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Application of LMI for design of digital control systems

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Abstract. The paper considers a digital design of time-invariant systems in the case of step-invariant (ZOH), bilinear (Tustin's) and fractional order hold (FROH) discretization methods. The design problem is formulated as linear matrix inequalities (LMI). A closed-loop stability of the digitally designed control systems is discussed.

Key words: digital control, LMI.

1. Introduction

The equivalence between continuous-time and digital systems takes place when the responses of both systems are closely matched for the same inputs and initial conditions. The digital design approach presented in this paper called digital redesign consists in design a suitable analogue controller first and then convert the obtained analogue controller to the equivalent digital controller maintaining the properties of the original analogously controlled system. Obviously, the stability of redesigned system is required for the practical application of of the digital redesign technique.

The objective of this paper is to present and compare the digital redesign methods when the step-invariant, bilinear and fractional order hold transformations are used. The first two methods are most commonly used methods of discretization while the third one is used to obtain the model with zeros as stable as possible.

The design method is based on the linear matrix inequality (LMI) technique according to the approach presented in [1]. When the optimisation problem is convex, it can be solved effectively and fast with the use of LMI algorithms [2, 3]. It can be said that once the problem is formulated as a set of LMIs, it can be treated as solved, and used software (such as LMI Toolbox, Yalmip, etc) enables one to find the exact optimum.

The result of this paper is the enlargement of the redesign method presented in [1] to the case of bilinear and fractional order hold discretization methods.

Performance of the considered design method applied with considered transformations is illustrated by simulation study of fourth-order system and linearized model of chemical reactor for the standard LQR control problem. The asymptotic properties of digital closed-loop systems, i.e. when the sampling period tends to zero are also discussed.

2. Control problem formulation

Consider a linear time-invariant continuous-time system described by

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t), \quad x_c(0) = x_0, \tag{1}$$

$$y_c(t) = Cx_c(t), \tag{2}$$

where $x_c(t)$ is the n-dimensional state vector, $u_c(t)$ is the m-dimensional control vector, $y_c(t)$ is the p-dimensional output vector, and A, B, C are matrices of appropriate dimensions. The continuous-time controller is given by

$$u_c(t) = -K_c x_c(t) + E_c r, \tag{3}$$

where K_c is the state feedback gain matrix, E_c is the feedforward gain matrix and r is the p-dimensional reference vector.

The system (1) and the controller (3) yields the continuous-time closed-loop control system

$$\dot{x}_c(t) = (A - BK_c)x_c(t) + BE_c r, \quad x_c(0) = x_0, \tag{4}$$

where the output remains as in (2).

2.1. Step-invariant transformation. The discrete-time model of the closed-loop system (3), (4) obtained with step-invariant (ZOH) method is given by

$$x_c(kT + T) = G_c x_c(kT) + H_c E_c r, \tag{5}$$

$$y_c(kT) = Cx_c(kT), \tag{6}$$

where T is the sampling period, and

$$G_c = \exp((A - BK_c)T), \tag{7}$$

$$H_c = (G_c - I)(A - BK_c)^{-1}B. \tag{8}$$

It is to be stressed that, whenever identity matrix is used, its size results from matrix algebra.

The corresponding matrix transfer function is

$$K_{si}(z) = C(zI - G_c)^{-1}H_c E_c. \tag{9}$$

The state Eq. (1) with a digital control input can be given as

$$\dot{x}_d(t) = Ax_d(t) + Bu_d(t), \quad x_d(0) = x_0, \tag{10}$$

$$y_d(t) = Cx_d(t), \tag{11}$$

where

$$u_d(t) = u_d(kT) = -K_d x_d(kT) + E_d r, \quad kT \le t < kT + T, \tag{12}$$

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where K_d and E_d denote the digital feedback and feedforward gains, respectively. The resulting closed-loop system is

$$\dot{x}_d(t) = Ax_d(t) - BK_d x_d(kT) + BE_d r, \tag{13}$$

and the discrete-time model of the closed-loop system (13) at $t = kT + T$ is given by

$$x_d(kT+T) = (G - HK_d)x_d(kT) + HE_d r, \tag{14}$$

$$y_d(kT) = Cx_d(kT), \tag{15}$$

where

$$G = \exp(AT), \tag{16}$$

$$H = (G - I)A^{-1}B. \tag{17}$$

If A is singular, the matrix H can be computed as [1]

$$H = \sum_{i=1}^{\infty} \frac{1}{i!}(AT)^{i-1}BT.$$

Now the problem is formulated as follows: for a well designed gains K_c, E_c in continuous-time controller (3), determine the digital control gains K_d, E_d in (10) such that digitally controlled system (12) is stable and the output of discrete-time system (14), (15) matches the output of continuous-time system (4), (2) as closely as possible.

2.2. Bilinear transformation. The discrete-time model of the closed-loop system (4) obtained with bilinear transformation (Tustin's method) is given analogously to (5), (6) by

$$x_c(kT+T) = G_c x_c(kT) + H_c E_c r, \tag{18}$$

$$y_c(kT) = C_c x_c(kT) + D_c E_c r, \tag{19}$$

where [2]

$$G_c = \left(I - \frac{T}{2}(A - BK_c)\right)^{-1}\left(I + \frac{T}{2}(A - BK_c)\right), \tag{20}$$

$$H_c = \frac{T}{2}\left(I - \frac{T}{2}(A - BK_c)\right)^{-1} B, \tag{21}$$

$$C_c = 2C\left(I - \frac{T}{2}(A - BK_c)\right)^{-1}, \tag{22}$$

$$D_c = \frac{T}{2}C\left(I - \frac{T}{2}(A - BK_c)\right)^{-1} B. \tag{23}$$

The matrix transfer function for (18), (19) is

$$K_{bl}(z) = [C_c(zI - G_c)^{-1}H_c + D_c]E_c. \tag{24}$$

The matrices G and H in the corresponding discrete-time model of the closed-loop system (14) are now given by

$$G = \left(I - \frac{T}{2}A\right)^{-1}\left(I + \frac{T}{2}A\right), \tag{25}$$

$$H = \frac{T}{2}\left(I - \frac{T}{2}A\right)^{-1} B. \tag{26}$$

The following asymptotic property of step-invariant and bilinear transformations has been proved in [4]: suppose that in the stable closed-loop system (4) the matrix $A - BK_c$ is diagonalizable with all real eigenvalues, then

$$\lim_{T \to 0^+} ||K_{si} - K_{bl}||_{\ell_p} = 0 \tag{27}$$

for all $1 \leq p \leq \infty$, where $||\cdot||_{\ell_p}$ denotes the ℓ_p-induced norm.

2.3. Fractional order hold transformation. The discrete-time model of the closed-loop system (4), (2) obtained with fractional order hold (FROH) method is given by [5]

$$\begin{bmatrix} x_c(kT+T) \\ x_1(kT+T) \end{bmatrix} = \begin{bmatrix} G_c & \beta\gamma^- \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_c(kT) \\ x_1(kT) \end{bmatrix} + \begin{bmatrix} \gamma - \beta\gamma^- \\ 1 \end{bmatrix} E_c r, \tag{28}$$

$$y_c(kT) = Cx_c(kT), \tag{29}$$

where G_c is given by (7) and

$$\gamma = \int_0^T exp((A - BK_c)s)dsB = H_c, \tag{30}$$

$$\begin{aligned}\gamma^- &= \int_0^T \frac{s}{T}\exp((A - BK_c)s)dsB \\ &= (A - BK_c)^{-1}[G_c - \frac{1}{T}(G_c - I)(A - BK_c)^{-1}]B.\end{aligned} \tag{31}$$

It can be noticed that $\gamma = H_c$ where H_c is given by (8). The parameter β is the device adjustable gain, and in the cases of ZOH and FOH (first-order hold) considered as particular cases of FROH, we have $\beta = 0$, $\beta = 1$, respectively. The corresponding matrix transfer function is

$$K_{fh}(z) = [C \quad 0]\left(zI - \begin{bmatrix} G_c & \beta\gamma^- \\ 0 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} \gamma - \beta\gamma^- \\ 1 \end{bmatrix}. \tag{32}$$

The corresponding discrete-time model of the closed-loop system (14) is now given by

$$\begin{bmatrix} x_d(kT+T) \\ x_{1,d}(kT+T) \end{bmatrix} = \begin{bmatrix} G - (\gamma' - \beta\gamma^{--})K_d & \beta\gamma^- \\ -K_d & 0 \end{bmatrix} \begin{bmatrix} x_d(kT) \\ x_{1,d}(kT) \end{bmatrix} + \begin{bmatrix} \gamma' - \beta\gamma^{--} \\ 1 \end{bmatrix} E_d r, \tag{33}$$

$$y_d(kT) = Cx_d(kT), \tag{34}$$

where G is given by (16) and

$$\gamma' = \int_0^T \exp(As)dsB = (G - I)A^{-1}B, \tag{35}$$

$$\gamma^{--} = \int_0^T \frac{s}{T}\exp(As)dsB = A^{-1}\left[G - \frac{1}{T}(G - I)A^{-1}\right]B. \tag{36}$$

From (33) the following equation can be obtained

$$\begin{aligned} x_d(kT+T) &= [G - (\gamma' - \beta\gamma^{--})K_d]x_d(kT) \\ &+ (\gamma' - \beta\gamma^{--})E_d r + \beta\gamma^{--}x_{1,d}(kT), \end{aligned} \tag{37}$$

where the corresponding matrix H is now

$$H = \gamma' - \beta\gamma^{--}. \tag{38}$$

It can be noticed that if $\beta = 0$ then $\gamma' = H$ where H is given by (17).

3. Design method

In this section, the method for determination of the gain matrices K_d, E_d proposed in [1] will shortly be described. According to the theorem given in [1] if there exist a symmetric positive definite matrix Γ, a matrix F, and a scalar $\alpha > 0$ such that the following generalized eigenvalue problem (GEVP) [2] is solved, then the digital control law (12) with $E_d = 0$ satisfies the given design objective formulated in the problem stated in Section 2.1.

$$\min_{\Gamma, F} \alpha \quad \text{s.t.} \tag{39}$$

$$\begin{bmatrix} -\alpha\Gamma & * \\ G_c\Gamma - G\Gamma + HF & -\alpha I \end{bmatrix} < 0, \tag{40}$$

$$\begin{bmatrix} -\Gamma & * \\ G\Gamma - HF & -\Gamma \end{bmatrix} < 0 \tag{41}$$

where $F = K_d\Gamma$ and $*$ denotes the transposed element in the symmetric positions. The feedback gain matrix K_d is given then by

$$K_d = F\Gamma^{-1}. \tag{42}$$

The feedforward gain matrix E_d can be obtained by equalizing the steady-state outputs of the original and redesigned systems. For the case of step-invariant transformation, this yields

$$E_d = ((I - (G - HK_d))^{-1}H)^{\dagger}(I - G_c)^{-1}H_cE_c \tag{43}$$

where $(\cdot)^{\dagger}$ denotes the Moore-Penrose pseudo-inverse, and the matrices G, H are given by (16), (17), respectively. A separate determination of E_d is done in order to avoid solving the bilinear matrix inequality (BMI) problem.

In the case of bilinear transformation, similarly to step-invariant transformation, the feedforward gain matrix E_d can again be obtained by equalizing the steady-state output of the original system

$$y_c(\infty) = K_{bl}(1)r = [C_c(I - G_c)^{-1}H_c + D_c]E_cr, \tag{44}$$

and the steady-state output of redesigned sytem

$$y_d(\infty) = C_c[(I - (G - HK_d))^{-1}H + D_c]E_dr, \tag{45}$$

This gives

$$E_d = [C_c(I - (G - HK_d))^{-1}H + D_c]^{\dagger} [C_c(I - G_c)^{-1}H_c + D_c]E_c, \tag{46}$$

where G, H are given by (25), (26).

In the case of fractional order hold transformation, the steady-state output of the original system

$$y_c(\infty) = C(I - G_c)^{-1}\gamma E_cr, \tag{47}$$

and the steady-state output of redesigned sytem

$$y_d(\infty) = C[I - G + (\gamma' - \beta\gamma^{--})K_d]^{-1} [(\gamma' - \beta\gamma^{--})E_dr + \beta\gamma^{--}x_{1,d}], \tag{48}$$

This gives

$$E_d = [(I - (G - \gamma' K_d))^{-1}\gamma']^{\dagger}(I - G_c)^{-1}\gamma E_c, \tag{49}$$

so it corresponds to (43).

4. Simulation tests

Performance of the described method for both transformations is illustrated through the example of a fourth-order unstable system with the following numerical values [1]:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & -17.15 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & -53.9 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ -1 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The feedback and feedforward gains K_c, E_c for the standard LQR control problem are

$$K_c = (3.1623, 2.8864, -14.9723, -4.3837), \tag{50}$$

$$E_c = 3.1623, \tag{51}$$

for the constant reference $r = 1$, and where the design parameters in the LQR cost function are $Q = 10I$, $R = 1$.

For all cases $\beta = 0.5$ is assumed. In the first simulation test, the sampling period was taken as $T = 0.02$ s, then the digital control gains of the proposed method are obtained

$$K_d = (2.9822, 2.6388, -12.2563, -4.2663), \tag{52}$$

$$E_d = 2.9822, \tag{53}$$

for the step-invariant transformation,

$$K_d = (5.9714, 5.2869, -24.5851, -8.5237), \tag{54}$$

$$E_d = 2.9857, \tag{55}$$

for the bilinear transformation, and

$$K_d = (3.9692, 3.5119, -16.3123, -5.6794), \tag{56}$$

$$E_d = 3.9692, \tag{57}$$

for the fractional order hold transformation.

Figure 1 shows the errors in between original $y_c(kT)$ and redesigned $y_d(kT)$ system step responses for step-invariant, bilinear and fractional order hold transformations and $T = 0.02$ s plotted in discrete time instants.

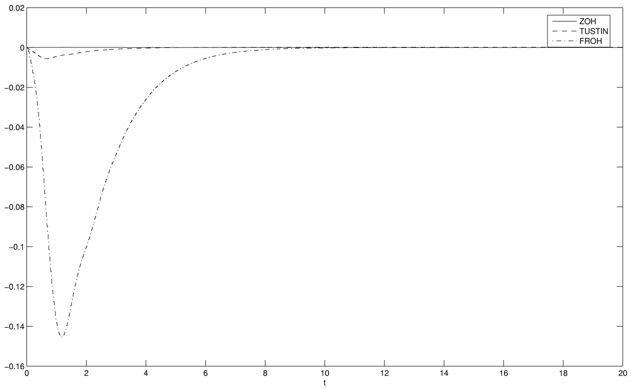

Fig. 1. Errors in step responses between original and redesigned control systems for $T = 0.02$ s

In the second simulation test, the sampling period was taken as $T = 0.2$ s, then the digital control gains of the proposed method are obtained

$$K_d = (1.6595, 0.8028, 10.1263, -2.3909), \quad (58)$$

$$E_d = 1.6595, \quad (59)$$

for the step-invariant transformation,

$$K_d = (4.1747, 2.8683, 4.2287, -5.3643), \quad (60)$$

$$E_d = 2.0996, \quad (61)$$

for the bilinear transformation, and

$$K_d = (2.1564, 1.0975, 11.9938, -2.9455), \quad (62)$$

$$E_d = 2.1564, \quad (63)$$

for the fractional order hold transformation. Figure 2 shows the corresponding errors in step responses.

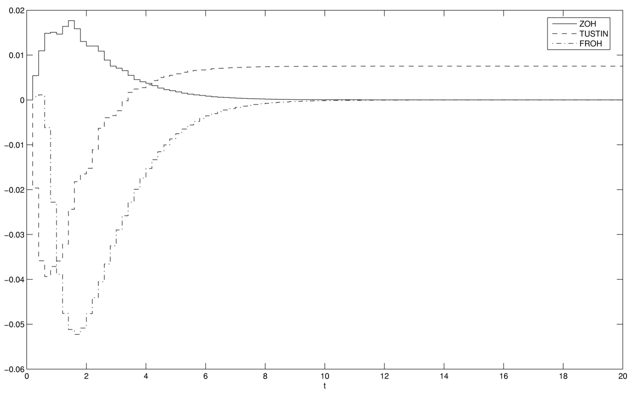

Fig. 2. Errors in step responses between original and redesigned control systems for $T = 0.2$ s

In Fig. 3, the plot of matching error δ versus sampling period T is shown for all transformations, where

$$\delta = \frac{1}{N} \sum_{k=1}^{k=N} |y_c(kT) - y_d(kT)|.$$

It can be observed that for small T the performance of step-invariant transformation is the best, and for $T \gg 0.7$ s the closed-loop system tends to instability for all transformations.

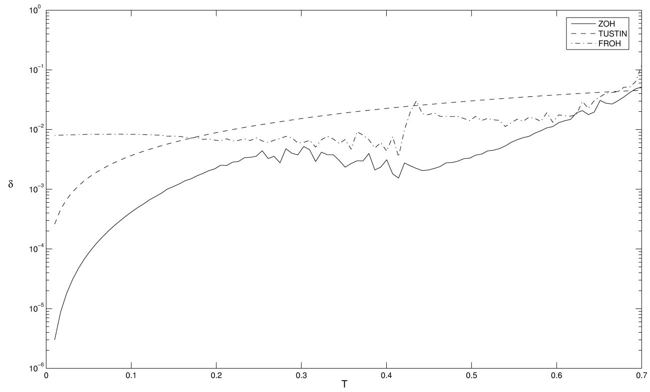

Fig. 3. Plot of matching error δ versus sampling period T

In the second example, the chemical reactor for waste management was taken for consideration. The model was linearized around the equilibrium point yielding the following matrices

$$A = \begin{bmatrix} -0.0752 & -0.0671 & 0 \\ -0.1346 & -0.1421 & 0 \\ 0.1346 & 0.1343 & -0.0079 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0.0052 \\ 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The feedback and feedforward gains K_c, E_c for the standard LQR control problem are

$$K_c = (32.1061, 15.5303, 37.1927), \quad (64)$$

$$E_c = 32.1061, \quad (65)$$

for the constant reference $r = 1$, and where the design parameters in the LQR cost function are

$$Q = \begin{bmatrix} 0 & 0 & 21.8 \\ 0 & 0 & 10 \\ 0 & 0 & 15.6 \end{bmatrix}, \quad R = 0.01.$$

The weights Q and R are chosen such that the control signal (flow rate) that is applied to the actuator stays within the given range. The continuous-time control system has been discretized for $T = 1$ s. then the digital control gains are obtained by the proposed method

$$K_d = (31.0169, 15.1827, 35.5270), \quad (66)$$

$$E_d = 30.7931, \quad (67)$$

for the step-invariant transformation,

$$K_d = (62.0733, 30.3620, 71.1505), \quad (68)$$

$$E_d = 30.8367, \quad (69)$$

for the bilinear transformation, and

$$K_d = (41.0075, 20.1313, 46.9949), \quad (70)$$

$$E_d = 39.9564, \quad (71)$$

for the fractional order hold transformation.

Figure 4 shows the errors in step responses between the original and redesigned systems for step-invariant, bilinear and fractional order hold transformations. The plot of matching error δ versus sampling period T is shown in Fig. 5 for all transformations. It is worthy to note that in the large range of sampling period the redesigned closed-loop control systems remain stable.

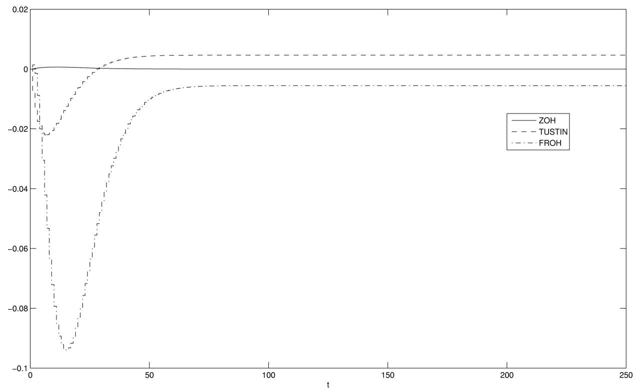

Fig. 4. Errors in step responses between original and redesigned control systems for $T = 1.0$ s

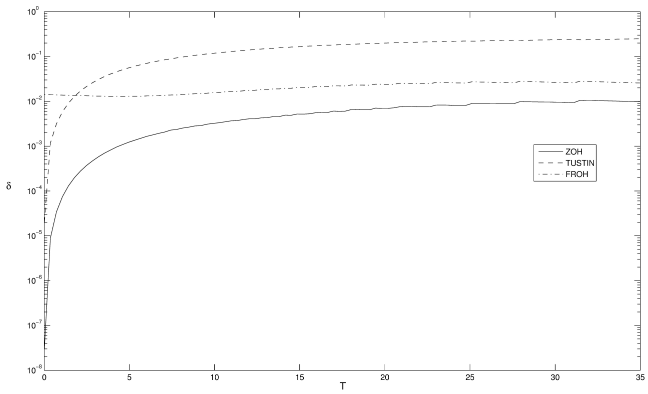

Fig. 5. Plot of matching error δ versus sampling period T

5. Conclusions

This paper presents an approach toward a digital redesign of linear time-invariant systems when step-invariant, bilinear and fractional order hold discretization methods are used. The comparative simulation examples of fourth-order unstable system and third-order stable model of chemical reactor are given for the standard LQR control problem.

To see the differences between methods more observable, the matching error between the unit step response of the original system and redesigned digital systems was calculated. In this respect, the performance for all transformations is comparable, however for small values of sampling period the performance of redesigned control system with ZOH discretization is superior to that with Tustin or FROH discretizations.

The proposed method ensures the stability of the all redesigned control systems for large range of sampling periods.

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