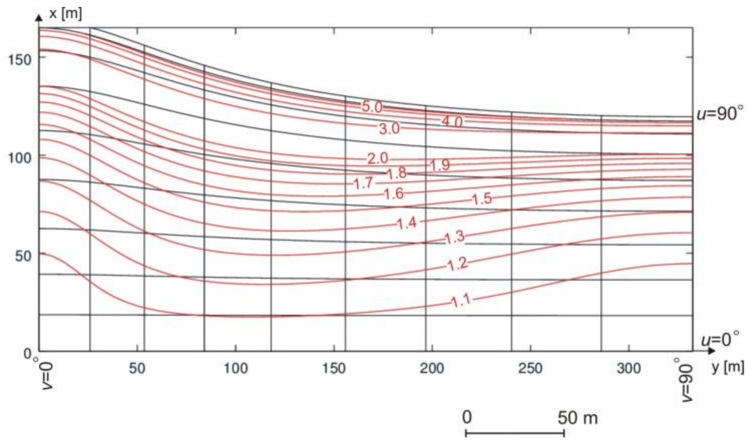

Fig. 8. and isograms for extreme linear scales in a cylindrical equal-area map projection of a tri-axial ellipsoid with the equator projected without distortion

4.2. Analysis of results in azimuthal map projections

Figure 9 presents the graticule of reduced coordinates for half of a tri-axial ellipsoid in azimuthal equal-area projection (40).

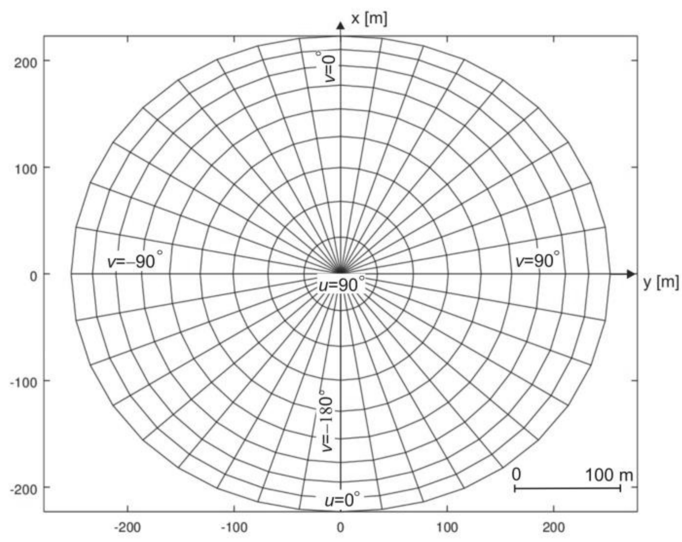

Fig. 9. The graticule of the northern half of a tri-axial ellipsoid in azimuthal equal-area projection

In the developed map projection images of meridians are straight lines outgoing radially from the central point. Angles between those lines are equal. It is not consistent with the original because on an ellipsoid meridians $v = \text{const}$ converge in the pole but the angles between them are not equal despite maintaining the same intervals of coordinate v.

Based on the developed map projection (40), maps were created showing the distortion distribution. Tissot's ellipses were presented in the first map (Figure 10). The second map (Figure 11) contains isograms for maximum angular deformation at 5° interval and in the range (30°, 35°) at 1°. The third one (Figure 12) shows isograms of extreme linear scales. The interval is equal to 0.1 and in the range (1.3–1.4) it is equal to 0.01.

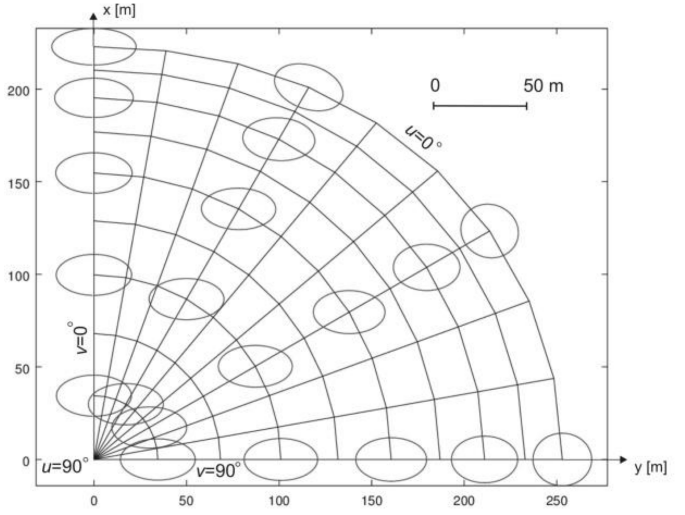

Fig. 10. Tissot's indicatrices in an azimuthal equal-area map projection of a tri-axial ellipsoid

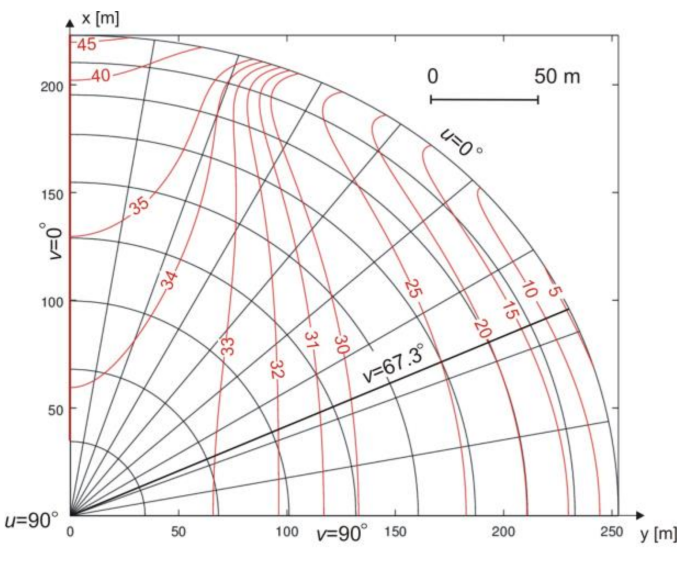

Fig. 11. Isograms for maximum angular deformation in an azimuthal equal-area map projection of a tri-axial ellipsoid

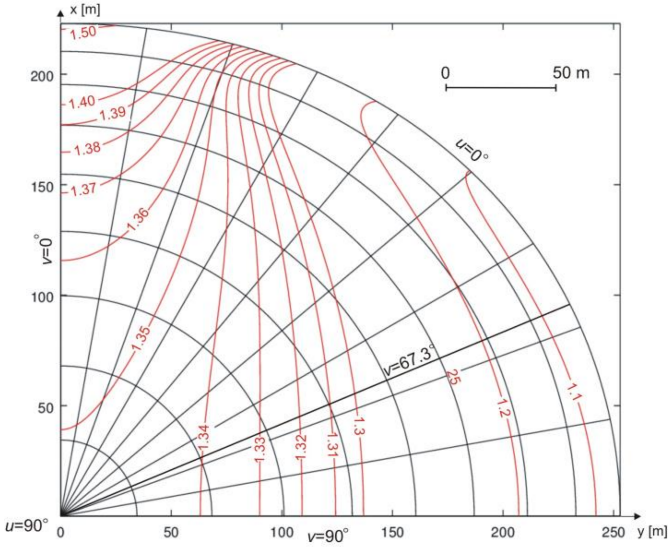

Fig. 12. Isograms of extreme linear scales in an azimuthal equal-area map projection of a tri-axial ellipsoid

Based on the analysis of the obtained results, an interesting property was detected, i.e. the smallest distortion occurred at a point located on the equator near the meridian 67.3°.

Using the developed map projections (48), a graticule of reduced coordinates was developed (Figure 13).

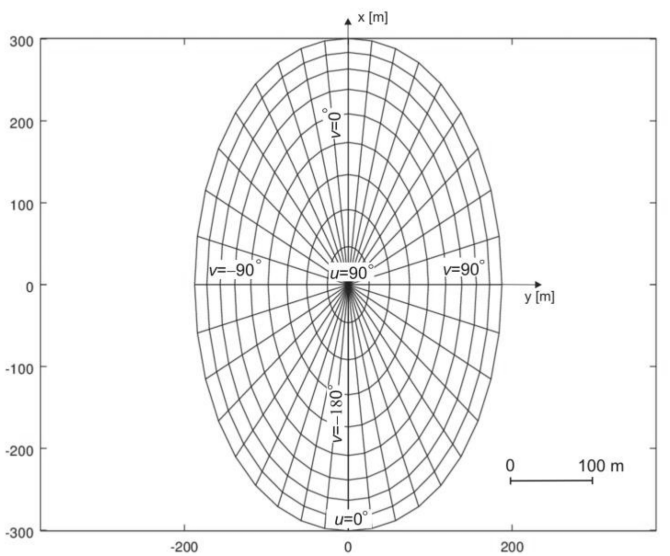

Fig. 13. The graticule of the northern half of a tri-axial ellipsoid in an azimuthal equal-area projection

Figure 13 clearly demonstrates that this time the images of meridians $v = \text{const}$ are not equally spaced but they are equal to the angles on an ellipsoid. Meridians are projected as straight lines and parallels as ellipses.

Based on the developed map projection (48), maps were created showing the distribution of distortion. Tissot's ellipses were presented in the first map (Figure 14). The second map (Figure 15) contains isograms for maximum angular deformation at 5° inter-

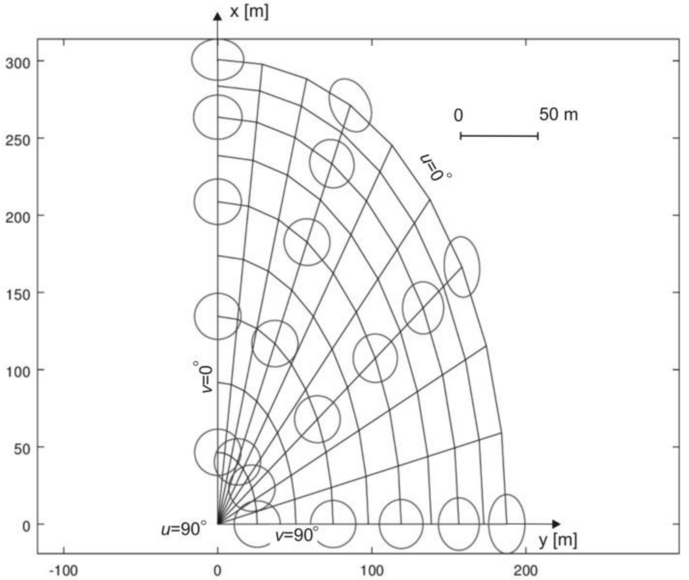

Fig. 14. Tissot's indicatrices in azimuthal equal-area map projection of a tri-axial ellipsoid

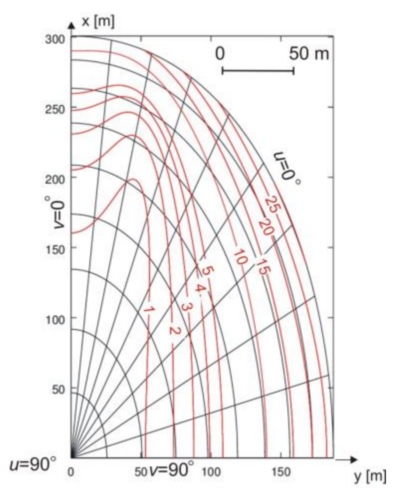

Fig. 15. Isograms for maximum angular deformation in azimuthal equal-area map projection of a tri-axial ellipsoid

val and in the range (0°, 5°) at 1°. The third one (Figure 16) shows isograms of extreme linear scales. Interval is equal to 0.1 and in the range (1.0–1.1) it is equal to 0.01.

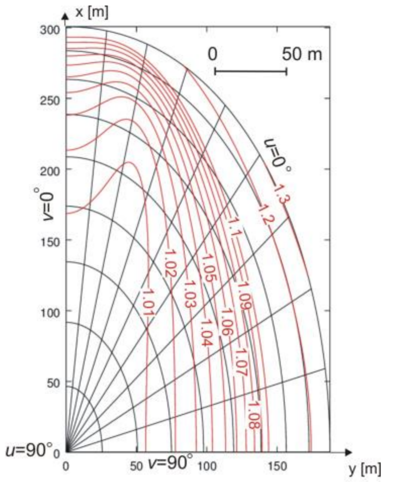

Fig. 16. Isograms of extreme linear scales in azimuthal equal-area map projection of a tri-axial ellipsoid

Based on the analysis of obtained results, it can be noticed that the distortion decreases with latitude. The shape of the obtained isolines is very interesting.

5. Conclusions

The article presents the method of construction of cylindrical and azimuthal equal-area map projections of a tri-axial ellipsoid expressed by means of reduced coordinates. This is an alternative method to other methods described in subject literature. The functions in the developed map projections are presented by means of elliptic integrals and Jacobian elliptic functions. The algorithms for calculating coordinates in those projections require a solution of the second kind of the elliptic integral in the normal form and the integrals G1 and G3, which are expressed by Jacobian elliptic functions, and those in turn by means of trigonometric and elementary functions, the solution of which should not cause much trouble. The article also presents the method of determining distortion. Partial derivatives are also expressed by similar functions as equations for the calculation of planar coordinates. The calculation of the elliptic integral of the second kind is required when calculating the coordinates in the developed map projections. Methods for solving such integrals have been described in many publications, including (Byrd and Friedmann, 1954).

The developed method can be used in planetary cartography for mapping irregular objects, for which tri-axial ellipsoids have been accepted as reference surfaces. It can also be used to calculate the surface areas of regions located on these objects. The paper presents graticules of reduced coordinates. Such graticules differ significantly from the graticules of other coordinates used on a tri-axial ellipsoid, i.e. planetocentric and planetographic ones. Therefore, when creating maps in developed projections, it is important to remember to use the appropriate coordinate graticules.

Acknowledgements

The research presented in this paper has been supported by statutory subsidies at the Faculty of Geodesy and Cartography of Warsaw University of Technology.

References

Balcerzak, J. and Panasiuk, J. (2005). *Wprowadzenie do kartografii matematycznej*. Oficyna Wydawnicza Politechniki Warszawskiej.

Berthoud, M.G. (2005). An equal-area map projection for irregular objects. *Icarus*, 175(2), 382–389.

Bugaevsky, L.M. (1987). K voprosu o poluchenii izometricheskikh koordinat i ravnougol'noy tsilindricheskoy proyektsii trekhosnogo ellipsoida. *Izvestiya Vysshikh Uchebnykh Zavedeniy. Geodeziya i Aerofotosyemka*, 4, 79–90.

Bugaevsky, L.M. (1991). Izometricheskiye koordinaty, ravnougol'noy tsilindricheskoy, konicheskoy i azimutal'noy proyektsii trekhosnogo ellipsoida. *Izvestiya Vysshikh Uchebnykh Zavedeniy. Geodeziya i Aerofotosyemka*, 3, 144–152.

Bugaevsky, L.M. (1999). *Teoria kartograficheskikh proyektsiy riegularnykh povierkhnostiey*. Zlatoust.

Byrd, P.F. and Friedmann, M.D. (1954). *Handbook of elliptic integrals for engineers and phisicists*. Springer-Verlag, Berlin-Gottingen-Heidelberg.

Fleis, M.E., Nyrtsov, M.V. and Borisov, M.M. (2013). Cylindrical Projection Conformality of Tri-axial Ellipsoid. *Doklady Earth Sciences*, 451, Part 1, 787–789. DOI: 10.1134/S1028334X13070234.

Nyrtsov, M.V., Fleis, M.E. and Borisov, M.M. (2012). Kartografirovaniye asteroida 433 Eros v ravnopromezhutochnykh vdol' meridianov tsilindricheskoy i azimutal'noy proyektsiyakh trekhosnogo ellipsoida. *Izvestiya Vysshikh Uchebnykh Zavedeniy. Geodeziya i Aerofotosyemka*, 1, 54–61.

Nyrtsov, M. V., Fleis, M. E., Borisov, M. M. and Stooke, P. J. (2013). Equal-area projections of the tri-axial ellipsoid: first time derivation and implementation of cylindrical and azimuthal projections for small solar system bodies. *The Cartographic Journal*, 52(2), 114–124. 10.1080/00087041.2015.1119471.

Nyrtsov, M.V., Fleis, M.E., Borisov, M.M., Stooke, P.J. (2014). Jacobi Conformal Projection of the Triaxial Ellipsoid: New Projection for Mapping of Small Celestial Bodies. In: M. Buchroithner et al. (eds.). *Cartography from Pole to Pole, Lecture Notes in Geoinformation and Cartography*. Springer-Verlag Berlin Heidelberg. DOI: 10.1007/978-3-642-32618-9_17.

Pędzich, P. (2017). Equidistant map projections of a tri-axial ellipsoid with the use of reduced coordinates. *Geodesy and Cartography*, 66(2), 271–290. DOI: 10.1515/geocart-2017-0021.

Pędzich, P. (2014). *Podstawy odwzorowań kartograficznych z aplikacjami komputerowymi*. Oficyna Wydawnicza Politechniki Warszawskiej.

Snyder J.P. (1985). Conformal mapping of the tri-axial ellipsoid. *Survey Review*, 28(217), 130–148.