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Stability regions of closed loop system with time delay inertial plant of fractional order and fractional order PI controller

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Abstract. The paper presents the stability problem of control systems composed of a fractional-order PI controller and a inertial plant of a fractional order with time delay. Simple and efficient computational method for determining stability regions in the controller and plant parameters space is given. Knowledge of these regions permits tuning of the fractional-order PI controller. The method proposed is based on the classical D-partition method.

Key words: PID controllers, fractional system, stability, delay, D-partition method.

1. Introduction

Proportional – Integral – Derivative (PID) controllers are widely applied, because of their simple structure. PID-control has been the subject of many publications (see for example [1–4]). Many methods of tuning PID controllers for satisfactory behaviour have been proposed in the literature [3]. These methods are based on knowledge of the mathematical description of the process. The first order-plant with time delay is the most frequently used model for tuning PID controller [1, 3].

The asymptotic stability of a closed-loop system is the basic requirement. Several methods for determining the asymptotic stability regions in the controller parameter space have been presented [2, 4–6]. These approaches are based on the Pontryagin's theory [2, 6] and generalization of the Hermite-Biehler Theorem [4, 5]. An alternative approach to the problem of stabilizing the first order-plant with time delay and multi-inertial plant with time delay was described in [7–9]. Using the classical D-partition method, simple and efficient computational methods of computing the asymptotic stability and D-stability region in the parameter space are presented.

In recent years, considerable attention has been paid to control systems whose processes and/or controllers are of a fractional order (see for example [10–14]). The fractional PID controllers, namely PI$^\lambda$D$^\mu$ controllers, including an integrator of a λ order and a differentiator of a μ order have been proposed in [13]. Several design methods of tuning the PI$^\lambda$D$^\mu$ controllers for systems without time delay have been presented [13-16]. It has been shown that the PI$^\lambda$D$^\mu$ controller which has five degrees of freedom enhances the system control performance when used for control systems with IO (integer order) plants and FO (fractional order) plants. A computation method of stabilizing fractional-order PI$^\lambda$D$^\mu$ controllers for fractional-order time delay systems was presented in [17].

In this paper a simple method of determining the stability region in the parameter space of a inertial plant of a fractional order with time delay and a fractional-order PI controller is given. Using this region, a very fast and simple way of calculating the stabilizing values of PI$^\lambda$ controllers is obtained. Four cases for closed-loop control systems are analysed: 1) IO controller with IO plant; 2) IO controller with FO plant; 3) FO controller with IO plant and 4) FO controller with FO plant. The integer order PI controller and a plant are special cases of the fractional PI controller and a plant. The method proposed is based on the D-partition method and an approach given in [7, 18].

2. Problem formulation

Consider the feedback control system shown in Fig. 1 in which the process to be controlled is described by an inertial plant with time delay

$$G(s) = \frac{Ke^{-sh}}{1 + s^\alpha T}, \tag{1}$$

where K, T, h are positive real numbers and the order α can be integer ($\alpha = 1$) or fractional with $0 < \alpha < 1$.

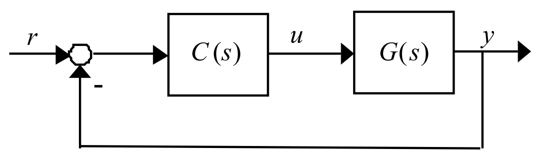

Fig. 1. Feedback control system

Let the controller $C(s)$ be of the fractional PI with the transfer function [13]

$$C(s) = k_p + \frac{k_i}{s^\lambda}, \tag{2}$$

where k_p and k_i denote the proportional and integral gains of the controller, and λ is the fractional order of the integrator (the order may assume positive real noninteger values). Clearly, on selecting $\lambda = 1$, a classical PI controller can be obtained.

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The characteristic function of the closed-loop system with plant (1) and controller (2) is given by

$$w(s) = K(k_p s^\lambda + k_i)\exp(-sh) + (1 + Ts^\alpha)s^\lambda. \quad (3)$$

The closed-loop system in Fig. 1 is said to be bounded-input bounded-output stable if and only if all the zeros of the characteristic function (3) have negative real parts. It is noted that (3) is a quasi-polynomial which has an infinite number of zeros. This makes the problem of analysing the stability of the closed-loop system difficult. There is no general algebraic methods available in the literature for the stability test of quasi-polynomials. In general, the stability test of a quasi-polynomial is performed with a graphical method, e.g., the Mikhajlov criterion. Algebraic methods are only known for a particular class of quasi polynomials [6, 19, 20]. The next problem of synthesis of the closed-loop system is how to choose such a fractional order λ of the integrator that the closed-loop system will be stable.

3. Main results

Multiplying quasi-polynomial (3) by $\exp(sh)$ we obtain a new quasi-polynomial in the form

$$w(s) = K(k_p s^\lambda + k_i) + s^\lambda(1 + Ts^\alpha)\exp(sh), \quad (4)$$

which has exactly the same zeros as quasi-polynomial (3). Substituting $z = sh$ in quasi-polynomial (4) after transformations we obtain the quasi-polynomial

$$w(z) = Xz^\lambda + Y + z^\lambda(1 + pz^\alpha)\exp(z), \quad (5)$$

where $X = Kk_p$, $Y = Kk_i h^\lambda$, $p = T/h^\alpha$.

Using the D-partition method [2] the asymptotic stability region in the parameter plane (X, Y) may be determined and the parameters can be specified. The plane (X, Y) is decomposed by the so-called boundaries of D-partition into finite number regions $D(k)$. Any point in $D(k)$ corresponds to such values of X and Y that quasi-polynomial (5) has exactly k zeros with positive real parts. The region $D(0)$, if it exists, is the stability region of quasi-polynomial (5). The D-partition boundaries are curves on which each point corresponds to quasi-polynomial (5) having zeros on the imaginary axis. It may be the real zero boundary or the complex zero boundary. It is easy to see that quasi-polynomial (5) has zero $z = 0$ if $Y = 0$ (the real zero boundary). The complex zero boundary corresponds to the pure imaginary zeros of (5). We obtain this boundary by solving the equation

$$w(j\omega) = X(j\omega)^\lambda + Y + (j\omega)^\lambda[1 + p(j\omega)^\alpha]\exp(j\omega) = 0, \quad (6)$$

which we get by substituting $z = j\omega$ in quasi-polynomial (5) and equating to 0. The term of j^λ which is required for equation (6) can be expressed by

$$j^\lambda = \cos\left(\frac{\pi}{2}\lambda\right) + j\sin\left(\frac{\pi}{2}\lambda\right). \quad (7)$$

Using (7) the complex equation (6) can be rewritten as a set of real equations in the form

$$\begin{aligned} &p\omega^{\lambda+\alpha}\cos\left(\frac{\pi}{2}(\lambda+\alpha)+\omega\right) + \omega^\lambda\cos\left(\frac{\pi}{2}\lambda+\omega\right) \\ &\quad + X\omega^\lambda\cos\left(\frac{\pi}{2}\lambda\right) + Y = 0, \end{aligned} \quad (8)$$

$$\begin{aligned} &p\omega^{\lambda+\alpha}\sin\left(\frac{\pi}{2}(\lambda+\alpha)+\omega\right) + \omega^\lambda\sin\left(\frac{\pi}{2}\lambda+\omega\right) \\ &\quad + X\omega^\lambda\sin\left(\frac{\pi}{2}\lambda\right) = 0. \end{aligned} \quad (9)$$

Finally, by solving the Eqs. (8) and (9) we obtain

$$X = \frac{-1}{\sin\left(\frac{\pi}{2}\lambda\right)}\left[p\omega^\alpha\sin\left(\frac{\pi}{2}(\lambda+\alpha)+\omega\right) + \sin\left(\frac{\pi}{2}\lambda+\omega\right)\right], \quad (10)$$

$$Y = \frac{1}{\sin\left(\frac{\pi}{2}\lambda\right)}\left[p\omega^{\lambda+\alpha}\sin\left(\frac{\pi}{2}\alpha+\omega\right) + \omega^\lambda\sin(\omega h)\right]. \quad (11)$$

Equations (10) and (11) determine the complex zero boundary in plane (X, Y). The real zero boundary and the complex zero boundary for $\omega > 0$ decompose plane (X, Y) into regions $D(k)$. The stability region $D(0)$ is chosen by testing an arbitrary point from each region and checking the stability of the quasi-polynomial (5) using the methods proposed in [21].

First, we analyse the asymptotic stability regions of quasi-polynomial (5) for case 1 (IO controller with IO plant). This was derived in [7]. Figure 2 shows the asymptotic stability regions of quasi-polynomial (5) for $\lambda = 1$, $\alpha = 1$ and different values of $p = T/h$. Any point from these regions ensures the stability of the control system considered. It is observed from Fig. 2 that the increasing value of p results in larger stability regions, because of the increase ranges of $X = Kk_p$ and $Y = Kk_i h$.

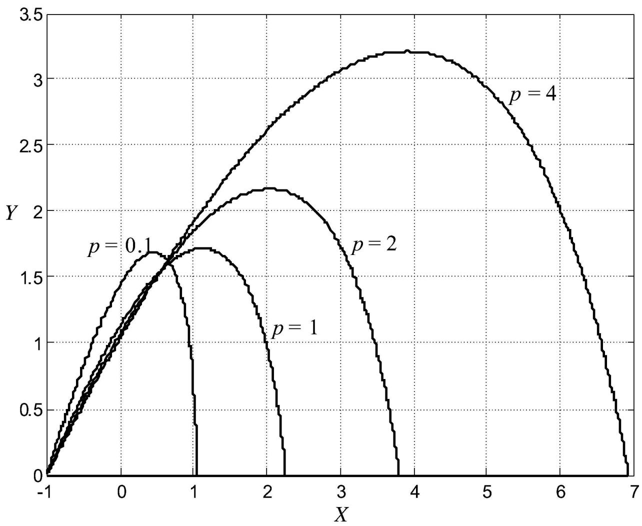

Fig. 2. Stability regions for quasi-polynomial (5) for $\lambda = 1$, $\alpha = 1$

Next we consider the case with IO controller and FO plant. Figure 3a shows the stability regions of quasi-polynomial (5) for $\lambda = 1$ and $p = 4$ and different values of the fractional order of plant $\alpha \in (0, 1)$. We can see that the stability regions in this case are larger than in the previous one. The use of the fractional order of a plant causes an increase in the stability regions. The increasing value of p entails increasing stability regions (Fig. 3b).

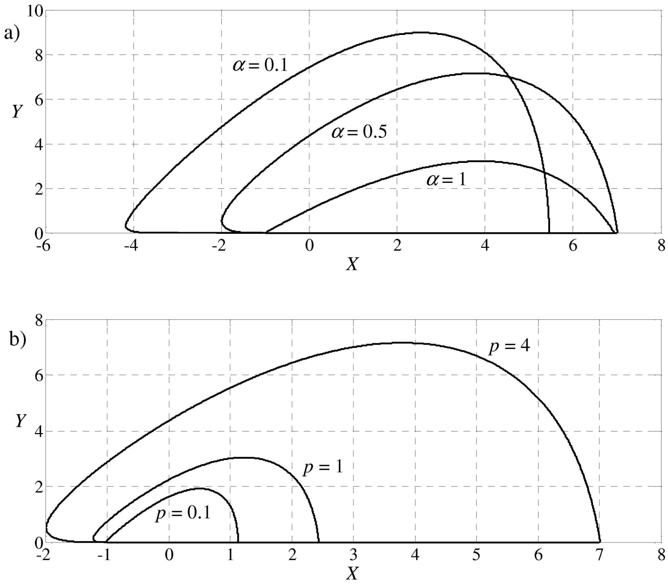

Fig. 3. Stability regions for quasi-polynomial (5) for $\lambda = 1$; a) $p = 4$, b) $\alpha = 0.5$

The stability regions for the case with FO controller and IO plant is shown in Fig. 4. The figure shows that for $\lambda < 1$ the stability regions are larger than for $\lambda = 1$. An increase in the value of λ to over one initially results in an increase in the stability region after which it begins to decrease. The value of λ at which the stability region disappears is $\lambda = 2$. The result of an increase in the value of p is the same as in case 1 (Fig. 4b).

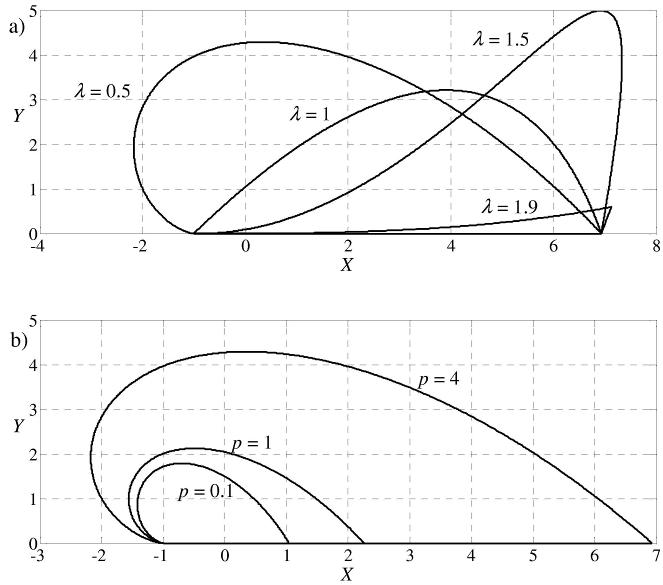

Fig. 4. Stability regions for quasi-polynomial (5) for $\alpha = 1$; a) $p = 4$, b) $\lambda = 0.5$

Figure 5 shows the stability regions of quasi-polynomial (5) for the case with FO controller and FO plant. We can see that the stability regions in this case are larger than in all the previous ones. Like the case with FO controller and IO plant, an excessive increase in the value of λ results in the disappearance of the stability region.

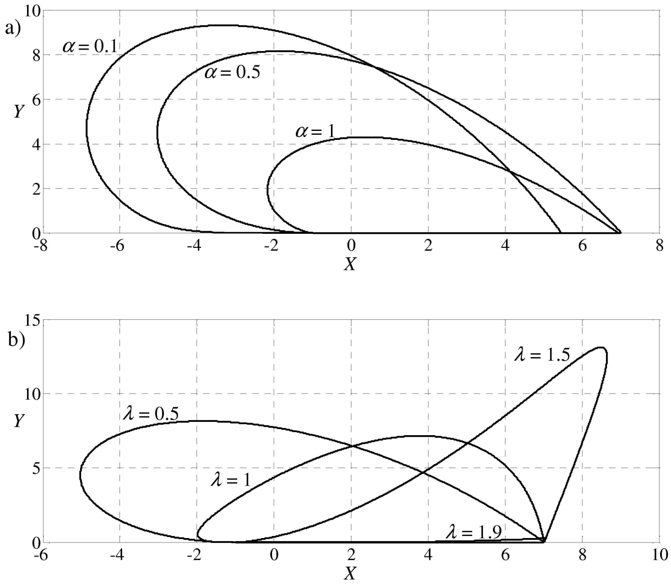

Fig. 5. Stability regions for quasi-polynomial (5) for $p = 4$; a) $\lambda = 0.5$, b) $\alpha = 0.5$

To sum up, we can see that increasing the value of p causes an increase in the stability regions with IO plant and with FO plant. The fractional order of the controller entails increasing stability regions both with IO plant and FO plant when $\lambda < 1$. An excessive value of λ results in the disappearance of the stability region. The value of λ at which the stability region disappears is $\lambda = 2$.

4. Example

Consider the feedback control system shown in Fig. 1 in which the process to be controlled is described by transfer function (1) where $K = 1$, $h = 0.5$, $T = 2$.

First we analyse the closed-loop system with IO controller and IO plant (case 1). The stability regions are shown in Fig. 2. In the example we have $p = T/h = 4$, thus the stability region is limited by the line $Y = 0$ and the curve corresponds to $p = 4$. Choosing an arbitrary point from this region, e.g. $X = 2$, $Y = 1$, we obtain $Kk_p = 2$, $Kk_ih^\lambda = 1$. By computation based on the above expressions the following stabilizing controller parameters $k_p = 2$, $k_i = 2$ are obtained. In a similar manner we can determine the transfer function of the controller (2) in others cases:

2) IO controller with FO plant $\lambda = 1$, $\alpha = 0.5$, $X = 2$, $Y = 4$, $C(s) = 2 + \dfrac{8}{s}$,
3) FO controller with IO plant $\lambda = 0.5$, $\alpha = 1$, $X = 2$, $Y = 2$, $C(s) = 2 + \dfrac{2.828}{s^{0.5}}$,
4) FO controller with FO plant $\lambda = 0.5$, $\alpha = 0.5$, $X = 2$, $Y = 4$, $C(s) = 2 + \dfrac{5.657}{s^{0.5}}$.

It is noted that point $X = 2$, $Y = 4$ lies outside the stability region for the case with IO controller and IO plant (Fig. 2). In case 3 (FO controller, IO plant) this point also lies outside the stability region for the value $\lambda = 0.5$. In order to ensure that this point lies in the stability region we have to decrease the value of λ. That point lies in the stability region for the case with FO controller and FO plant for $\lambda = 0.5$, $\alpha = 0.5$ and for the case with IO controller and FO plant for $\alpha = 0.5$.

5. Concluding remarks

In this paper, the stability problem of control systems composed of a fractional-order PI controller and a inertial plant of fractional order with time delay is examined. Four cases of closed-loop control systems have been analysed: 1) IO controller with IO plant; 2) IO controller with FO plant; 3) FO controller with IO plant and 4) FO controller with FO plant. The integer order PI controller and plant are special cases of the fractional PI controller and plant. On the basis of the D-partition method, analytical forms expressing the D-partition boundaries of stability regions in the parameter space were determined. Knowledge of stability regions permits tuning of the fractional PI type controller. This study shows that a PI^{λ} controller which has three degrees of freedom increases the stability regions for $\lambda < 1$ when used for control systems with IO and FO process models with time delay.

The studies can be extended for the problem of stabilization using a fractional-order PID controller.

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