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Optimal real-time control for dynamical systems under uncertainty

R. GABASOV[1] and F.M. KIRILLOVA[2*]

[1]Belorussian State University, 4 Nezalezhnasty Ave, Minsk, Belarus
[2]Institute of Mathematics, National Academy of Sciences of Belarus, 11 Surganov St., Minsk, Belarus

Abstract. The synthesis problem for optimal control systems in the class of discrete controls is under consideration. The problem is investigated by reducing to a linear programming (LP) problem with consequent use of a dynamic version of the adaptive method of LP. Both perfect and imperfect information on behavior of control system cases are studied. Algorithms for the optimal controller, optimal estimators are described. Results are illustrated by examples.

Key words: optimal synthesis problem, open-loop control, uncertainty, feedback, on-line control.

1. Introduction

The first problems of optimal control stated by engineers in 40's of the last century were aimed at the synthesis of closed-loop control systems with optimal feedbacks. These problems were investigated for determined mathematical models under the assumption that exact values of state variables had been known in the course of control processes. In addition, in optimal control problems constraints on controls were of a great importance. At present, optimal control theory reached a high level of development including series of outstanding results among which Pontryagin's maximum principle and Bellman's dynamic programming are the most important and generally recognized. However, the problem of synthesis of optimal systems which served as the initial impulse to origin of optimal control theory has remained not completely investigated. The suitable for synthesis of optimal feedbacks dynamic programming encounters enormous computational difficulties at synthesizing optimal systems of high order due to the known "the curse of dimensionality" phenomenon. To avoid "the curse of dimensionality" at constructing optimal feedbacks under unknown but bounded initial states and disturbances we use the on-line control principle. As a result, in every particular process of control an implementation of optimal feedbacks can be constructed in real time.

Automatic control in real time (on-line control) represents one of the results of modern scientific and technological revolution. This principle is turned out to be an effective supplement to the classical closed-loop control principle. Unlike the latter, the on-line control principle does not demand feedbacks to be synthesized before a control process, it uses an idea of calculating feedbacks current values in the course of control processes.

In the paper results on constructing implementations of optimal state and output feedbacks in real time are presented. Under discussion are: open-loop and closed-loop solutions of optimal control problems, linear extremal problems accompanying observation processes, optimal on-line controls.

2. Optimal open-loop solutions

Consider a linear optimal control problem

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(t_*) = x_0, \quad (1)$$
$$Hx(t^*) = g, \quad |u(t)| \leq 1, \quad t \in T = [t_*, t^*].$$

Here $x = x(t) \in R^n$ is a state vector of (1), $u = u(t) \in R$ is a control function, $g \in R^m$, $\mathrm{rank} H = m < n$; $A(t)$, $b(t)$, $t \in T$, are given piecewise continuous matrix and vector function; t_*, t^*, c, x_0 are fixed. Problem (1) is the simplest problem of optimal control theory but it is nontrivial and includes principal elements of general optimal control problems.

At first, we consider the problem of constructing optimal open-loop solutions in the class of discrete controls. Let us give standard notions.

Definition 1. A function $u(t)$, $t \in T$, is said to be discrete (with a given quantization period $h = (t^* - t_*)/N$, N is a positive integer) if it has the form $u(t) = u(t_* + kh)$, $\quad t \in [t_* + kh, t_* + (k+1)h[$, $\quad k = \overline{0, N-1}$.

Definition 2. A discrete control $u(t)$, $t \in T$, is called admissible if it together with a trajectory $x(t)$, $t \in T$, of system (1) satisfies the constraint $|u(t)| \leq 1$, $\quad t \in T$, and the terminal condition $x(t^*) \in X^* = \{x \in R^n : Hx = g\}$.

Definition 3. An admissible control $u^0(t)$, $t \in T$, is said to be an optimal open-loop control of problem (1) if the corresponding trajectory $x^0(t)$, $t \in T$, satisfies the equality $c'x^0(t^*) = \max c'x(t^*)$ where the maximum is taken over all admissible controls.

*e-mail: kirill@nsys.minsk.by

Let $\tau \in T_h = \{t_*, t_*+h, \ldots, t^*-h\}$, $z \in R^n$. We imbed problem (1) into the family of problems

$$c'x(t^*) \to \max, \quad \dot{x} = A(t)x + b(t)u, \quad x(\tau) = z, \qquad (2)$$
$$Hx(t^*) = g, \quad |u(t)| \le 1, \quad t \in T(\tau) = [\tau, t^*],$$

depending on the position (τ, z). Let $u^0(t|\tau, z)$, $t \in T(\tau)$, be an optimal open-loop control of problem (2), X_τ be a set of $z \in R^n$ for which problem (2) has solutions at fixed $\tau \in T_h$.

DEFINITION 4. A function

$$u^0(\tau, z) = u^0(\tau|\tau, z), \quad z \in X_\tau, \quad \tau \in T_h, \qquad (3)$$

is said to be an optimal (discrete) control of feedback type to problem (1) (the positional solution).

Close system (1) by feedback (3) and consider its behaviour under a piecewise continuous disturbance $w(t)$, $t \in T$:

$$\dot{x} = A(t)x + b(t)u^0(t, x) + w(t), \quad x(t_*) = x_0. \qquad (4)$$

We define a trajectory $x(t)$, $t \in T$, of (4) as a continuous solution of the linear equation $\dot{x} = A(t)x + b(t)u^*(t) + w(t)$, $x(t_*) = x_0$, where $u^*(t) = \underline{u^0(t_* + kh, x(t_* + kh))}$, $t \in [t_* + kh, t_* + (k+1)h[$, $k = \overline{0, N-1}$.

The classical statement of the optimal synthesis problem assumes the construction of function (3) in an explicit form and before the process starting. To pass to on-line control we consider a particular process and assume that optimal feedback (3) has been constructed. Consider the behaviour of closed system (4) with an unknown piecewise continuous disturbance $w^*(t)$, $t \in T$. Let $x^*(t)$, $t \in T$, be the corresponding trajectory of the closed system

$$\dot{x}^*(t) = A(t)x^*(t) + b(t)u^0(t, x^*(t)) + w^*(t), \qquad (5)$$
$$t \in T, \quad x(t_*) = x_0^*.$$

In the course of process the signals $u^*(t) = u^0(t, x^*(t))$, $t \in T$, are fed to the input of the control system. As it is obvious from (5), there is no need to calculate optimal feedback (3) in all the domain of definition, it is sufficient to obtain its values along the isolated trajectory $x^*(t)$, $t \in T$. Moreover, it is sufficient at each current instant $\tau \in T_h$ to calculate only its current values $u^*(\tau) = u^0(\tau, x^*(\tau))$ for the time which does not exceed h (in real-time mode). We call these signals an implementation of the optimal feedback.

A device which is able to fulfill this work is called Optimal Controller (OC).

Thus, the optimal synthesis problem is reduced to constructing an algorithm for OC. In the class of discrete controls the problem (1) is equivalent to the linear programming problem

$$\sum_{t \in T_h} c(t)u(t) \to \max, \qquad (6)$$
$$\sum_{t \in T_h} d(t)u(t) = g_0, \quad |u(t)| \le 1, \quad t \in T_h.$$

Here

$$c(t) = \int_t^{t+h} \psi_c'(\vartheta)b(\vartheta)d\vartheta, \quad d(t) = \int_t^{t+h} G(\vartheta)b(\vartheta)d\vartheta,$$
$$g_0 = g - Hx_0(t^*),$$

$x_0(t)$, $t \in T$, is a trajectory of (1) with $u(t) = 0$, $t \in T$; $\psi_c(t)$, $t \in T$, is a solution to the adjoint equation

$$\dot{\psi} = -A'(t)\psi, \quad \psi(t^*) = c; \qquad (7)$$

$G(t)$, $t \in T$, is a given $m \times n$-matrix function

$$\dot{G} = -GA(t), \quad G(t^*) = H.$$

At small h the matrix of the condition of problem (6) $(d(t), t \in T_h)$ is large and has a high density. For that reason traditional methods of LP for (6) are not quite effective if h is small. To solve problem (6) a dynamic version of the adaptive method was elaborated [1]. A new parameterization of the optimal open-loop controls (optimal programs) is the base of the dynamic version of the adaptive method. If in Pontryagin's maximum principle an optimal control problem is parameterized by an initial state of the adjoint system then in the suggested approach switching points of optimal programs are chosen as parameters. At use of the classical parameterization it is necessary to integrate primal and adjoint systems on the whole inteval of control. A new parametrization allows to do with integrating the adjoint systems (7) on small time interval where switching points are shifted. As in optimal control to dynamic systems the main time expense is connected with the integration of differential equations, then the suggested method of the solution of (6) quickly obtains the switching points of a new optimal program $u^0(t|\tau, x^*(\tau))$, $t \in T(\tau)$, using the information on switching points of the optimal program $u^0(t|\tau - h, x^*(\tau - h))$, $t \in T(\tau - h)$.

The main tool of the method is a base T_b. This is a set $\{t_1, \ldots, t_m\}$ consisting of m moments of the discrete interval T_h.

DEFINITION 5. A set T_b is said to be a base if the matrix $D_b = \Big(d(t), t \in T_b\Big)$ is nonsingular. The matrix D_b is called the basic matrix.

The basic matrix can be constructed using one integration of primal (1) or adjoint (7) systems if m processors are used. Every base T_b is accompanied by

1) the vector of Lagrange's multipliers $\nu = \nu(I)$, $I = \{1, 2, \ldots, m\}$ that is a solution of the equation $\nu' D_b = c_b'$, $c_b = (c(t), t \in T_b)$,
2) co-trajectory $\psi(t)$, $t \in T$, that is a solution of

$$\dot{\psi} = -A'(t)\psi, \quad \psi(t^*) = c - H'\nu,$$

3) co-control

$$\Delta(t) = \int_t^{t+h} \psi'(\vartheta)b(\vartheta)d\vartheta, \quad t \in T_h,$$

4) pseudocontrol $\omega(t)$, $t \in T$. Nonbase values $\omega(t)$, $t \in T_{nonb} = T_h \backslash T_b$, are defined as

$$\omega(t) = -1 \text{ at } \Delta(t) < 0; \quad \omega(t) = 1 \text{ at } \Delta(t) > 0;$$

$$\omega(t) \in [-1,1] \text{ at } \Delta(t) = 0, \quad t \in T_{nonb}.$$

Base values $\omega(t)$, $t \in T_b$, of the pseudocontrol satisfy the equation

$$\sum_{t\in T_b} d(t)\omega(t) + \sum_{t\in T_{nonb}} d(t)\omega(t) = g_0.$$

5) quasicontrol $\tilde{\omega}(t)$, $t \in T$: $\tilde{\omega}(t) = \omega(t)$, $t \in T_{nonb}$; $\tilde{\omega}(t) =$sat $\omega(t)$, $t \in T_b$.

Any quasicontrol $\tilde{\omega}(t)$, $t \in T$, produces a discrepancy of endpoint constraints $\tilde{g}(T_b) = \|g - H\tilde{x}(t^*)\|$ where $\tilde{x}(t)$, $t \in T$, is a trajectory of system (1) generated by $\tilde{\omega}(t)$, $t \in T$.

A base T_b is said to be regular if $\Delta(t) \neq 0$, $t \in T_{nonb}$. A base T_b is called optimal if it is accompanied by some pseudocontrol which satisfies the condition

$$|\omega(t)| \leq 1, \quad t \in T_b. \tag{8}$$

In case (8), $\omega(t)$, $t \in T$, is the optimal open-loop control.

On the base of introduced concepts several methods of constructing optimal open-loop controls for problem (1) were justified [1 – 4].

Example 1. Open-loop controls. Consider the optimal control problem

$$\int_0^{25} u(t)dt \longrightarrow \min, 0 \leq u(t) \leq 1, \ t \in [0, 25[,$$

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4,$$
$$\dot{x}_3 = -x_1 + x_2 + u, \quad \dot{x}_4 = 0.1x_1 - 1.02x_2 \tag{9}$$
$$x_1(0) = x_2(0) = 0, \ x_3(0) = 2, \ x_4(0) = 1,$$
$$x_1(25) = x_2(25) = x_3(25) = x_4(25) = 0.$$

If we interpret $u(t)$ as a fuel consumption per second, then problem (9) is to damp oscillations of a two-mass system with the minimal fuel consumption. The problem (9) was solved with the quantization period $h = 25/1000 = 0.025$. As an initial base, a set $T_b = \{5, 10, 15, 20\}$ was taken.

The problem of effectiveness of methods is principal in constructive theory of optimal control. It is not always reasonable to estimate the effectiveness of methods by the amount of iterations because different methods consist of different types of iterations. It is more natural to use as the measure of effectiveness the quantity of full integrations of the primal and the adjoint systems which are used to construct the optimal solution [5]. In the case, as a unit of complexity, one can take one integrations on the whole control interval T.

In problem (9) the complexity of the algorithm turned out to be equal to 2.41 (one integration was used to construct D_b, one to test optimality conditions and only 0.43 of the complete integration to obtain the optimal open-loop control).

3. Optimal positional solutions

Suppose that OC has been acting during the time $\{t_*, t_* + h, \ldots, \tau\}$ producing control signals $u^*(t_*)$, $u^*(t_*+h)$, $\ldots$, $u^*(\tau)$. These signals and realized disturbances $w^*(t_*)$, $w^*(t_* + h)$, $\ldots$, $w^*(\tau)$ transfer the dynamical system at $\tau + h$ in the state $x^*(\tau + h)$. The task of OC at $\tau + h$ is to calculate a current signal $u^*(\tau+h) = u^0(\tau+h, x^*(\tau+h))$. By assumption, at the previous moment τ OC calculated $u^*(\tau)$, i.e. it solved the problem

$$\sum_{t\in T_h(\tau)} c(t)u(t) \to \max, \tag{10}$$

$$\sum_{t\in T_h(\tau)} d(t)u(t) = g - G(\tau)x^*(\tau),$$

$$|u(t)| \leq 1, \ t \in T_h(\tau),$$

and it knows the optimal base $T_b^0(\tau)$. Problem (10) at the initial moment $\tau = t_*$ coincides with (1) and can be solved by OC before the beginning of the real control process to get $T_b^0(t_*)$, $u^*(t_*)$. The current state $x^*(\tau + h)$ is connected with the previous state $x^*(\tau)$ by the Cauchy formula

$$x^*(\tau + h) = F(\tau + h)F^{-1}(\tau)x^*(\tau)+$$
$$\int_\tau^{\tau+h} F(\tau + h)F^{-1}(t)b(t)dtu^*(\tau)+$$
$$+\int_\tau^{\tau+h} F(\tau + h)F^{-1}(t)w^*(t)dt.$$

Here $F(t), t > t_*$, is fundamental matrix of (1), $u(t) \equiv 0$. Because the value $w^*(\tau)$ is bounded, the difference between two states $x^*(\tau)$ and $x^*(\tau + h)$ is small. Therefore the problem

$$\sum_{t\in T_h(\tau+h)} c(t)u(t) \to \max,$$

$$\sum_{t\in T_h(\tau+h)} d(t)u(t) = g - G(\tau + h)x^*(\tau + h),$$

$$|u(t)| \leq 1, \ t \in T_h(\tau + h),$$

that OC has to solve at $\tau + h$ differs from (10) a little if h is small. At such situation the dual method under consideration [1] is very effective. OC uses $T_b^0(\tau)$ as an initial base $T_b(\tau + h)$, constructs the optimal base for $T_b^0(\tau + h)$ and calculates $u^*(\tau + h)$.

Example 2. Positional control. Let the perturbed system (9) be closed by the optimal feedback $u^* = u^0(x_1, \ldots, x_n)$,

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -x_1 + x_2 + u, \tag{11}$$
$$\dot{x}_4 = 0.1x_1 - 1.02x_2 + w$$

where the unknown for OC disturbance has the form

$$w^*(t) = 0.3 \sin 4t, \ t \in [0, 9.75[; \quad w^*(t) \equiv 0, \ t \geq 9.75.$$

In this example the complexity of the correction of current bases by the dual method does not exceed 0.02. It means that the computer calculated $u^*(\tau)$ for the time less than 0.02α where α is the computer time for one integration of system (11) on T.

4. Optimal on-line control for dynamical system under uncertainty

On the interval $T = [t_*, t^*]$ consider the system

$$\dot{x} = A(t)x + B(t)u + M(t)w. \quad (12)$$

Here $A(t) \in R^{n\times n}$, $B(t) \in R^{n\times r}$, $M(t) \in R^{n\times n_w}$, $t \in T$, are given piecewise continuous matrix functions; $x = x(t) \in R^n$ is a state of the control system at an instant t; $u = u(t) \in U \subset R^r$ is a value of a discrete control with a quantization period h: $u(t) \equiv u(\tau), t \in [\tau, \tau+h[, \tau \in T_h = \{t_*, t_*+h, \ldots, t^*-h\}$, $(h = (t^*-t_*)/N$, $(N > 0)$, $U = \{u \in R^r : u_* \leq u \leq u^*\}$ is a bounded set ; $w = w(t)$, $t \in T$, is a disturbance function $w(t) = \Lambda(t)v, t \in T$, where $\Lambda(t) \in R^{n_w \times n_v}$ is a given piecewise continuous matrix function; $v \in R^{n_v}$ is a vector of disturbance parameters taking values from a bounded set $V = \{v \in R^{n_v} : w_* \leq v \leq w^*\}$.

Assume that the initial state $x(t_*)$ of system (12) is an unknown element of a bonded set $X_0 \subset R^n$. Let

$$X_0 = x_0 + GZ,$$

where $x_0 \in R^n$, $G \in R^{n\times n_z}$ are an known vector and a matrix; $Z = \{z \in R^{n_z} :\ d_* \leq z \leq d^*\}$ is a bounded set of unknown parameters z of the initial state $x(t_*).(d_*, d^*$ are given)

Let the measurement device be of the form

$$y(\theta) = \int_{\theta-h}^{\theta} C(t)x(t)dt + R(\theta)\xi(\theta),\ \theta \in T_h \setminus t_*, \quad (13)$$

where $C(t) \in R^{q\times n}$, $t \in T$, is a given piecewise continuous matrix function; $R(\theta) \in R^{m\times n_\xi}$, $\theta \in T_h \setminus t_*$; $y(\theta) \in R^q$, $\theta \in T_h \setminus t_*$, is an output signal of (16) ; $\xi(\theta) \in R^{n_\xi}$, $\theta \in T_h \setminus t_*$, are measurement errors satisfying

$$\xi_* \leq \xi(\theta) \leq \xi^*, \theta \in T_h \setminus t_*; \quad 0 < \|\xi^* - \xi_*\| < \infty.$$

Discrete closed-loop control of system (12) is performed in the following way. On the interval $[t_*, t_*+h[$ the control function $u(t) \equiv u(t_*)$, $t \in [t_*, t_*+h[$, is fed to the input of system (12) where $u(t_*) \in U$ is chosen upon a priori information. At the instant $\tau = t_*+h$ measurement device (13) obtains the first signal $y(t_*+h)$, generated by the realized initial state $x(t_*)$, the error $\xi(t_*)$ and the disturbance $w(t)$, $t \in [t_*, t_*+h[$. Using the signal $y(t_*+h)$, a vector $u(t_*+h) = u(t_*+h,\ y(t_*+h)) \in U$ is chosen following the rules selected in advance (before the process starting). The control function $u(t) \equiv u(t_*+h)$ is fed into system (12) for $t \in [t_*+h, t_*+2h[$. This control and the realized disturbance $w(t)$, $t \in [t_*+h, t_*+2h[$, transfer the system into the state $x(t_*+2h)$, and together with the error $\xi(t_*+2h)$ generate the output signal $y(t_*+2h)$. At arbitrary moment $\tau \in T_h \setminus t_*$, based on the measured output signal $y(\tau)$, a vector $u(\tau) = u(\tau, y_\tau(\cdot)) \in U$ is chosen and the control function $u(t) \equiv u(\tau)$, $t \in [\tau, \tau+h[$, is fed into the control system.

Here $y_\tau(\cdot) = (y(\theta),\ \theta \in T_h(\tau))$, $T_h(\tau) = \{t_*+h, t_*+2h, \ldots, \tau\}$.

Let Y_τ be a set of all output signals $y_\tau(\cdot)$ of (13) that can be obtained by the moment τ.

DEFINITION 6. A functional

$$u = u(\tau, y_\tau(\cdot)),\ y_\tau(\cdot) \in Y_\tau,\ \tau \in T_h \setminus t_*, \quad (14)$$

and the control functions $u(t, y_t(\cdot)) \equiv u(\tau, y_\tau(\cdot)) \in U$, $t \in [\tau, \tau+h[$, $\tau \in T_h \setminus t_*$, generated by (14), are said to be a feedback for system (12) under uncertainty.

Let $X(t^*|u_{t^*}(\cdot, y_{t^*}(\cdot)))$ be a set of all terminal states of the closed system

$$\dot{x} = A(t)x + B(t)u(t_*) + M(t)w,\ t \in [t_*, t_*+h[;$$

$$\dot{x} = A(t)x + B(t)u(t, y_t(\cdot)) + M(t)w,\ t \in [t_*+h, t^*];$$

with all possible initial states $x(t_*)$, disturbances $w(t)$, $t \in T$, and errors $\xi(\tau)$, $\tau \in T_h \setminus t_*$, able to generate the signal $y_{t^*}(\cdot)$.

Introduce a terminal set $X^* = \{x \in R^n : g_* \leq Hx \leq g^*\}$, where $H \in R^{m\times n}$, $g_* < g^*$ are given. Feedback control (14) is called admissible if $X(t^*|u_{t^*}(\cdot|y_{t^*}(\cdot)) \subset X^*$. Evaluate the quality of admissible control by the functional

$$J(u) = \min c'x, x \in X(t^*|u_{t^*}(\cdot|y_{t^*}(\cdot)) \quad (c \in R^n).$$

DEFINITION 7. An admissible feedback $u^0(\tau, y_\tau(\cdot))$, $y_\tau(\cdot) \in Y_\tau$, $\tau \in T_h \setminus t_*$, is said to be optimal if $J(u^0) = \max J(u)$, where the maximum is calculated over all admissible feedbacks (14).

According to the definition, the introduced optimal feedback provides the best result under the worst conditions (optimal guaranteed feedback).

4.1. Optimal on-line control. Now we describe the optimal on-line control principle for a concrete control process where a signal $y^*(\theta)$, $\theta \in T_h$, would be realized. The control process starts at the moment $\tau = t_*$ with the control $u^{**}(t) = u^0(t_*)$, $t \geq t_*$, where $u^0(t)$, $t \in T$, is an optimal open-loop control constructed on the a priori information. At the instant $\tau = t_*+h$ the measurement signal $y^*(t_*+h)$ is obtained. Using it, a control signal $u^*(t_*+h)$ is calculated in time $s(t_*+h) < h$. The control function $u^{**}(t) = u^0(t_*)$ is fed into the control system on the interval $[t_*+h,\ t_*+h+s(t_*+h)[$. Starting from the moment $t_*+h+s(t_*+h)$, the control function switches on $u^{**}(t) = u^*(t_*+h)$.

At an arbitrary τ the control function

$$u^{**}(t) = u^0(t_*),\ t \in [t_*, t_*+h+s(t_*+h)[;$$

$$u^{**}(t) = u^*(\vartheta),\ t \in [\vartheta+s(\vartheta), \vartheta+h+s(\vartheta+h)[,$$

$$\vartheta \in \{t_*+h, t_*+2h, \ldots, \tau-2h\};$$

$$u^{**}(t) = u^*(\tau-h),\ t \in [\tau-h+s(\tau-h), \tau[;$$

has been fed into the input of (12) and a current measurement $y^*(\tau)$ is obtained. The calculation of the control signal $u^*(\tau) = u^{00}(\tau, y^*_\tau(\cdot))$ is required to be made in time $s(\tau) < h$. Before it is calculated the previous signal $u^*(\tau-h)$ is fed into the system.

To describe the rules for the calculation of $u^*(\tau)$, we will present the signal $y^*(\tau)$ in the form

$$y(\tau) = \int_{\tau-h}^{\tau} C(t)(x_w(t) + x_u(t))dt + R(\tau)\xi(\tau),$$

where $x_w(t)$, $t \in [t_*, \tau]$, is a trajectory for

$$\dot{x} = A(t)x + M(t)w, \; x(t_*) = Gz, \tag{15}$$

$x_u(t)$, $t \in [t_*, \tau]$, is a trajectory for

$$\dot{x} = A(t)x + B(t)u, \; x(t_*) = x_0, \tag{16}$$

with $u(t) \equiv u^{**}(t), t \in [t_*, \tau[$. Omitting the known part of the trajectory $x_u(t)$, $t \in [\tau - h, \tau]$, from the signal $y^*(\tau)$, we obtain

$$y_0^*(\tau) = y^*(\tau) - \int_{\tau-h}^{\tau} C(t)x_u(t)dt.$$

Thus, the signal $y_{0\tau}^*(\cdot) = (y_0^*(\theta), \theta \in T_h(\tau)\backslash t_*)$ is available by τ. It coincides with the signal that would be obtained by measurement device (13) for (15) and represents additional information about the parameter vector realized in the process. This information is contained in a posteriori distribution.

A set $\hat{\Gamma}(\tau) = \hat{\Gamma}(\tau; y_{0\tau}^*(\cdot))$ is called a posteriori distribution of parameters (z, v) if and only if it consists of vectors $\gamma = (z, v) \in \Gamma = Z \times V$, to which there correspond the initial state $x(t_*) = Gz$ of (15) and the disturbance $w(t) = \Lambda(t)v$, $t \in [t_*, \tau[$, able together with some errors $\xi(\theta)$, $\theta \in T_h(\tau)$, to generate $y_{0\tau}^*(\cdot)$.

A control $u^\tau(\cdot) = (u(t), t \in [\tau, t^*])$ is called an admissible open-loop control if for every $\gamma \in \hat{\Gamma}(\tau)$ at the moment t^* it together with $u^{**}(t)$, $t \in [t_*, \tau[$, transfer system (12) to X^*, i.e.

$$g_{*i} \le \min h'_{(i)}(x_w(t^*) + x_u(t^*)); \tag{17}$$
$$\max h'_{(i)}(x_w(t^*) + x_u(t^*)) \le g_i^*; \; i = \overline{1, m};$$

where $h_{(i)}$ is the i-th row of the matrix H, g_i^*, g_{*i} are the i-th components of g^*, g_*; $x_u(t^*)$ is a terminal state of (16) under $u(t) = u^\tau(t)$, $t \in [\tau, t^*[$. Let $\hat{X}_w^*(\tau)$ be a set of all terminal states $x_w(t^*)$ of (15) generated by $(z, v) \in \hat{\Gamma}(\tau)$. The problems arising in (17)

$$\chi_i^*(\tau) = \max h'_{(i)}x, \; x \in \hat{X}_w^*(\tau), \; i = \overline{1, m}; \tag{18}$$
$$\chi_{*i}(\tau) = \min h'_{(i)}x, \; x \in \hat{X}_w^*(\tau), \; i = \overline{1, m};$$

are called extremal problems accompanying the optimal control problem under uncertainty.

Thus, for the control $u^\tau(\cdot)$ to be admissible for $(\tau, y_{0\tau}^*(\cdot))$ it is necessary and sufficient that at $\tau = t^*$ it moves determined system (16) with the initial condition $x(\tau) = x_u(\tau)$ to the set $X^*(\tau) = \{x \in R^n : g_*(\tau) \le Hx \le g^*(\tau)\}$, where $g_*(\tau) = g_* - \chi_*(\tau)$, $g^*(\tau) = g^* - \chi^*(\tau)$. Let the quality of the admissible control $u^\tau(\cdot)$ is evaluated by $I(u) = \min c'x(t^*)$, $\gamma \in \hat{\Gamma}(\tau)$. Then the optimal open-loop control $u^{\tau 0}(\cdot) = u^0(t|\tau, y_\tau^*(\cdot))$, $t \in [\tau, t^*]$, is a solution to the problem

$$c'x(t^*) \to \max, \dot{x} = A(t)x + B(t)u, \; x(\tau) = x_u(\tau), \tag{19}$$
$$x(t^*) \in X^*(\tau), \; u(t) \in U, t \in [\tau, t^*].$$

We call (19) a determined problem of optimal control accompanying the optimal control problem under uncertainty. Let $u^*(\tau) = u^0(\tau|\tau, y_\tau^*(\cdot))$. On $[\tau, \tau + h[$ the following control function is fed into the input of (12): $u^{**}(t) = u^*(\tau - h)$, $t \in [\tau, \tau + s(\tau)[; u^{**}(t) = u^*(\tau)$, $t \in [\tau + s(\tau), \tau + h[$.

The optimal open-loop control $u^0(t), t \in T$, introduced above, is a solution to

$$c'x(t^*) \to \max, \dot{x} = A(t)x + B(t)u, \; x(t_*) = x_0,$$
$$g_* - \chi_{*i}(t_*) \le Hx(t^*)g^* - \chi_i^*(t_*);$$

where

$$\chi_{*i}(t_*) = \min h'_{(i)}x, \; x \in X_w^*(t_*), i = \overline{1, m};$$
$$\chi_i^*(t_*) = \max h'_{(i)}x, \; x \in X_w^*(t_*), i = \overline{1, m};$$

$X_w^*(t_*)$ is a set of all terminal states $x_w(t^*)$ of system (15) for all possible parameters $(z, v) \in \Gamma$.

According to the scheme presented, to construct a control signal $u^*(\tau)$ one has to solve:
1) 2m accompanying extremal problems (18);
2) one determined problem of optimal control (19).

A device solving the accompanying extremal problems is called Optimal Estimator (OE). If the time $s(\tau)$ needed to OE and OC to solve problems (18) and (19) is less than h, then we say that they are suitable for optimal on-line control for the system under uncertainty.

Consider the problem

$$\chi_*(\tau) = \min p'x, \; x \in \hat{X}_w^*(\tau), \tag{20}$$

which includes accompanying extremal problems (18). Problem (20) is equivalent to the linear programming problem

$$p'_z z + p'_v v \to \max, \xi_* \le y_0^*(\theta) - D(\theta)z - H(\theta)v \le \xi^*, \tag{21}$$
$$\theta \in T_h(\tau), d_* \le z \le d^*, \quad w_* \le v \le w^*;$$

where $p'_z = p'F(t^*)$, $p'_v = p'P(t^*)$,

$$D(\theta) = \int_{\theta-h}^{\theta} C(t)F(t)dt, \; H(t) = \int_{\theta-h}^{\theta} C(t)P(t)dt;$$

$$F(t), t \in T : \dot{F} = A(t)F, \; F(t_*) = G; P(t), t \in T :$$
$$\dot{P} = A(t)P + M(t)\Lambda(t), P(t_*) = 0.$$

Problem (21) has $(\tau - t_*)/h + 1$ general constraints and $n_z + n_v$ variables. Taking into account that number of the general constraints tends to infinity at $h \to 0$, one can call problem (21) a semi-large extremal problem. The algorithm for the suitable OE is based on the dual adaptive method of LP. The main operations of the method follow the scheme proposed in [6].

OC and OE can work in parallel when before the control process starting, they solved problems (18), (19) using a priori information and preserving the results. At an arbitrary τ they use the information obtained for $\tau - h$. And the problem of on-line control is reduced to the fast cor-

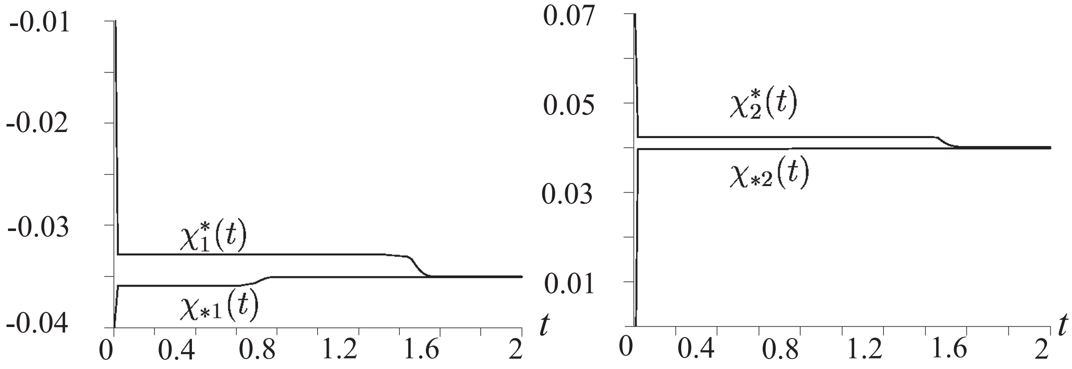

Fig. 1. The linear estimates

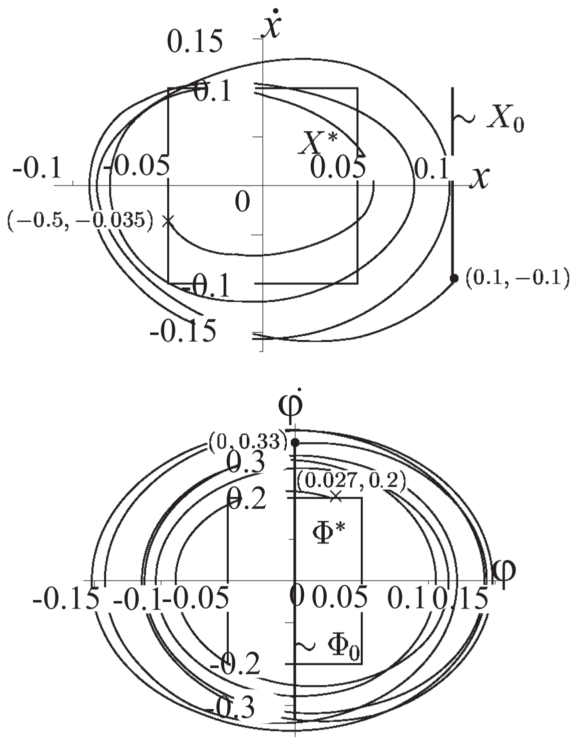

Fig. 2. Projections of optimal trajectories on the planes $x\dot{x}$ and $\phi\dot{\phi}$

rections of the solution of the problems (18), (19). Another way of on-line control consists in calculating $\chi_{*i}(t_*), \chi_i^*(t_*), i \in I$, on $[t_*, t_{*+h}]$ and only after the finishing of these calculations at $t_* + \max s_i(t_*), i \in I$, the optimal controller is involved to the process. As a result in this case OC has a lag of value h with respect to E.

Example 3. Consider the control system which represents a mathematical half-model of the car

$$\ddot{x} = -2.1x + 0.31\varphi - u_1 + u_2 + w_1, \qquad (22)$$
$$\ddot{\varphi} = 0.93x + 6.423\varphi + 1.1u_1 + 0.9u_2 + w_2, t \in T = [0, 15],$$

with $x(0) = 0.1$, $\varphi(0) = 0$ and unknown $\dot{x}(0) = z_1$, $\dot{\varphi}(0) = z_2$: $(z_1, z_2) \in Z = \{z \in Z : |z_1| \leq 0.1, |z_2| \leq 0.33\}$, and disturbances of the form $w_1(t) = v_1 \sin(4t)$, $w_2(t) = v_2 \sin(3t)$, $t \in T$: $(v_1, v_2) \in V = \{v \in R^2 : |v_i| \leq 0.01, i = 1, 2\}$. Let the sensor at moments $t \in T_h = \{0, h, \dots, 15 - h\}$, $h = 0.02$, satisfies

$$y_1 = -x + l_1\varphi + \xi_1, \; y_2 = x + l_2\varphi + \xi_2,$$

where $\xi_i = \xi_i(t)$, $|\xi_i(t)| \leq 0.01$, $t \in T_h$, are bounded errors.

The aim of control is to transfer system (22) at $t^* = 15$ to the sets $X^* = \{x \in R^2 : |x_1| \leq 0.05, |x_2| \leq 0.1\}$; $\Phi^* = \{\varphi \in R^2 : |\varphi_1| \leq 0.05, |\varphi_2| \leq 0.2\}$; $(0 \leq u_i(t) \leq 0.02)$, $i = 1, 2$, $t \in T$; minimizing the functional

$$J(u) = \int_0^{15} (u_1(t) + u_2(t))dt.$$

Let in a concrete control process $z_1^* = -0.1$; $z_2^* = 0.33$; $v_1^* = -0.005; v_2^* = 0.01$; $\xi_1^*(t) = 0.01\cos(2t)$, $\xi_2^*(t) = -0.01\cos(4t), t \in T_h$. The optimal value of the cost function turned out to be equal to 0.104. The complexity of iterations did not exceed 0.043.

The Figure.1 presents the plots for the linear estimates $\chi_{*i}(\tau)$, $\chi_i^*(\tau)$, $i = 1, 2$; $t \in [0, 2]$. When $t > 1.52$ values $\chi_{*i}(\tau)$, $\chi_i^*(\tau)$, $i = 1, 2$, almost coincide. The projections of optimal trajectories on the planes $x\dot{x}$ and $\varphi\dot{\varphi}$ are given in Figure 2.

Remark. The problem of guaranted control of dynamical systems under not finite parametric disturbances with set-membership constraints generates several type of optimal feedbacks. In dependence on the used information unclosable, one-time-closable, multi-closable feedbacks are usually investigated [7].

5. Conclusion

An approach to the synthesis problem of optimal control of feedback type is discussed. Algorithms for controllers which calculate values of optimal feedbacks during each particular control process in real-time are considered. A linear systems with unknown (but bounded) initial states, disturbances and errors of the measurement device are investigated. Linear extremal problems are introduced which generate in real-time estimates used by OC.

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