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# Stochastic dynamics and reliability of degrading systems

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**Abstract.** In this paper the basic methodology of the coupled response-degradation modelling of stochastic dynamical systems is presented along with the effective analysis of selected problems. First, the general formulation of the problems of stochastic dynamics coupled with the evolution of deterioration process is given. Then some specific degrading oscillatory systems under random excitation are analyzed with a special attention on the systems with fatigue-induced stiffness degradation. Both, the general discussion and the analysis of selected exemplary problems indicate how the reliability of deteriorating stochastic dynamical systems can be assessed.

**Key words:** stochastic dynamics, fatigue induced stiffness degradation, hysteretic systems, systems with varying/random structure.

## 1. Introduction

In the last decades the response of linear and non-linear dynamical systems to random excitation (parametric and external) has been studied extensively. These studies, stimulated by application problems of control theory as well as by the analysis of mechanical and structural systems, have resulted in the theory constituting at present the subject of stochastic dynamics, or, in a narrower sense, random vibration theory. The methods elaborated allow to characterize a stochastic response process in a variety of important situations and at the same time they provide the information for the reliability estimation [1–3].

However, as it is known, dynamic excitation of engineering systems (including random varying excitation) causes irreversible changes in the material structure and results in decreasing the system ability to carry the intended loading. Damage caused by vibrations manifests itself primarily in the stiffness degradation of the components and systems. Examples of engineering systems with simultaneous stiffness deterioration are: (i) vibrating mechanical/structural elements with fatigue process taking place in them; (ii) mechanical systems with elasto-plastic properties in which, in addition to elastic deformation, a plastic deformation occurs as well; (iii) systems composed of many elements with varying bearing capacity; global mechanical and reliability characteristics (e.g. stiffness) of such systems depend on the failure of some of the elements.

In the last years an increasing amount of research efforts has been directed towards stochastic modelling of various deterioration (or degradation) processes in mechanical/structural components. Because of the practical importance of fatigue damage and fracture in various engineering structures, stochastic models of fatigue accumulation have been a subject of special interest (cf. Sobczyk and Spencer [4] and references therein). It should be underlined, however that though the fatigue process is inherently associated with vibrations of mechanical/structural systems the research in random vibration theory, and in modeling of fatigue has been conducted without a proper mutual coupling. Stochastic analysis of dynamics of mechanical/structural systems has been focused on the characterization of the response (and its unsafe states, e.g. instability regions, first-passage probabilities), whereas the analysis of fatigue deterioration has been concentrated on the fatigue crack growth analysis assuming that the characteristics of the response (e.g. stresses) are given.

It is clear that a more adequate approach should account for the joint (coupled) treatment of both the system dynamics and deterioration process (e.g. fatigue accumulation). Such an analysis allows to account the effect of stiffness degradation during the vibration process on the response and, at the same time, gives the actual stress values for estimation of fatigue. It seems that in stochastic dynamics the coupled analysis of the response and degradation had been treated first in the context of elasto-plastic (hysteretic) systems [5,6]. In the articles cited a degradation of the system is defined in terms of the hysteretic energy dissipation. As far as the joint analysis of random vibrations and fatigue degradation is concerned, one should mention the paper [7] containing the model in which fatigue crack growth equation is coupled with the equation for the amplitude of the response (obtained via the averaging method – cf. [1,2]), and more extensive studies published in papers [8,9]. The objective of this paper is to expound the basic methodology of the coupled stochastic dynamics-degradation modelling along with concise exposition of the effective analysis of selected problems. First, a general formulation of various problems of stochastic dynamics coupled with the evolution of deterioration process is presented. Afterwards, some specific oscillatory systems subjected to random excitation, such as hysteretic systems and systems with fatigue-induced deterioration of stiffness are considered. The analysis presented also shows how the reliability of deteriorating stochastic dynamical systems can be assessed.

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## 2. Stochastic response-degradation models

**2.1. General formulations.** Stochastic governing equations for many engineering dynamical systems should be represented in the form which accounts for both – the system dynamics and degradation process taking place in the system. In the case of mechanical/structural systems these are, above all, the elastic-plastic vibratory systems (under severe random loadings) in which the restoring force has a hereditary nature (cf. [3,6]) and elastic systems with stiffness degradation due to fatigue damage.

In general, a coupled response-degradation model for non-linear vibratory systems with random excitation (parametric and/or external) can be formulated in the following vectorial form:

$$\boldsymbol{M}\ddot{\boldsymbol{Y}}(t)+\boldsymbol{C}\dot{\boldsymbol{Y}}(t) + \boldsymbol{R}\left[\boldsymbol{Y}(t),\dot{\boldsymbol{Y}}(t),\boldsymbol{Z}(t),\boldsymbol{X}_1(t,\gamma)\right]=\boldsymbol{X}_2(t,\gamma) \tag{1}$$

$$\boldsymbol{Q}\left\{\boldsymbol{Z}(t),\ \dot{\boldsymbol{Z}}(t),\ \boldsymbol{Y}(t),\ \dot{\boldsymbol{Y}}(t)\right\}=\boldsymbol{0} \tag{2}$$

$$\boldsymbol{Y}(t_0)=\boldsymbol{Y}_0,\quad \dot{\boldsymbol{Y}}(t_0)=\boldsymbol{Y}_{1,0},\quad \boldsymbol{Z}(t_0)=\boldsymbol{Z}_0 \tag{3}$$

where $\boldsymbol{M}$ and $\boldsymbol{C}$ represent the constant mass and damping matrices, respectively, $\boldsymbol{Y}(t)=[Y_1,\ Y_2,\ \ldots,\ Y_n]$ is an unknown response vector process, $\boldsymbol{R}$ characterizes a nonlinear restoring force depending on $\boldsymbol{Y}$ and $\dot{\boldsymbol{Y}}$, and on the process $\boldsymbol{Z}(t)=[Z_1,\ Z_2,\ \ldots,\ Z_m]$, $m<n$, which characterizes a process responsible for degradation phenomena; $\boldsymbol{X}_1(t,\ \gamma)$, $\boldsymbol{X}_2(t,\ \gamma)$ are given random processes symbolizing parametric and external excitations, respectively. The variable $\gamma$ is an element of the space of elementary events in the basic scheme $(\Gamma, F, P)$ of probability theory (cf. [1]). $\boldsymbol{Q}\{\cdot\}$ denotes a relationship between degradation and response processes; its specific mathematical form depends on the particular physical/mechanical situation. It is clear that $\boldsymbol{Y}_0$, $\boldsymbol{Y}_{1,0}$, $\boldsymbol{Z}_0$ are given initial values of the response $\left[\boldsymbol{Y}(t),\ \dot{\boldsymbol{Y}}(t)\right]$ and degradation $\boldsymbol{Z}(t)$ processes, respectively.

It should be noted, that in the cases when the original system is of a continuous type (e.g. beam, plate, shell) governed by partial differential equations, the model (1)–(3) is a spatially discretized version (e.g. via Galerkin or finite element methods) of the original equations and it describes the system response-degradation (as a function of time) at fixed spatial points. It is also worth noticing that the meaning of $\boldsymbol{Q}\{\cdot\}$ in (2) can be quite different in specific situations; it can be a differential operator, and also a functional defined on $\left[\boldsymbol{Y}(t),\ \dot{\boldsymbol{Y}}(t)\right]$.

It is natural to assume that $\boldsymbol{Z}(t_0)=\boldsymbol{0}$. During the dynamical process vector $\boldsymbol{Z}(t)$ approaches, as time increases, the unsafe state symbolized by the boundary $\tilde{B}$; each$\boldsymbol{Z}\in\tilde{B}$ denotes a critical level of degradation. Set $B$ of the admissible values of $\boldsymbol{Z}(t)$ – being a part of the first quadrant – constitutes a quality space. Therefore, the reliability of the system in question is defined as the probability that process $\boldsymbol{Z}(t)$ will belong to $B$, i.e.

$$R(t)=P\{\boldsymbol{Z}(\tau)\in B,\quad \tau\in[t_0,t]\} \tag{4}$$

The most common case of model (1)–(3) is obtained if relationship (2) takes the form of a differential equation, i.e.

$$\dot{\boldsymbol{Z}}(t)=\boldsymbol{G}\left[\boldsymbol{Z}(t),\ \boldsymbol{Y}(t),\ \dot{\boldsymbol{Y}}(t)\right] \tag{5}$$

where $\boldsymbol{G}$ is the appropriate non-negative function specifying the evolution of degradation; its mathematical form is inferred from the elaboration of empirical data, or it is derived from the physics of the process. It is clear that the model (1), (5) has a feedback mechanism: large displacements and velocities cause weakening of the restoring force, which in turn allows for larger deflections. Although we do not introduce here the notions of slow and fast variables, one should keep in mind that the deterioration $\boldsymbol{Z}(t)$ is a slow process, so $\boldsymbol{Y}$ is a fast variable compared to $\boldsymbol{Z}$; this fact can be used for substantiation of some approximations in analysis of specific problems.

Although a degradation process $\boldsymbol{Z}(t)$ in the most cases can be taken as a scalar process, its vectorial nature $\boldsymbol{Z}(t)=[Z_1(t),\ldots,Z_m(t)]$ in Eqs. (2) and (5) allows one to account for various types of degradation phenomena. The components $Z_i(t)$, $i=1,2,\ldots,m$, can be interpreted as different interacting damage processes or internal damage variables influencing a response process $\boldsymbol{Y}(t)$, e.g. thermal or chemical degradations.

In various degradation problems, especially – in the analysis of response of vibrating system with the stiffness degradation due to fatigue accumulation, it is natural to quantify the process $\boldsymbol{Z}(t)$ in (2) by a scalar degradation quantity $D(t)$ and adopt as a coupling equation (2) one of the "kinetic" equations for fatigue crack growth. These equations, however, contain the stress intensity factor range, so the degradation rate $\dot{D}(t)$ depends on the quantity related to $Y_{\max}-Y_{\min}$. In this situation, equation (2) has the form

$$\dot{D}(t)=H\left[D(t),\ Y_{\max}-Y_{\min}\right] \tag{6}$$

where $Y_{\max}$ and $Y_{\min}$ are usually the maximal and minimal values of a scalar process $Y(t)=Y_1(t)$; $H$ is a suitable non-negative (generally nonlinear) function identified from the fatigue studies.

Another version of an equation for $D(t)$ in the coupled response-degradation problem is obtained if functional relationship (2) does not include $\dot{Z}(t)$, and the degradation $\boldsymbol{Z}(t)=D(t)$ depends on some functionals defined on the response process $\left[Y(t),\ \dot{Y}(t)\right]$, i.e. (2) takes the form ($\mathcal{F}$ denotes here the appropriate functional)

$$D(t)=\mathcal{F}\left\{Y(t),\ \dot{Y}(t)\right\}. \tag{7}$$

Important examples of Eq. (7) include randomly vibrating systems in which a degradation process depends on the time length which the response $Y(t)$ spends above some critical level $y^*$ (or, $D(t)$ depends on the number of crossings of the level $y^*$ by the trajectories of the process $Y(t)$ within a given interval $[0,\ T]$). This is the case of an elastic-plastic oscillatory system with $D(t)$ interpreted as accumulated plastic deformation generated by the "excursion" of the response process $Y(t)$ into plastic domain (in this situation $y^*$ may be regarded as the yield limit of the material component in question, cf. [10].

This is also situation of randomly vibrating plate with fatigue-induced stiffness degradation; in this case $D(t)$ is interpreted as accumulated fatigue damage due to the exceeding the fatigue limit by the response process.

The analysis of the stochastic response-degradation problem for elastic-plastic vibratory systems and for the system with fatigue-induced degradation can also be analyzed by the more explicit cumulative model for degradation $D(t)$. We mean the situation in which relationship (2) in which $\boldsymbol{Z}(t) = D(t)$ is represented as follows (in scalar form)

$$D(t) = D_0 + \sum_{i=1}^{N(t)} \Delta_i (Y,\ \gamma) \tag{8}$$

where $\Delta_i = \Delta_i (Y,\ \gamma)$ are random variables characterizing the elementary degradations taking place in the system; the magnitude of $\Delta_i$ depends on the characteristics of the process $Y(t)$ above a fixed (critical) level $y^*$. Process $N(t)$ is a stochastic counting process characterizing number of degrading events in the interval $[t_0, t]$. In the case of elastic plastic oscillator (cf. [10]) $\Delta_i (Y, \gamma)$ are the yielding increments taking place in a single yielding duration $\tau_Y$ which is related to the time interval which the response process spends above the yield level during a single excursion or during a single clump of excursions. In the case of fatigue $\Delta_i$, $i = 1, 2, \ldots, N(t)$ can be regarded as the magnitudes of elementary (e.g. within one cycle) crack increments (cf. [4,11]).

Therefore, as the discussion above indicates, the general stochastic response-degradation model (1)–(3) includes (in specific practical situations) various untypical systems of equations in which ordinary differential equations are coupled with not necessarily differential evolution models for degradation. In the next sections we will show how such coupled models can be treated effectively in practice.

### 2.2. Specific deteriorating systems

**Daniels systems.** Consider a system containing $n$ brittle, parallel fibers fixed in their upper ends and carrying a mass (attached to their lower ends); this system is subjected to a random load process $S(t)$, $t \geq 0$ (cf. Fig. 1). Let us assume that the fibers have the same stiffness $k$ and the damping parameter $c$, but random, independently distributed resistances $R_i$ $(i = 1, 2, \ldots, n)$. Such a system may carry a given load in various damage states $D(t)$ characterized by a fraction of failed fibers at time $t$. The load (at each time instant $t$) is assumed to be equally distributed among the unfailed fibers according to the rule known as the equal load sharing. It is assumed, that $D(t) = 0$ as long as the displacement of a mass $Y(t)$ does not exceed some "smallest" value $y_0$. During a random vibration motion (generated by $S(t)$), the displacement $Y(t)$ changes in time and causes failures/damage of some fibers. The system goes through various degradation states $l$, $l = n, n-1, \ldots, 0$. The Daniels system collapses when the degradation state $l = 0$ (all fibers are damaged).

Let $\tau_l$ denote a time period in which the system remains in its degradation state $l$. Reliability (or, life-time) of the system is characterized by the probability

$$P_S(\tau) = P\left(\sum_{l=1}^{n} \tau_l > \tau\right) \tag{9}$$

that time to ultimate failure exceeds a fixed service life-time $\tau$. Probability (9) depends on the probability distributions of the lengths of time intervals $\tau_l$. These probabilities, in turn, depend on the relationships between the system displacements in degradation states $l$ and random resistances of fibers; fiber $i$ fails at time $t$ if the displacement process exceeds the limit value, say $\Delta_i$, of the fiber elongation for the first time ($\Delta_i = R_i/k$, $i = 1, 2, \ldots, n$). The determination of probability (9) poses significant difficulties due to the coupling between system response and sequential degradation states of the system. A detailed analysis can be found in the papers [12,13].

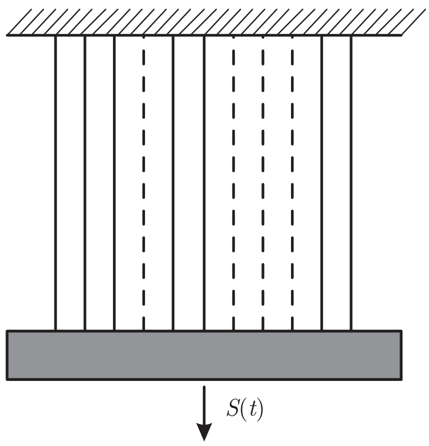


Fig. 1. Illustration of Daniels system with n fibers ($n = 11$); $m$ fibers unfailed ($m = 7$) and $n - m$ failed fibers

**Elastic-plastic/hysteretic vibratory systems.** In engineering design of structures which have to withstand severe random loading there is a strong need for taking into account the yielding behaviour of real structural components. This is the case of structures subjected to earthquakes, strong wind loadings, etc. Usually it is required to estimate the probability of structural failure within a given interval of time, by using the appropriate models of the system and excitation. However, a model of the system has to account for plastic deformations which may happen during the system dynamics and which degrade its performance. Such a model is known as an elastic-plastic oscillatory system.

Let us consider a vibrating system in which, in addition to elastic deformations, some plastic deformations may occur occasionally (in the form of "excursions" of the system response into the plastic region – cf. Fig. 2). Such dynamics causes accumulation of plastic deformations which, in turn, induce a deterioration of the system properties. For example, the initial stiffness of the system $k$ decreases to, say, $\alpha(L_1)\, k = k_1$, where $L_1$ denotes the length of time interval in which the response is in the plastic domain for the first time, and $\alpha(\cdot)$ is a specified non-negative monotonically decreasing function with $\alpha(0) = 1$. Assuming that the plastic degradation $D(t)$ is scalar quantity and equal to the sum of the values of all plastic partial deformations which took place within time interval $[0, t]$, we can represent $D(t)$ in the form of a random sum of random increments $\Delta D_i$:

$$D(t) = \sum_{i=1}^{N(t)} \Delta D_i \tag{10}$$

where $N(t)$ is the appropriate stochastic counting process characterizing a number of "degrading events" within interval $[0, t]$, whereas $\Delta D_i$, $i = 1, 2, \ldots, N(t)$ are random variables which quantify magnitudes of partial plastic deformations. Since each plastic deformation degrades the system stiffness, the magnitude of each increment $\Delta D_i$ depends on the response of the system with its "stiffness state" $k_{i-1}$; this fact can be symbolized as

$$\Delta D_i = \Delta D_i \left(Y(t);\ k_{i-1}\right). \tag{11}$$

In the case of rare plastic "events" we can assume that $\Delta D_i$ are independent random variables and that $N(t)$ is a Poisson random process. For a detailed analysis a reader is referred to [10]; (cf. also [14]).

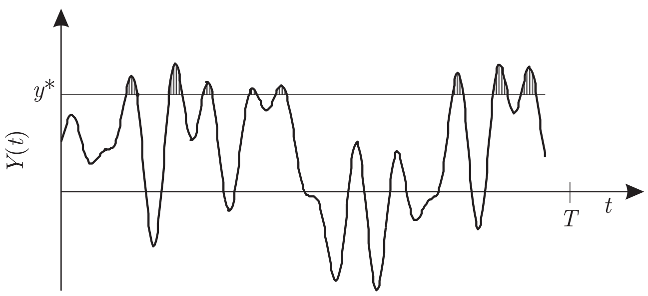


Fig. 2. Random 'excursions' of the response into plastic region; cumulative degradation

In the majority of situations deformations occurring in elastic-plastic oscillatory systems are characterized by the appropriate model of elastic force with hysteresis (containing additional component governed by independent differential equation). In such a model the restoring force depends not only on the instantaneous deformation but also on the past history of deformation. More explicitly, the system dynamics is modelled by the equation (in scalar representation)

$$\ddot{Y}(t) + h\left(Y, \dot{Y}\right) + kF(Y, Z) = X(t, \gamma) \tag{12}$$

where $k$ is a stiffness coefficient, $h\left(Y, \dot{Y}\right)$ represents arbitrary nonlinear damping, whereas $F(Y, Z)$ has a primary elastic component (with slope $\alpha$) and a secondary plastic component. In the special case where the damping is linear, i.e. $h\left(Y, \dot{Y}\right) = 2\zeta\dot{Y}$ equation (12) can be represented in the explicit bilinear form

$$\ddot{Y}(t) + 2\zeta\dot{Y}(t) + \alpha k Y(t) + (1-\alpha) kZ = X(t, \gamma) \tag{13}$$

where $\zeta$ is the usual damping coefficient for a linear system, $k$ is the pre-yielding stiffness, $\alpha$ is the ratio of post-yielding stiffness to pre-yielding stiffness, and $(1-\alpha) kZ$ is the hysteretic part of the restoring force, in which $Z = Z(t)$ characterizes the hysteretic loop and varies in time according to the general evolution equation (cf. equation (5))

$$\dot{Z}(t) = G\left[Y, \dot{Y}, Z\right] \tag{14}$$

A variety of hysteresis laws have been used in the existing analyses. A possible form is as follows (cf. [15])

$$\dot{Z}(t) = \dot{y}\left[1 - H(\dot{y}) H(z-1) - H(-\dot{y}) H(-z-1)\right] \tag{15}$$

where $H(\cdot)$ is the Heaviside unit step function. A smooth hysteretic evolution equation often used has the form of Bouc and Wen (cf. [6])

$$\dot{Z}(t) = -A\left|\dot{Y}\right| |Z|^{n-1} Z - B\dot{Y}\left|\dot{Z}\right|^{n} + C\dot{Y} \tag{16}$$

where constants $A, B, C, n$ characterize the amplitude, shape of the hysteretic loop, and transition from elastic to inelastic ranges.

In the situation considered a degradation of the restoring force has been defined in terms of the total hysteretic energy dissipation characterizing the cumulative effect of severe response (cf. [6]), that is

$$D(t) = \varepsilon_T(t) = (1-\alpha) k \int_0^t \dot{Y}(\tau) Z(\tau)\, d\tau \tag{17}$$

Therefore, the coupled response-degradation problem for randomly excited vibratory hysteretic systems is governed by equations: (12), (13), (14), (17). This is a complicated system of stochastic differential equations which can only be treated by use of approximate methods; some of such methods have been developed in the last decades (cf. [3,5,16,17]).

**Systems with fatigue-induced stiffness degradation.** An important example of a coupled response-degradation problem is concerned with stiffness degradation in randomly vibrating structural/mechanical components due to fatigue accumulation. For a wide class of such systems (in general, nonlinear) a coupled response-degradation model can be represented in the following form:

$$\ddot{Y}(t) + h\left(Y, \dot{Y}\right) + k\left[A(t)\right] Y(t) = X(t, \gamma) \tag{18}$$

$$\dot{A}(t) = f(A, \Delta S, R) \tag{19}$$

where $k(A)$ is a given decreasing function, $f$ is a non-negative, empirically identified function, $A(t)$ – the fatigue crack size at time $t$ – is governed by its evolution equation known in mechanics of fatigue fracture (cf. Fig. 3); $\Delta S$ is the stress range, i.e. $\Delta S = S_{\max} - S_{\min}$, where $S(t)$ is a random applied stress process, $R$ is the stress ratio, i.e. $R = S_{\min}/S_{\max}$. The well-known Paris-Erdogan model makes Eq. (19) more specific, namely

$$\dot{A}(t) = f_1(A)\,\Delta S, \quad \Delta S \sim Y_{\max} - Y_{\min} \tag{20}$$

where $f_1(A)$ is a non-negative function whose functional form depends on the geometries of the crack and the structural component in which the crack is growing. If stress $S(t)$ is a linear function of the displacement $Y(t)$, then it is also governed by

Eq. (18); in nonlinear case the model requires the appropriate extension. Of course, Eq. (19) is a stochastic one since it includes a random process being the amplitude or envelope of random response $Y(t)$.

Stochastic analysis of systems with fatigue-induced stiffness degradation has attracted some attention in the last years (cf. [7–9]) but there are still open problems to be solved. In Section 4 a possible approach to the problem sketched above will be presented in detail.

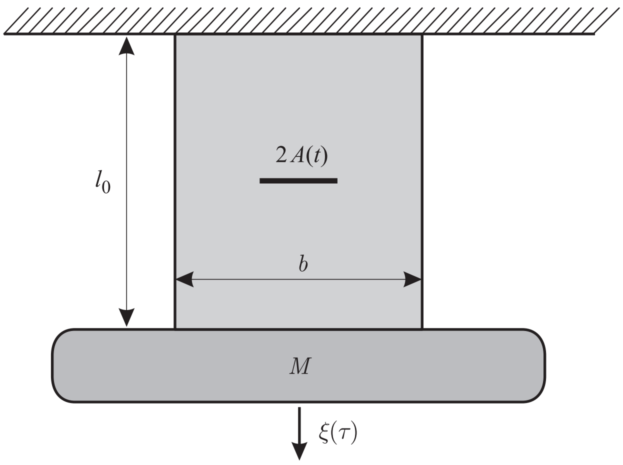


Fig. 3. Diagram of a vibratory system with stiffness degradation due to fatigue crack propagation

## 3. Stochastic systems with stiffness degradation: simple cases

**3.1. Dynamics of defective systems; remarks.** A special class of vibration problems related in a sense with degradation of system properties consists in modelling and analysis of defective vibratory systems when a "stationary" or "frozen" degradation occurs in the form of some structural defects. The basic objective of this type of research is to find out the relationships between some dynamic characteristics of a vibrating system (e.g. its natural frequencies) and the size of a fixed defect. Since many years such studies have attracted much attention in the context of vibration diagnosis of structural/mechanical systems (cf. papers [18,19]).

The most common structural defect is the existence of a crack. A crack in an elastic structural/mechanical element introduces considerable local flexibility due to the strain energy concentration in the vicinity of the crack tip under load. To take this effect into account an equivalent spring, a local compliance, was used in many articles. It allowed to correlate the crack depth to the change in natural frequencies of the first three harmonics of the structure for known crack position. In paper [19] a crack disturbance function $f(x,z)$ was introduced to characterize the change in stress, strain and displacement distribution due to the crack. This function directly affects the characteristic equation for natural frequencies of the cracked beam. In this way one can identify the size of crack defects via measurements of the natural frequencies. However, all these studies of cracked vibratory structural/mechanical components (e.g. beams, shafts) do not account for a mutual coupling of the response amplitude and crack growth during the vibration process.

**3.2. First order systems; stochastic dynamics with aging and wear-out.** Let us consider a first order system with random white noise excitation of the following form (cf. [20])

$$\dot{Y}(t) = -a(t)\,Y(t) + \sqrt{\varepsilon}\xi(t,\gamma) \tag{21}$$

$$\dot{a}(t) = -\beta h(a) \tag{22}$$

where $a(t), \beta$ and $h(\cdot)$ are positive. Function $a(t)$ characterizes changes (weakening) of the restoring mechanism (e.g. due to aging) and $\xi(t,\gamma)$ is an uncorrelated random process (a white noise), $\varepsilon > 0$ quantifies the noise intensity. The aging function $a(t)$ is governed by its own evolution equation (uncoupled with $Y(t)$). In fact, this equation could be solved and the result substituted to (21). The main Eq. (21) would be still linear in $Y$. However, it is beneficial to look for the joint solution process $[Y(t), A(t)]$ which is a diffusion Markov process (cf. [1]).

Let $f(x,a)$ be a joint probability density of $[Y(t), a(t)]$ in its stationary state. The Fokker-Planck-Kolmogorov equation associated with system (21), (22) is

$$a(t)\,y\frac{\partial f}{\partial y} + \frac{\varepsilon}{2}\frac{\partial^2 f}{\partial y^2} - \beta h\frac{\partial f}{\partial a} = 0 \tag{23}$$

We are interested in characterization of the exit of the process from the region $B = \{(Y,a'), a_{cr} \le a' \le a\}$ at the boundary point $a_{cr}$. Therefore, the conditions which have to be satisfied are

$$\begin{gathered} f(y,a_{cr}) = \delta(y - y^*) \\ \lim_{y\to\pm\infty} f(y,a') = 0, \quad a_{cr} \le a' \le a \end{gathered} \tag{24}$$

With conditions (24) $f(y,a)$ gives the probability density of exit of process $[Y(t), a(t)]$ from the interval $(y^*, a_{cr})$ starting from $(y^*, a)$. The problem described here was first treated in paper [20] where the probability density has been found (via separation of variables) in the form of a series expansion in terms of the parabolic cylinder functions.

Another situation discussed in [20] which can be regarded as a simple coupled problem with wearout is described by the system of equations

$$\dot{Y}(t) = AY + \sqrt{\varepsilon}\xi(t) \tag{25}$$

$$\dot{A}(t) = -\beta\,h(Y) \tag{26}$$

where function $h$ determines how wear-out depends on the response (displacement, stress); large displacements cause weakening of the restoring force, which in turn allows for larger deflections.

System of Eqs. (25), (26) is a special case of a more general system of equations in which one variable (say, $y$) changes much faster than the second variable (say, $D$). The $y$ variable is called a fast variable and the $D$ variable is a slow variable. One may construct the governing stochastic equations for these two variables in nonlinear form

$$\dot{Y}(t) = \eta f_1(Y,D) + \sqrt{\eta} f_2(Y,D)\,\xi_Y(t) \tag{27}$$

$$\dot{D}(t) = g_1(Y,D) + g_2(Y,D)\,\xi_D(t) \tag{28}$$

where $\xi_Y$ and $\xi_D$ are the white noise disturbances acting on the response process $Y(t)$ and degradation process $D(t)$, respectively. Parameter $\eta$ is introduced in Eq. (27) to indicate the different time scales in the above equations. Making use of this fact, the 'fast' variable can be eliminated (via the so called adiabatic elimination procedure originating from the quantum physics, cf. [21]). According to this procedure one is looking for solutions of (27), (28) in the limit $\eta \to \infty$, consistent with the assumption that $Y$ will decay very rapidly to an equilibrium state $Y_{eq}$. It is assumed that Eq. (27) without random noise has a stable equilibrium state $y_{eq}$ which can be obtained by solving the equation $f_1(y_{eq}, D) = 0$. The equilibrium value $y_{eq}$ depends generally on the slow variable $D$. When the noise term in (27) is taken into account one is looking for an equilibrium probability distribution of the $Y$ variable. If one is interested only in the time scale large compared to the decay time ($\sim \eta^{-1}$) of the fast variable $Y$, the process $[Y(t), D(t)]$ governed by Eqs. (27), (28) can be approximately described only by the motion of the slow variable $D$. In this sense one can say that the slow variable is subordinated to the fast variable.

**3.3. Vibratory systems with "empirical" stiffness degradation.** As we have discussed in Section 2, the evolution of the degradation in time can be specified in various ways depending on its physical nature as well as on the availability of the empirical information. If the available information on the degradation process comes from experiments, then a rational way to include it into the response analysis can be based on the following formulation:

$$\ddot{Y}(t) + 2\zeta\dot{Y}(t) + q(D(t))\,Y(t) = \xi_1(t,\gamma), \tag{29}$$

$$D(t) = \phi(t) \tag{30}$$

where $\phi(t)$ is a given, empirical function quantifying the growth of the degradation in time. If degradation is generated by the fatigue crack growth and $D(t)$ is identified with the crack size $A(t)$, then $\phi(t)$ can be regarded as a function which characterizes the averaged crack growth for $t \geq 0$ and incorporates the "averaged" contribution of the stress range $\Delta S$. In a special case of a narrow-band stress response process with zero mean $\Delta S$ may be approximated, for example, by the root mean square of the stress process $S(t)$ generated by the displacement $Y(t)$.

In the above formulation, the problem is reduced to the analysis of oscillator (29) with a given time-varying stiffness coefficient, which is a composition of the monotonically decreasing function $q$ and degradation $D(t) = \phi(t)$. Generally, the function $q$ representing the stiffness dependence on the degradation measure $D(t)$ (in particular, on the dominant crack size) is often taken in the form of a polynomial

$$q(D) = \sum_{i=1}^{M} \beta_i D^i. \tag{31}$$

However, it can be also approximated by the exponential function [8]

$$q(D) = (1-\alpha_1)\exp(-\alpha_2 D^{\alpha_3}) \tag{32}$$

where $\alpha_1, \alpha_2, \alpha_3$ are positive constants. The values of the empirical parameters $\alpha_i, \beta_i$ should be identified from experimental data. The probability distribution of $Y(t)$ of the system (29), (30) is Gaussian with time-dependent mean $m_Y(t)$ and variance $\sigma_Y^2(t)$. These functions satisfy the differential equations derivable from Eq. (29). Introducing the variables $Y_1 = Y$, $Y_2 = \dot{Y}_1$ we have the system

$$\begin{aligned} \dot{Y}_1 &= Y_2 \\ \dot{Y}_2 &= -q(D(t))\,Y_1 - 2\zeta Y_2 + \xi_1(t,\gamma). \end{aligned} \tag{33}$$

The equations for the second order moments $m_{ij}(t) = \langle Y_i(t)\, Y_j(t)\rangle$, $i,j = 1,2$, are as follows:

$$\begin{aligned} \dot{m}_{11}(t) &= 2m_{12}(t) \\ \dot{m}_{12}(t) &= -\left[q(D(t))\,m_{11}(t) + 2\zeta m_{12}(t)\right] + m_{22}(t) \\ \dot{m}_{22}(t) &= -2\left[q(D(t))\,m_{12}(t) + 2\zeta m_{22}(t)\right] + 2\zeta. \end{aligned} \tag{34}$$

Equations (34) along with the initial conditions $m_{ij}(t_0)$ (which follow from the initial conditions for $Y_1(t)$ and $Y_2(t)$ when $t = t_0$) determine the moments $m_{ij}(t)$. Because $\langle Y(t)\rangle = 0$ the variance $\sigma_Y^2(t)$ is equal to $m_{11}(t)$.

If we wish to account for the statistical dispersion of the degradation process we may consider $D(t) = \phi(t, C(\gamma))$ instead of $\phi(t)$, where $C(\gamma)$ is a suitably defined random variable. In such a case the probability distribution of $Y(t)$ is no longer Gaussian; it is a mixture of Gaussian distribution and the probability distribution of $C(\gamma)$. It can be obtained by the integration of the Gaussian (conditional) distribution with respect to the distribution of $C(\gamma)$.

It is worth to add that the vibration equation can be supplemented by the degradation process which takes place at a much earlier stage than growth of a dominant crack. For example, a significant degradation mechanism that affects the reliability of various engineering components is the pitting corrosion [22]. It has been found empirically that the pit volume in low-alloy steels under cyclic loading increases linearly with time $t$.

Mathematically, model (29), (30) is uncoupled; only $Y(t)$ is explicitly affected by the degradation of stiffness ($D(t)$ is assigned a priori). However, if the characteristics of the degradation process $D(t)$ are determined from the measurements of $D(t)$ during the vibration process, they are affected implicitly by the response amplitudes.

## 4. Stochastic dynamics with fatigue-induced stiffness degradation

**4.1. Description of underlying model.** As we have already mentioned, the degradation of stiffness due to fatigue accumulation during the vibration process is the phenomenon of great importance in practice. In this section we show how the coupled response-fatigue degradation problem can be treated effectively.

Let us consider a thin rectangular plate of size $l_0 \times b$ with an initial central crack of length $2A_0$ and supporting a rigid heavy mass $M$ at its end (cf. Fig. 3). The plate is made of homogeneous and isotropic elastic material with linear viscous damping. The mass is subjected to a random Gaussian white noise excitation $\xi(t)$ perpendicular to the crack. Thus, the crack extends in a direction normal to the applied stress at a rate depending on the system geometry and the intensity of random applied stress. Let us denote by $Y(t)$, $2A(t)$, and $q(D(t))$ the plate displacement, the crack size and the degrading plate stiffness due to crack growth at time $t > 0$. The displacement process $Y(t)$ satisfies the following stochastic equation

$$m\ddot{Y}(t) + c\dot{Y}(t) + q(D(t))\,Y(t) = \xi(t) \tag{35}$$

where $c$ denotes the system damping and is assumed to be time invariant. If stress is a linear function of the displacement, then it is also governed by Eq. (35) with suitably rescaled process $\xi(t)$. Dividing both sides of Eq. (35) by $m$ (and then introducing simple transformation of variables: $\tilde{Y} = Y/\sigma_Y$, $\tau = \omega_0 t$, where $\sigma_Y$ is a standard deviation of the stationary solution of (35) without degradation, i.e. when $q(A) = \omega_0^2$) one obtains a dimensionless form of (35). It is convenient to represent this equation as the system of two first order equations for the process $[Y_1(t),\, Y_2(t)] = \left[Y(t),\, \dot{Y}(t)\right]$

$$\begin{aligned} \dot{Y}_1(t) &= Y_2(t) \\ \dot{Y}_2(t) &= -2\zeta Y_2(t) - q(D(t))\,Y_1(t) + \xi(t) \end{aligned} \tag{36}$$

where $\zeta$ is a damping coefficient. Function $q(\cdot)$ characterizes the stiffness dependence on the degradation due to fatigue. This stiffness function can be regarded as empirical one, but it can also be inferred from the solutions of vibration problems with "frozen" cracks. The best known representation of the stiffness dependence on the crack size $a$ has the form of a polynomial, i.e.

$$k(x) = k(0)\sum_{i=1}^{K}\beta_i x^i, \qquad x = \frac{2a}{b} \tag{37}$$

where coefficients $\beta_i$ are suitably identified constants. For example (cf. [7]):

$$\begin{aligned} k(x) = k(0)\,[1 &- 1.708x^2 + 3.081x^4 \\ &-7.036x^6 + 8.928x^8 - 4.266x^{10}] \end{aligned} \tag{38}$$

The stiffness function can also be approximated by the following exponential function (cf. [8]):

$$k(x) = \alpha_1 + \alpha_2 \exp\left(-\alpha_3 x^{\alpha_4}\right) \tag{39}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are positive constants, such that $\alpha_1 + \alpha_2 = q_0$ to have $q(a = 0) = q_0$.

In order to obtain a complete response-degradation model when degradation is due to the fatigue crack growth the appropriate "kinetic" equation for crack size $A(t)$ should be constructed. The physics of fatigue crack growth is a complex phenomenon taking place on different scales (starting from nucleation of micro-cracks, through their coalescence and growth, and the growth of a macroscopic/dominant crack). This process depends crucially on the material microstructure and the macroscopic material properties; it depends also on external conditions such as external loading acting on mechanical/structural element, temperature, chemical environment etc. (cf. [4,23]). Here we restrict our attention to the growth of macroscopic crack in elastic material. In this case, the main driving force for a crack growth is the stress intensity factor $K$; it characterizes the stress intensity in the crack tip, and depends on the crack length, i.e. $K = K(a)$. In general

$$K = B(a)\,S\sqrt{\pi a} \tag{40}$$

where $S$ represents the far-field stress resulting from the applied load and $B(a)$ is a factor which accounts for the shape of the specimen and the crack geometry. The crack growth rate ($\Delta a$, or $da/dt$) is understood as the crack extension of a crack of length $a$ during unit of time, or in one stress cycle.

The investigations of fatigue crack growth in elastic materials have shown that the most suitable quantity for characterizing the fatigue crack growth is the stress intensity, or – more specifically – the stress intensity factor range $\Delta K = K_{\max}(a) - K_{\min}(a)$. Therefore, the crack growth equations can be represented in the general form

$$\begin{aligned} \frac{dA}{dN} &= f(\Delta K) \\ &= f\left(B(A)\sqrt{\pi a}\Delta S\right) \end{aligned} \tag{41}$$

where $\Delta S = S_{\max} - S_{\min}$ is the stress range, $N$ denotes a number of stress cycles and $f$ is a suitable non-negative function. One of the most common equations belonging to class (41) is the following Paris-Erdogan equation (for constant amplitude loading)

$$\begin{aligned} \frac{dA}{dN} &= C(\Delta K)^m, \qquad \Delta K > 0 \\ &= CB^m(A)\left(\sqrt{\pi A}\right)^m (\Delta S)^m \end{aligned} \tag{42}$$

where $\Delta S$ is the stress range associated with the response process $Y(t)$, and $C$ and $m$ are empirical constants (cf. [4,23]).

Therefore, a coupled model of the response-fatigue degradation process is represented by Eqs. (36), (42) along with the initial conditions $[Y_1(t_0), Y_2(t_0), A(t_0)]$ and formulae (37), (39). This is a system of three first order differential equations with random excitation. Because of large random scatter of fatigue crack trajectories and uncertainty in effects of many uncontrolled factors the evolution Eq. (42) can also be randomized and included into the analysis (such equation was proposed and analyzed in [24]).

To make the further analysis consistent with the model (which includes $D(t)$ ) we will deal not with $A$ directly but with the nonlinear transformation $\Psi(A)$ of $A$ defined as

$$\Psi(A) = \int_{A_0}^{A} \frac{dx}{B^m(x)\left(\sqrt{\pi x}\right)^m} \tag{43}$$

where $A_0$ is the initial crack size. Let us denote by $\Psi^*$ the value of $\Psi(A)$ for the critical crack length $A = A^*$, and define the degradation measure $D$ as

$$D = \frac{\Psi(A)}{\Psi^*}, \qquad \Psi^* = \Psi(A^*), \qquad D \in [D_0, 1] \tag{44}$$

Of course,

$$dD = \frac{1}{\Psi^*} d\Psi(A) = \frac{1}{\Psi^*} \frac{dA}{B^m(A)\left(\sqrt{\pi}\,A\right)^m} = \frac{1}{\Psi^*} C(\Delta S)^m dN \tag{45}$$

Therefore, the evolution equation for the fatigue crack induced degradation $D(t)$ defined by (44) takes the form

$$\frac{dD}{dN} = \frac{1}{\Psi^*} C\left(\Delta S_Y\right)^m \tag{46}$$

where $\Delta S_Y$ is the stress range generated by the response process $Y(t)$.

**4.2. Envelope approximation of stress range; moment equations.** In order to make the model (36), (46) effective some approximations have to be introduced. Here, we show how the random stress range $\Delta S$ occurring in Eq. (42) and in its transformed version (46) can be effectively characterized. Let us assume that the vibrating system is lightly damped and its response (the associated stress process) is a narrow band process. In such a process a single frequency dominates and its trajectories resemble harmonic oscillations. Such a process may be approximated by two times the envelope process (cf. [4,25]), i.e.

$$\Delta S_Y \approx 2\sqrt{Y_1^2(t) + Y_2^2(t)} \tag{47}$$

and the passage from cycles to time in Eqs. (42), (46) may be made with use of the relationship $dN \approx \mu dt$, where $\mu$ is the average number of zero crossings by process $Y(t)$. Therefore, Eq. (46) takes the form

$$\frac{dD}{dt} = C_1\left[Y_1^2(t) + Y_2^2(t)\right]^{m/2}, \tag{48}$$

where $D(t_0) = D_0,\ C_1 = C2^m\mu\left(\Psi^*\right)^{-1}$. After the approximations above the basic model is represented by the system of Eqs. (36), (48). It should be noticed that even though the vibrating system is linear, the coupled response-degradation problem is nonlinear. The exact analysis of the coupled system (36), (48) is involved. To make the further treatment of the problem possible we will take advantage of the fact that the envelope amplitude varies slowly in time. This agrees with the observation that the degradation process is slow in comparison with the response itself. Therefore, making the 'linear approximation' $D(t) = \dot{D}(t)\,t$ where $\dot{D}(t)$ is given by (48) we can regard $D$ occurring in Eq. (36) as explicitly expressed by $Y(t)$, $\dot{Y}(t)$ and $t$. Hence, Eq. (36) can be written in the form of the following Itô stochastic equations (cf. [8])

$$\begin{aligned} dY_1(t) &= Y_2(t)\,dt \\ dY_2(t) &= -\left[2\zeta Y_2(t) - g\left(Y_1, Y_2, t\right)\right]dt + 2\sqrt{\zeta}\,dW(t) \end{aligned} \tag{49}$$

where $W(t)$ is the standard Wiener process, and $g(Y_1, Y_2, t)$ is the nonlinear term accounting for the dependence of stiffness on the degradation, i.e.

$$\begin{aligned} g(y_1, y_2, t) &= y_1 q(D) = y_1 q(y_1, y_2, t) \\ &= C_1\left(y_1^2 + y_2^2\right)^{m/2} y_1 t \end{aligned} \tag{50}$$

In order to obtain quantitative results, one has to assume the specific function characterizing the dependence of the stiffness on the degradation due to fatigue. Here, the approximation (39) is used.

In order to obtain a probabilistic characterization of the response with the degradation of stiffness (39) the moment equations for the stochastic system (49) are generated (cf. [1]). If we denote $m_{ij}(t) = \left\langle Y_1^i(t)\,Y_2^j(t)\right\rangle$ where $\langle\cdot\rangle$ is the symbol of the probabilistic mean value, then we have

$$\begin{aligned} \dot{m}_{ij}(t) &= \frac{d}{dt}\left\langle Y_1^i Y_2^j\right\rangle = \Big\langle iY_1^{i-1}Y_2^{j+1} \\ &\quad - j\left[2\zeta Y_2 + g\left(Y_1, Y_2, t\right)\right] \\ &\quad \times Y_1^i Y_2^{j-1} + 2\zeta j\,(j-1)\,Y_1^i Y_2^{j-2}\Big\rangle \\ &\equiv \left\langle G_{ij}\left(Y_1, Y_2; t\right)\right\rangle, \quad \left\langle Y_1^i(t_0)\,Y_2^j(t_0)\right\rangle = m_{ij}(t_0) \end{aligned} \tag{51}$$

$i, j = 1, 2, \ldots$. The information on the behaviour of the system is taken in the form of five first equations from the above hierarchy of equations, i.e. equations for the first order and second order moments $(m_{10}, m_{20}, m_{01}, m_{02}, m_{11})$

$$\begin{aligned} &i=1,\ j=0: \ \frac{d}{dt}\langle Y_1\rangle = \langle Y_2\rangle \\ &i=2,\ j=0: \ \frac{d}{dt}\left\langle Y_1^2\right\rangle = 2\left\langle Y_2 Y_1\right\rangle \\ &i=0,\ j=1: \ \frac{d}{dt}\langle Y_2\rangle = -\left\langle 2\zeta Y_2 + g\left(Y_1, Y_2, t\right)\right\rangle \\ &i=0,\ j=2: \ \frac{d}{dt}\left\langle Y_2^2\right\rangle = -2\left\langle 2\zeta Y_2 + Y_2 g\left(Y_1, Y_2, t\right)\right\rangle + 2\zeta \\ &i=1,\ j=1: \ \frac{d}{dt}\left\langle Y_1 Y_2\right\rangle = \left\langle Y_2^2 - 2\zeta Y_1 Y_2 - Y_1 g\left(Y_1, Y_2, t\right)\right\rangle \end{aligned} \tag{52}$$

The approximate probability density $p(y_1, y_2; t)$ is determined via the modified maximum entropy method (cf. [26,27]). This density has the form

$$\begin{aligned} p(y_1, y_2, t) &= \frac{1}{\tilde{C}}\exp\Big\{-\big[\lambda_{10}y_1 + \lambda_{20}y_1 g(y_1, y_2, t) \\ &\quad + \lambda_{01}y_2 + \lambda_{02}y_2^2 + \lambda_{11}y_1y_2\big]\Big\} \\ &= \frac{1}{\tilde{C}}\tilde{p}(y_1, y_2, t) \end{aligned} \tag{53}$$

where $\lambda_{ij}$ are unknown Lagrange coefficients and

$$\tilde{C} = \int_{-\infty}^{+\infty} \tilde{p}(y_1, y_2, t)\,dy_1 dy_2 \tag{54}$$

is the normalizing constant parametrized by time. Let us discretize system (52) using (for instance) the Euler scheme (to make further equations more clear) with the step $\Delta t$. As the result, the system (52) can be rewritten as

$$\begin{aligned} m_{ij}(t_{k+1}) &= m_{ij}(t_k) \\ &\quad + \Delta t \int_{-\infty}^{+\infty} G_{ij}(y_1, y_2, t_k)\,p(y_1, y_2, t_k)\,dy_1 dy_2 \\ i, j &= 0, 1, 2 \qquad 0 < i + j \leq 2 \end{aligned} \tag{55}$$

with $m_{ij}(t_0)$ assumed to be given initial condition. In our consideration these initial moments are taken as the moments of stationary solution of the system (35) without degradation.

The Lagrange coefficients $\lambda_{ij}$ are determined at each discrete time $t_k$ numerically from the following system of algebraic nonlinear equations

$$m_{ij}(t_k) = \int_{-\infty}^{+\infty} y_1^i y_2^j p(y_1, y_2, t_k)\, dy_1 dy_2 \qquad (56)$$

$$i, j = 0, 1, 2 \qquad 0 < i + j \leq 2$$

Taking into account (53) this system can be written as

$$\int_{-\infty}^{+\infty} \left[ y_1^i y_2^j - m_{ij}(t_k) \right] \tilde{p}(y_1, y_2, t_k)\, dy_1 dy_2 = 0 \qquad (57)$$

$$i, j = 0, 1, 2 \qquad 0 < i + j \leq 2$$

and can be solved using (for example) the five dimensional Newton method.

In the calculations function (39) was used with the following values of parameters: $\alpha_1 = \alpha_2 = 0.5$, $\alpha_4 = 2.0$, $\zeta = 0.125$, $m = 3.0$, $C_1 = 1.3298 \cdot 10^{-5}$. It is assumed that the degradation starts when dimensionless system reaches the stationary state. In the absence of degradation ($q(D) = 1$) the response of the system (which is linear) is Gaussian. Degradation introduces nonlinear and time-dependent stiffness and, therefore leads – in general – to non-Gaussian behaviour of the system.

Of the results of numerical calculations we show here two illustrations. Figure 4 visualizes function $q(D)$ versus $D$ for selected values of parameters $\alpha_3$: for curve (1) $\alpha_3 = 0.1$, for curve (2) $\alpha_3 = 0.5$, for curve (3) $\alpha_3 = 1$. Generally, the form of degradation function (39) is very flexible and many kinds of possible types of degradation (from linear to strongly nonlinear) can be obtained. In practice, the values of parameters $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ should be estimated from experimental data.

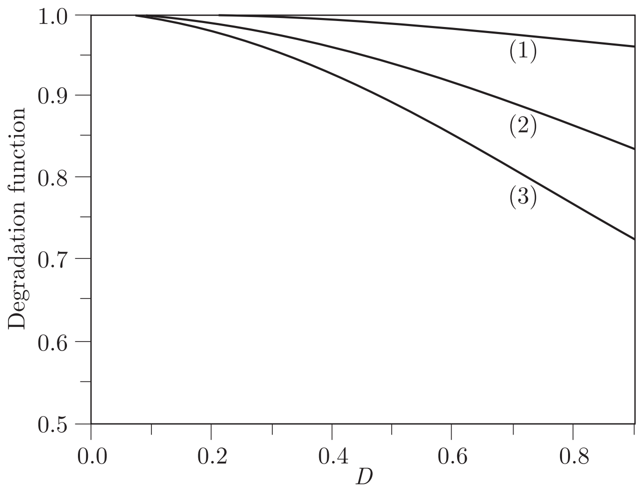


Fig. 4. Function of degradation $q(D)$ given in (39) for different values of parameter $\alpha_3$: (1) – $\alpha_3 = 0.1$; (2) – $\alpha_3 = 0.5$; (3) – $\alpha_3 = 1$. Values of other parameters in the text

Figure 5 shows the variance $\sigma_y^2$ of the displacement in the system with degradation for the same different values of parameters $\alpha_3$ as in Fig. 4. In the case of dimensionless system without degradation we have $\sigma_y^2 = 1$. Therefore, Fig. 5 displays the effect of stiffness degradation on the response of the system. The increasing of degradation causes the increase of the variance of displacement of the system.

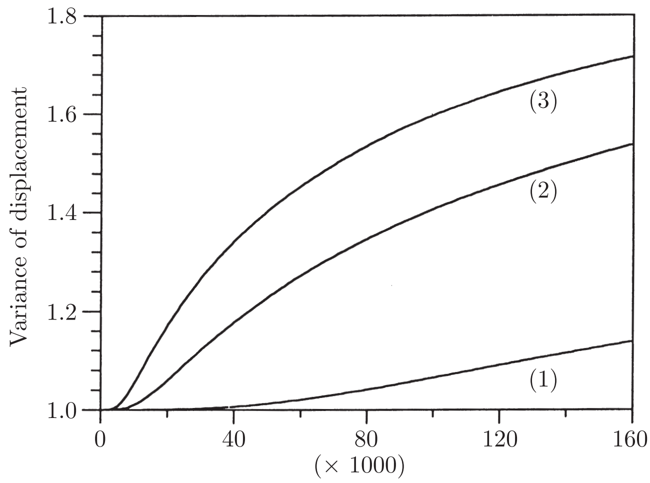


Fig. 5. The variance of the displacement in system with degradation for degradation functions corresponding to the Figure 4

**4.3. Analysis of response-degradation problem via conditioning.** In this subsection we wish to show another approach to effective analysis of the coupled response-stiffness degradation problem which accounts explicitly for the cumulative nature of the fatigue degradation process (cf. [9]). Let us notice that Eq. (46) indicates that the increment of the degradation measure $D$ in one equivalent cycle can be represented as

$$\Delta D_i = \frac{1}{\Psi^*} C\, (\Delta S_i)^m \qquad (58)$$

where $\Delta S_i$ is the stress range in the $i$-th cycle of loading. Let us represent $D(t)$ in the form of a sequence of random variables $D_N(\gamma) =: D(t_N)$, $N = 0, 1, \ldots, N^*$, where $D_N(\gamma)$ characterizes the state of the degradation process after $N$ cycles. Therefore

$$D_N(\gamma) = \sum_{i=1}^{N} \Delta D_i(\gamma) \qquad (59)$$

$$\Delta D_i(\gamma) = D_i(\gamma) - D_{i-1}(\gamma) \qquad (60)$$

The coupled computational response-degradation model has the form

$$\ddot{Y}(t) + 2\zeta\dot{Y}(t) + q(D_{N-1}(\gamma))\, Y(t) = \xi_1(t, \gamma) \qquad (61)$$

$$D_N(\gamma) = D_{N-1}(\gamma) + \Delta D_N(\gamma) \qquad (62)$$

where $\Delta D_N(\gamma)$ denotes the increment of the degradation process during the $N$-th cycle. It is defined by formula (58), in which $\Delta S_N$ is the stress range in the $N$-th cycle. Assuming that the degradation starts when response $Y(t)$ is in its stationary state and that the response is a narrow-band process ($2\zeta \ll 1$) we approximate $\Delta Y_i = Y_{\max,i} - Y_{\min,i}$ by two times the amplitude $H_i$ of the $Y(t)$, i.e. $2H_i$. Therefore, the stress range $\Delta S_i$ in the $i$-th cycle is

$$\Delta S_i = 2H_i \frac{E}{l_0} \tag{63}$$

where $l_0$ is the length of the elastic element (cf. Fig. 3) and $E$ is the Young modulus. Finally, the increment $\Delta D_N$ of the degradation process occurring in (62) has the form

$$\Delta D_N(\gamma) = C_1 H_N^m(\gamma) \tag{64}$$

where constant $C_1 = C2^m E^m \sigma_y^m / l_0^m \Psi^*$ is obtained during the transformation from dimensional to non-dimensional system (cf. [9]).

Eqs. (61), (62) along with (58) and (63), (64) constitute a complete sequential model for characterization of the response-degradation process $[Y(t), D(t)]$ in discretized time instants (cycles) $N = 0, 1, \ldots, N^*$. Because the degradation process is slow in comparison to the response itself and the degradation process $D$ starts when the system (61) reached its stationary state for initial stiffness $q(D_{N=0})$ generated by deterministic or random value of the initial damage measure $D_{N=0} = D_0$ we take the distribution of the magnitude $H_N$ given $D_{N-1}$ as being the Rayleigh distribution. In this model the response after $N$ cycles is affected by the stiffness degradation state after $N-1$ cycles, whereas the degradation process after $N$ cycles depends on the response amplitude $H_N$ at cycle $N$, given $D_{N-1}$.

The probabilistic characterization of the response-degradation process $[Y_N, D_N]$, where $Y_N = H_N$ (and $H_N$ is the amplitude of the process $Y$ at cycle $N$) has been performed in paper [9] via conditioning. Without going into details, the idea is as follows.

Let us denote by $H_N|D_{N-1}$ the (conditional) amplitude of the process $Y(t)$ at the $N$-th cycle given a fixed value of the stiffness in cycle $N$ – specified by degradation level at $(N-1)$-st cycle. The conditional probability density of $H_N$ is the Rayleigh distribution

$$\hat{f}_{H_N}(h|D_{N-1}) = \frac{h}{\sigma^2_{Y|D_{N-1}}} \exp\left(-\frac{h^2}{2\sigma_{Y|D_{N-1}}}\right) \tag{65}$$

where $\sigma^2_{Y|D_{N-1}}$ is the variance of the (conditional) stationary Gaussian response process $Y(t; D_{N-1})$ – evaluated from the analysis of oscillator (61) with given $q(D_{N-1})$. In the case considered (dimensionless oscillator, cf. [9])

$$\sigma^2_{Y|D_{N-1}} = \frac{1}{q(D_{N-1})} \tag{66}$$

and the probability density (65) characterizing the response amplitude at $N$ cycles is

$$\hat{f}_{H_N}(h|D_{N-1}) = hq(D_{N-1}) \exp\left(-\frac{1}{2}h^2 q(D_{N-1})\right). \tag{67}$$

In order to find probability distribution of $D_N$ defined by (62) we calculate first the probability density of $H_N^m$ occurring in (64) and then – the conditional density of $\Delta D_N$ when $D_{N-1}$ is fixed, i.e. $g_{\Delta D_N|D_{N-1}}(x|D_{N-1})$. To evaluate the probability density of the degradation $D_N$ at cycle $N$, we need, according to (62), the joint distribution of $D_{N-1}$ and $\Delta D_N$ which is represented as

$$f_{\Delta D_N, D_{N-1}}(x, y) = g_{\Delta D_N|D_{N-1}}(x|y) f_{D_{N-1}}(y) \tag{68}$$

Finally, the probability density of random variable $D_N$, being a sum of $\Delta D_N$ and $D_{N-1}$, is given in form of the following convolution

$$\begin{aligned} f_{D_N}(z) &= \int_0^z f_{\Delta D_N, D_{N-1}}(z-y, y)\, dy \\ &= \int_0^z g_{\Delta D_N|D_{N-1}}(z-y, y) f_{D_{N-1}}(y)\, dy \end{aligned} \tag{69}$$

where $g_{\Delta D_N|D_{N-1}}$ is known, i.e. it is evaluated earlier on the basis of formula (64); its explicit form is

$$\begin{aligned} g_{\Delta D_N|D_{N-1}}(x|D_{N-1}) &= \frac{1}{C_1 m} q(D_{N-1}) \left(\frac{x}{C_1}\right)^{(2-m)/m} \\ &\times \exp\left[-\frac{1}{2} q(D_{N-1}) \left(\frac{x}{C_1}\right)^{2/m}\right]. \end{aligned} \tag{70}$$

Therefore, the probability density $f_{D_N}(z)$ of the degradation process $D$ at the $N$-th cycle is expressed by the formula (69) in terms of the conditional density (70) and the density of the degradation process at $(N-1)$-st cycle. This integral recursive formula (69) can serve as a base for calculations. The probability distribution of the response process at cycle $N$, given the degradation at cycle $N-1$ is expressed by formula (67).

In order to show the effectiveness of the method described above the numerical calculations were performed assuming that the specimen and crack geometry function $B(a)$ in crack growth Eq. (42) has the form (cf. [28])

$$B(a) = \left[1 - 0.025\left(\frac{A}{b}\right)^2 + 0.06\left(\frac{A}{b}\right)^4\right] \left(\cos\frac{\pi A}{2b}\right)^{-1/2}. \tag{71}$$

The stiffness-fatigue degradation relationship $q(D)$, obtained with use of (44), (45) from the relation (38) for stiffness dependence on the crack size was represented as follows

$$\begin{aligned} q(D) &= 0.993283 - 0.0544954D + 0.24168D^2 \\ &- 2.82587D^3 + 11.1158D^4 - 23.1294D^5 \\ &+ 23.2367D^6 - 9.39197D^7 \end{aligned} \tag{72}$$

Damping coefficient in the system (36) was $\zeta = 0.01$, and the constants in crack growth Eq. (48): $C_1 = 4.7015$, $m = 3.0$.

The results of calculations according to the integral recursive formula (69) are presented in detail in the paper [9]. Here we show only two plots: one displaying the mean and standard deviation of the response (displacement) amplitude of the system without and with stiffness degradation (Fig. 6) and the second – showing probability density curves of the degradation measure for the system without and with simultaneous stiffness degradation (Fig. 7) for different numbers of cycles. These figures clearly indicate that stiffness degradation should play an important role in reliability analysis of vibrating systems. For example, for fixed level of degradation $D^* = 0.8$ and $N = 140$ thousands of cycles we have the probability of failure $P_F = 1 - P(D < D^*) \approx 0.05$ – for non-degraded system, and $P_F \approx 0.45$ for degraded system. The non-degraded system is understood as the system whose stiffness degradation is not taken into account.

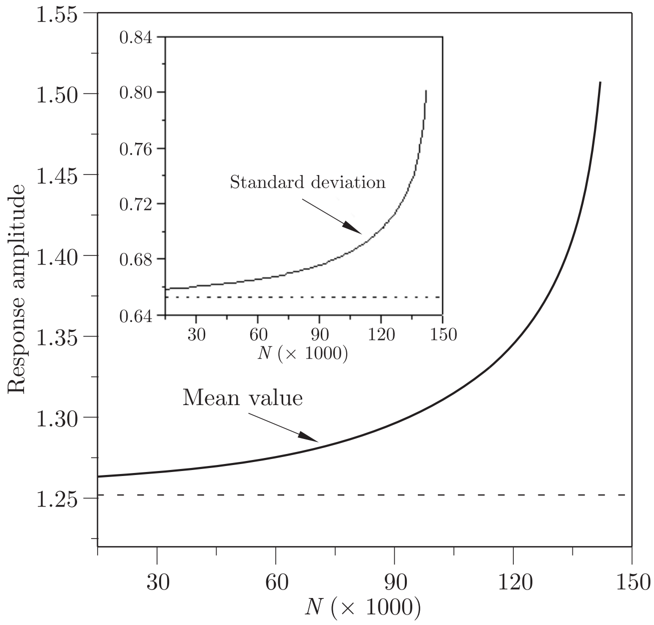


Fig. 6. Mean and standard deviation of the response amplitude: system with non-degraded stiffness (dashed line) and system with degraded stiffness (continuous line)

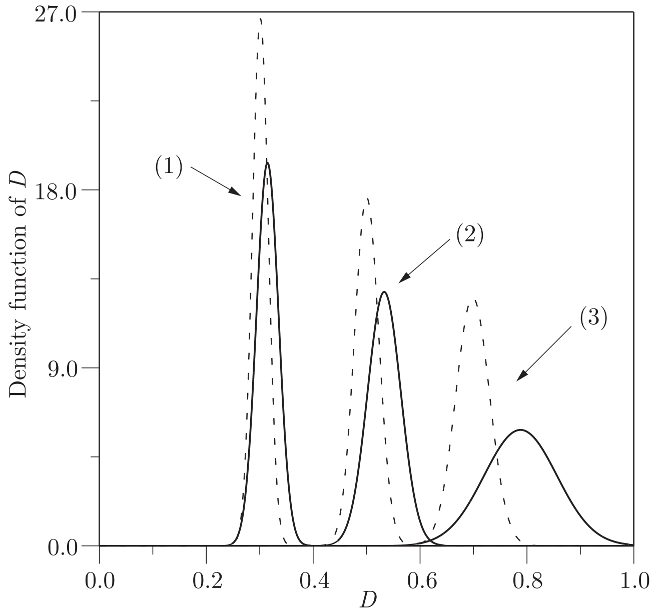


Fig. 7. Probability densities of the degradation measure $D$ for the system with non-degraded stiffness (dashed line) and degraded stiffness (continuous line) at different number of cycles: (1) – $N = 60$, (2) – $N = 100$, (3) – $N = 140$ (×1000)

**4.4. Nonlinear dynamics; remarks.** Although in the subsections above the system dynamics was analyzed within a linear model, it does not mean that linearity is a crucial restriction. Of course, the coupled response-degradation problem for nonlinear systems is more involved, but its effective analysis is still possible along the line described in this section, if suitable approximations are adopted (cf. [29]).

The first problem which arises is concerned with the construction of a consistent model of the evolution of fatigue-induced degradation. The model presented in this section uses (as the basic driving force of a crack growth) the stress intensity factor $K$. However, this factor is derived on the basis of the linear elasticity theory. So, the appropriate quantity is required to describe stress intensity in materials with nonlinear stress-strain behaviour. One might expect the $J$-integral of Rice (cf. [4] and references therein) to be applicable when plastic deformations occur. But, a possible use of $J$-integral for predicting fatigue crack growth is still not sufficiently understood.

Another problem in effective analysis of coupled response-degradation models for nonlinear dynamical systems lies in difficulties in obtaining analytical results for the stress range for the nonlinear response. It seems that the stochastic averaging method (cf. [1,2]) of stochastic dynamics should be useful since it works directly with the equation for the response amplitude; one should, however, keep in mind that this method has also its own limitations.

Paper [29] gives a possible treatment of the problem when a nonlinear stochastic response is governed by the equation

$$m\ddot{Y}(t) + cY(t) + k(D)\,g(Y; \eta_1, \ldots, \eta_M) = \xi(t) \qquad (73)$$

where function $g(y; \eta_1, \ldots, \eta_M)$ is non-decreasing function satisfying the condition $g(-y) = -g(-y)$, and the parameters $\eta_k$, $k = 1, \ldots, M$, are the coefficients in the approximate representation of $g$ in the form: $g(y) = \eta_1 y + \eta_2 y^2 + \ldots + \eta_M y^M$. Such a representation is associated with a nonlinear symmetric relationship between stress $S$ and strain $\varepsilon$ in uniaxial tension-compression deformation of metals (cf. [30]).

## 5. Other related problems

**5.1. Systems with varying structure.** The dynamical systems discussed in previous sections have possessed a feature that their structure (characterized by the appropriate parameters) changed during the motion. It has been assumed that these changes took place continuously in time and were governed by the explicit evolution equations. We have also assumed that the changes were degrading in nature, which means that the system structure at time $t_2$ was worse (in a suitably defined sense) than at time $t_1$, if $t_2 > t_1$, for each $t_1$, $t_2$ belonging to the time interval of interest. Such systems can be viewed as closely related to a wider class of dynamical systems with varying structure.

Saying ‘systems with varying/random structure’ we understand systems whose behaviour at random time intervals is characterized by different structures (and is governed by different equations). This class of systems includes, e.g. control systems in which relations between the system elements change depending on the system state. It is clear, that systems (of various physical nature) with possible faults or inefficiencies, which can occur at random instants of time, are also examples of systems with varying structure.

Each sub-structure of a system with varying structure is associated with its own sub-space in the state space of the system. A change of structure consists in the passage of the system from one sub-space of states to another. For large class of systems of practical interest governed by ordinary differential equations in $R^n$ this change takes place at hypersurfaces in $R^n$, and is associated with discontinuous change of the system parameters (switching systems). However, in general the change of structure can take place at arbitrary point (of the state space of the system) with some intensity, which depends on the state of the system and is defined from the physics of the problem.

For mechanical/structural vibratory systems the sub-structures can be specified by physical phenomena (processes) taking place in the system. For example, in the case of vibratory systems with dry friction the governing equations contain discontinuity expressed by signum function of the state variables (which naturally induces a division of the state space). In the case of degrading vibratory systems the changes in structure can be defined by requirement that a suitable functional defined on the states of the system reaches specific (critical) values.

Let $K$ denote a number of possible sub-structures of the dynamical system with variable structure under consideration. In general, the dynamics of such a system can be described by the extended state vector $[\boldsymbol{Y}(t), \boldsymbol{\Phi}(t)]$ where $\boldsymbol{Y}(t) \in R^n$ is a piecewise smooth stochastic response process (characterizing whole system), and $\boldsymbol{\Phi}(t)$ is a function describing the nature of transition of the system from $q$-th sub-structure to $r$-th sub-structure.

The dynamics of the system at its q-th sub-structure can be modelled in the form of the following vectorial Itô stochastic differential equation

$$d\boldsymbol{Y}^{(q)}(t) = \boldsymbol{A}^{(q)}[\boldsymbol{Y}(t), t]\, dt + \boldsymbol{B}^{(q)}[\boldsymbol{Y}(t), t]\, d\boldsymbol{W}^{(q)}(t)$$
$$\boldsymbol{Y}^{(q)}(t_0) = \boldsymbol{Y}_0^{(q)}, \quad q = 1, 2, \ldots, K \tag{74}$$

where $\boldsymbol{A}^{(q)}(\boldsymbol{y}, t)$ is a drift vector of the system in its q-th sub-structure, $\boldsymbol{B}^{(q)}(\boldsymbol{y}, t)$ is the diffusion matrix in q-th sub-structure, and $\boldsymbol{W}^{(q)}(t)$ is the vectorial Wiener random process.

Transition from the $q$-th structure to $r$-th structure is modelled as the annihilation or absorption of the trajectories of the process $\boldsymbol{Y}^{(q)}(t)$ and creation of the trajectories of the process $\boldsymbol{Y}^{(r)}(t)$. In general, this transition takes place on the boundary $\partial D_{qr}(\boldsymbol{y})$ between $q$-th structure and $r$-th structure (a localized change of structure). In order to characterize the intensity of annihilation and creation of the trajectories two local matrix-valued functions are introduced: $\boldsymbol{c}(\boldsymbol{y}, t) = \{c_{qr}(\boldsymbol{y}, t)\}$ – the annihilation function, and $\boldsymbol{d}(\boldsymbol{y}, t) = \{d_{rq}(\boldsymbol{y}, t)\}$ – the creation (or birth) function. These functions must be constructed and included into the (extended) Fokker-Planck-Kolmogorov equation for the probability density of the process $\boldsymbol{Y}(t)$. More detailed information on stochastic dynamics of systems with varying structure can be found in [31]; in the doctoral thesis [32] the mechanical vibratory systems with varying structure were investigated with use of maximum entropy method. Also, the switching stochastic systems were studied in the thesis [33].

**5.2. Spatially extended/continuous systems.** As one can expect, more general formulation of the response-degradation problems should take into account explicitly effect of spatial variability of the system. We mean here the systems which are governed by the partial differential equations of the form

$$m\ddot{U}(\boldsymbol{r}, t) + c\dot{U}(\boldsymbol{r}, t) + \boldsymbol{M}_{\boldsymbol{r},t}[\boldsymbol{U}, D(\boldsymbol{r}, t)] = \boldsymbol{X}(\boldsymbol{r}, t, \gamma) \tag{75}$$

where $\boldsymbol{U}(\boldsymbol{r}, t)$ is unknown vector field defined for $\boldsymbol{r} \in G \subset R^n$, $t \in [0, \infty)$, $D(\boldsymbol{r}, t)$ is the degradation (scalar or vectorial) depending on spatial and temporal variables; $\boldsymbol{r} = (x, y, z)$. On the right-hand side of Eq. (75) we have random external excitation characterized by a spatial-temporal random field $\boldsymbol{X}(\boldsymbol{r}, t, \gamma)$. The third term on the left-hand side $\boldsymbol{M}_{\boldsymbol{r},t}[\cdot]$ of Eq. (75) denotes an operator (in general, nonlinear) with respect to $\boldsymbol{r}$ and $t$ acting on the fields indicated. The degradation space-time field $D(\boldsymbol{r}, t)$ is governed by its own evolution equation of the general form

$$\dot{D}(\boldsymbol{r}, t) = N_{\boldsymbol{r}}[D, \boldsymbol{U}] \tag{76}$$

where $N_{\boldsymbol{r}}$ is a differential, nonlinear operator with respect to spatial variables. A coupled system of Eqs. (75), (76) includes a variety of dynamical processes taking place in material media whose physical/mechanical properties evolve in time.

Evolving material structures have attracted much attention in the last years. For example, modern electronic and phonic devices are solid structures of small feature size. During fabrication and use diffusion processes can relocate matter, so the structure evolves in time. Collective actions of atoms, electrons and photons contribute to the free energy, which in turn contributes to thermodynamical forces, and these drive the configurational change of structure. But, also on the macroscopic level various types of defects, like cracks or voids in the material change their shapes and properties. In general, an evolving structure is a dynamical system, which can be modelled via the appropriate generalized coordinates or internal variables (cf. [34,35]). Analysis of fracture of solid materials containing various defects which interact with main/dominant crack, and whose fractional volume changes is also related to this new and prospective field (cf. [36,37]). Stochastic analysis of such problems is open for future research.

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