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Small time local controllability of driftless nonholonomic systems in a task-space

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In this paper a small time local controllability, naturally defined in a configuration space, is transferred into a task-space. It was given its analytical characterization and practical implications. A special attention was put on singular configurations. Theoretical considerations were illustrated with two calculation examples. An extensive comparison of the proposed construction with the controllability defined in an endogenous configuration space approach was presented pointing out to their advantages and disadvantages.

Key words: nonholonomic system, small time local controllability, configuration space, task-space

1. Introduction

Controllability [10] is a desired feature of any controlled system. It states that there exist admissible controls to steer a system between any two points in a configuration space. In this paper, being an extended version of the conference paper [5], a sub-class of systems described by continuous ordinary differential equations will be considered, namely driftless nonholonomic ones. Those systems appear frequently in robotics at the kinematic level and include models of wheeled mobile robots [4], free-floating space robots [3], and also special manipulators – nonholonomic ones [12]. For driftless nonholonomic systems, a stronger version of controllability is defined, namely a small time local controllability (abbreviated as STLC). Later on this type of controllability will be called Q-STLC as it is defined in a configuration space $\mathbb{Q}$. Q-STLC is defined around any configuration (locally), but, contrary to the global controllability, guarantees not only a possibility to reach any configuration in its neighborhood but also to

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control a volume of the maneuver. It means that reaching a close point requires only small displacements in space. This feature is especially important when configurations of nonholonomic systems are constrained due to obstacles, i.e. there exist inaccessible areas in the configuration space. It should be pointed out that approaching towards any point is a direct consequence of ability to generate any direction of motion with controls. Therefore a primary space to deal with is a space of velocities (directions) rather than a positional space.

In many practical cases not all coordinates of a configuration vector are important from the perspective of a task to be solved. For example, to plan a motion of a wheeled mobile robot positions of its wheels are not particularly important although the positions influence the model of the robot. Sometimes, however, they should be known when odometry is used as a control strategy in a robot navigation [1]. Therefore a map between a configuration space $\mathbb{Q}$ and a task-space $\mathbb{X}$ is introduced. Frequently, a dimension of the task-space is smaller than a dimension of the configuration space.

In this paper it will be shown how to transfer Q-STLC to the task-space (X-STLC) based on some Lie-algebraic constructions and a generalized Campbell-Baker-Hausdorff-Dynkin formula, gCBHD [13]. It will be explained how to reliably select a minimal set of controls preserving X-STLC. As a side-effect of this construction singularities due to the mapping are considered.

The paper is organized as follows: in Section 2 mathematical preliminaries are introduced necessary to recall Q-STLC. In Section 3 Q-STLC is transferred into the task-space and singularities of the mapping are defined and discussed. Section 4 includes some illustrative computational examples. In Section 5 the proposed X-STLC is compared with controllability offered by the endogenous configuration space approach [14]. Section 6 concludes the paper.

2. Q-STLC preliminaries

Driftless nonholonomic systems are described by the equation

$$\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u} = \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{q})u_i, \tag{1}$$

where $\boldsymbol{q}$ is a configuration, $\dim \mathbb{Q} = n$, $\boldsymbol{u}$ are controls (at kinematic level having an interpretation of velocities), $\dim \boldsymbol{u} = m < n$; $\boldsymbol{g}_i(\boldsymbol{q})$ are C^∞ vector fields called generators of system (1). Algebraically Q-STLC is described by Chow's theorem [2]

$$\forall \boldsymbol{q} \in \mathbb{Q} \quad \operatorname{rank}(LA(\boldsymbol{G}(\boldsymbol{q}))) = n, \tag{2}$$

which states that the Lie algebra LA spanned by generators, i.e. columns of the matrix $\boldsymbol{G}(\boldsymbol{q})$, is of the full rank at any configuration. The Lie algebra is constructed by an iterative application of the Lie bracket operation. The Lie bracket produces a new vector field from two vector fields according to the formula

$$[\boldsymbol{A}, \boldsymbol{B}] = \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{q}}\boldsymbol{A} - \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{q}}\boldsymbol{B}. \tag{3}$$

The iterative procedure is initialized with generators and follows with previously generated vector fields. To each vector field generated with this procedure a degree can be assigned and equal to the number of generators appearing in a symbolic description of the vector field (alternatively and equivalently: a number of Lie bracket operations used to generate a given vector field plus one). Vector fields sharing the same i-th degree are grouped into the i-th layer of vector fields. Not all vector fields generated with this procedure are independent as for any $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ the Lie bracket satisfies the following properties

1) the anti-symmetry $[\boldsymbol{B}, \boldsymbol{A}] = -[\boldsymbol{A}, \boldsymbol{B}]$,
2) the Jacobi identity $[\boldsymbol{A}, [\boldsymbol{B}, \boldsymbol{C}]] + [\boldsymbol{C}, [\boldsymbol{A}, \boldsymbol{B}]] + [\boldsymbol{B}, [\boldsymbol{C}, \boldsymbol{A}]] = \boldsymbol{0}$.

In order to avoid the redundancy in vector fields, and not to lose any desired properties, instead of using the Lie algebra LA in Eq. (2), it is advised to exploit its basis. There are at least three such bases due to Lyndon, Chibrikov, and Ph. Hall. The last one will be used in this paper [7]. Processing the basis simplifies also analytical and numerical computations of ranks of some matrices as the matrices become smaller in their sizes.

It should be noted that vector fields have got an interpretation of velocities (when scaled with controls) as they impact the velocity $\dot{\boldsymbol{q}}$ thus show possible directions of motion. Therefore the Q-STLC can be re-read as a possibility to move the configuration in any direction at any configuration.

The considered systems are nonholonomic ones. For the systems, this property is decided by a construction of a small flag of distribution $\boldsymbol{D}_0$ spanned by generators of system (1). Next distributions are generated iteratively according to the rule

$$\boldsymbol{D}_{i+1} = \boldsymbol{D}_i \oplus [\boldsymbol{D}_0, \boldsymbol{D}_i], \quad i = 0, 1, \ldots, \tag{4}$$

where the Lie bracket of distributions should be interpreted as a set of Lie brackets composed from any pair of vector fields, when the first/second one belongs to the first/second distribution. At a given configuration $\boldsymbol{q}$, to each distribution its rank can be assigned

$$\dim \boldsymbol{D}_i(\boldsymbol{q}) = f_i^Q(\boldsymbol{q}), \quad i = 0, 1, \ldots \tag{5}$$

which is algebraically checked as a rank of the matrix with columns formed with vector fields spanning a given distribution and evaluated at a given configuration. A vector composed of $f_i^Q(\boldsymbol{q}), i = 1, \ldots$ is called a growth vector at the configuration $\boldsymbol{q}$. When the vector does not depend on configuration (is constant), it is just a growth vector. The minimal s for which $f_s^Q(\boldsymbol{q}) = n$ is a nonholonomy degree at $\boldsymbol{q}$, and when s does depend on $\boldsymbol{q}$ – a nonholonomy degree. With these notations introduced, we are in a position to define nonholonomic system as a system for which degree of a certain distribution reaches the dimension of the configuration space

$$\forall \boldsymbol{q} \in \mathbb{Q} \;\; \exists s \quad \dim(\boldsymbol{D}_s(\boldsymbol{q})) = f_s^Q(\boldsymbol{q}) = n. \tag{6}$$

When the condition (6) is not satisfied, a system is holonomic and can be steered within a lower dimensional sub-space of the configuration space. It can be noticed that the distribution $\boldsymbol{D}_i$ is composed of vector fields with degrees up to the $(i+1)$-st, inclusively.

A practical procedure to calculate a rank of a distribution at a given configuration is to generate appropriate vector fields, check whether the generated vector field is linearly dependent on previously considered. If so, it is excluded immediately, otherwise it is added to the set generated. After each addition, the rank condition (2) is checked. The rank may either grow by one or remain the same. In the second case, the just added vector field (an appropriate example will be given later on) can not be expressed as a linear combination of its predecessors but as a vector (vector field evaluated at a given configuration) becomes dependent on vectors obtained from the previously generated vector fields. Almost the same procedure applies while checking Q-STLC:

Step 1 The initial matrix used to check Q-STLC (it is referred to as a Q-matrix) is composed of columns of the matrix $\boldsymbol{G}(\boldsymbol{q})$ from Eq. (1), i.e. degree one vector fields which belong to layer $i = 1$.

Step 2 A layer number is increased $i \leftarrow i + 1$ and using the Lie bracket new vector fields are generated that belong to the i-th layer and to the Ph. Hall basis as well.

Step 3 If the new vector field does not depend on previously generated, it is added to the Q-matrix as its new column.

Step 4 The rank condition of the Q-matrix is determined.

Step 5 The procedure is completed when either the rank of the Q-matrix attains the dimension of the configuration space or it is known that it can not be reached (when none of vector fields from the i-th layer has not been added to the Q-matrix).

A sub-matrix of the Q-matrix composed of vector fields up to the i-th degree is denoted as $F_i^Q(\boldsymbol{q})$ and its rank at a given configuration $\boldsymbol{q}$ is equal to $f_i^Q(\boldsymbol{q})$. For example, the very first elements of the Ph. Hall basis for the system (1) spanned by two generators $\boldsymbol{g}_1, \boldsymbol{g}_2$ are the following

$$\mathbb{H} = \{\boldsymbol{g}_1,\; \boldsymbol{g}_2,\; [\boldsymbol{g}_1, \boldsymbol{g}_2],\; [\boldsymbol{g}_1,\; [\boldsymbol{g}_1, \boldsymbol{g}_2]],\; [\boldsymbol{g}_2,\; [\boldsymbol{g}_1, \boldsymbol{g}_2]], \ldots\}.$$

After evaluation at $\boldsymbol{q}$, to compute the rank $f_1^Q(\boldsymbol{q})$, it is necessary to take first two elements, to determine $f_2^Q(\boldsymbol{q})$ – three, and $f_3^Q(\boldsymbol{q})$ – five elements, etc.

3. X-STLC

Now we are in a position to extend the concepts introduced in the Q-space to a task-space. The model (1) described by differential equations is supplemented with a static output function

$$\boldsymbol{x} = \boldsymbol{k}(\boldsymbol{q}), \quad \dim(\mathbb{X}) = r \leqslant n, \tag{7}$$

mapping a configuration into a point within the task-space $\mathbb{X}$. Vector fields collected in the matrix $\boldsymbol{G}(\boldsymbol{q})$, cf. Eq. (1), interpreted as velocities, are transferred into the tasks-space via Jacobian $\boldsymbol{J} = \partial\boldsymbol{k}/\partial\boldsymbol{q}$ matrix of the output function

$$\dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{k}(\boldsymbol{q})}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q})\boldsymbol{u}. \tag{8}$$

Analogously to presented constructions in the configuration space, $\mathbb{Q}$, matrices $\boldsymbol{F}_i^X(\boldsymbol{q})$ and their ranks f_i^X at configuration $\boldsymbol{q}$ are defined in the task-space as follow

$$\boldsymbol{F}_i^X(\boldsymbol{q}) = \boldsymbol{J}(\boldsymbol{q}) \cdot \boldsymbol{F}_i^Q(\boldsymbol{q}), \quad f_i^X(\boldsymbol{q}) = \operatorname{rank}(\boldsymbol{F}_i^X(\boldsymbol{q})), \quad i = 1, \ldots. \tag{9}$$

Now, it is easy to define that the system (1), (7) is X-STLC at the configuration $\boldsymbol{q}$ if only

$$\exists\, \hat{p}(\boldsymbol{q}) \in \mathbb{N} : \quad f_{\hat{p}}^X(\boldsymbol{q}) = r \tag{10}$$

and to formulate an analogon of the Chow's theorem in the task-space: the system given by Eqns. (1), (7) is X-STLC if only it is X-STLC at any configuration $\boldsymbol{q} \in \mathbb{Q}$. Obviously, the natural number $\hat{p}$, if only exists, should be minimal and may depend on $\boldsymbol{q}$. Its maximal/minimal value is called maximal/minimal X-layer

$$\hat{p}_{\max} = \max_{\boldsymbol{q}\in\mathbb{Q}} \hat{p}(\boldsymbol{q}), \quad \hat{p}_{\min} = \min_{\boldsymbol{q}\in\mathbb{Q}} \hat{p}(\boldsymbol{q}). \tag{11}$$

In a typical case

$$\hat{p}_{\max} - \hat{p}_{\min} = 1.$$

From a theoretical standpoint, an interesting question can be posed whether there exist degenerated systems with the difference higher than one. If $\hat{p}_{\max} = \hat{p}_{\min} = \hat{p}$, then $\hat{p}$ is called a nonholonomy degree in the task-space by analogy to the term defined in the configuration space (this name is slightly unfortunate as the nonholonomy is originally defined in Q-space and is tightly related with differential equations but in the X-space it is only mirrored from the Q-space via Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ in Eq. (8)) into the X-space.

It is worth of noticing that for any (controllability) property defined in the task-space, the evaluation of vector fields is still performed at a given $\boldsymbol{q}$ in the configuration space and then the velocity is moved into the point $\boldsymbol{x} = \boldsymbol{k}(\boldsymbol{q})$ via Jacobian of the output mapping. Theoretically, there exist two possibilities to release dependence of properties in the X-space from the Q-space. The first (weak) one requires to find any configuration $\boldsymbol{q}^\star$ where the property is satisfied at $\boldsymbol{x}^\star \in \mathbb{X}$ and $\boldsymbol{k}(\boldsymbol{q}^\star) = \boldsymbol{x}^\star$. The second (strong) one rely on finding a sub-space $\{\boldsymbol{q} : \boldsymbol{k}(\boldsymbol{q}) = \boldsymbol{x}^\star\} = \mathbb{Q}_{\boldsymbol{x}^\star} \subset \mathbb{Q}$ and a given property has to be satisfied at any configuration in the sub-space. None of the two approaches is universal and the second one is particularly difficult to check.

3.1. Singular configurations

Let us start with recalling some terms and definitions valid for stationary manipulators. A manipulator is described by its forward kinematics and the Jacobian matrix

$$\boldsymbol{x} = \boldsymbol{k}(\boldsymbol{q}), \quad \boldsymbol{J}(\boldsymbol{q}) = \partial \boldsymbol{k}/\partial \boldsymbol{q}, \tag{12}$$

with $\dim \mathbb{X} = r$. At singular configurations of manipulators, a motion at some directions is impossible. Algebraically, it means that a rank of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ of forward kinematics is smaller than the maximal attainable (equal to the dimension of the task-space) [15]. Equivalently, a rank of the manipulability matrix [11]

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q}) \tag{13}$$

is smaller than r. For nonholonomic systems, this interpretation of singularities in the Lie-algebraic setting can not be applied directly as there are only m generators-directions of motion so without maneuvering the system can move within the space $span_{\mathbb{R}}\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_m\}$ which is clearly only a sub-space of n dimensional space. Consequently, according to the definition of singularities valid for manipulators, any configuration would be singular. A motion in directions outside the sub-space is realized as a net-motion composed of some elementary sub-motions. Therefore, for nonholonomic systems a new definition of singular configurations is required which covers not only those directions directly realizable but also those produced as a net motion. Consequently, at a singular configuration $\boldsymbol{q}$ (corresponding to the output function (7)), the rank of matrix in definition (9) is not maximal possible

$$f_i^X(\boldsymbol{q}) < r, \quad i = 1, 2, \ldots. \tag{14}$$

It is known from the matrix analysis [9] that

$$\operatorname{rank}(\boldsymbol{J}(\boldsymbol{q}) \cdot \boldsymbol{F}_i^Q(\boldsymbol{q})) \leqslant \min\left(\operatorname{rank}(\boldsymbol{J}(\boldsymbol{q})), \operatorname{rank}(\boldsymbol{F}_i^Q(\boldsymbol{q}))\right). \tag{15}$$

By definition, the considered system (1) is nonholonomic one thus, for some $p^\star$, the matrix $\boldsymbol{M}^Q_{p^\star}(\boldsymbol{q})$ is of the full rank and

$$\operatorname{rank}(\boldsymbol{J}(\boldsymbol{q}) \cdot \boldsymbol{F}^Q_{p^\star}(\boldsymbol{q})) = \operatorname{rank}(\boldsymbol{J}(\boldsymbol{q})). \tag{16}$$

There are two consequences of (16):

1) nonholonomic systems (1) do not introduce singular configurations, thus all singular configurations are due to singularity of mapping (7),

2) the system (1), (7) not always is X-STLC, even when the system (1) is Q-STLC. However, it is X-STLC in all non-singular configurations of mapping (7).

For some special, but practical, forms of mapping (7) (known as projections, i.e. a proper ($r < n$) subset of coordinates $\boldsymbol{q}$ is selected), the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ is constant and of the full rank. For this sub-class of output functions, according to (10), (16), Q-STLC implies X-STLC, and the matrix $\boldsymbol{F}^X_i(\boldsymbol{q})$ is composed of selected rows of the matrix $\boldsymbol{F}^Q_i(\boldsymbol{q})$. Similarly to manipulators, also in systems (1), (7) singularities can be derived via mobility matrices (corresponding to manipulability matrix (7) but obtained for mobile robots)

$$\boldsymbol{M}^Q_i(\boldsymbol{q}) = \boldsymbol{F}^Q_i(\boldsymbol{q})\boldsymbol{F}^Q_i(\boldsymbol{q})^T, \qquad \boldsymbol{M}^X_i(\boldsymbol{q}) = \boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{M}^Q_i(\boldsymbol{q})\,\boldsymbol{J}^T(\boldsymbol{q}). \tag{17}$$

3.2. Realization of vector fields with controls

The previous considerations dealt with some vector fields and ranks of matrices composed of the vector fields without taking into account how they should be produced with real controls. An appropriate tool relating controls with vector fields is the (generalized) Campbell-Baker-Hausfdorff-Dynkin formula [13]. At a given configuration $\boldsymbol{q}$, the formula maps controls into a series of infinitesimal displacements (thus velocities) in a configuration space expressed as a linear combination of vector (vector fields evaluated at $\boldsymbol{q}$) with control dependent coefficients, cf. Fig. 1. A standard method used in the Lie algebraic motion planning

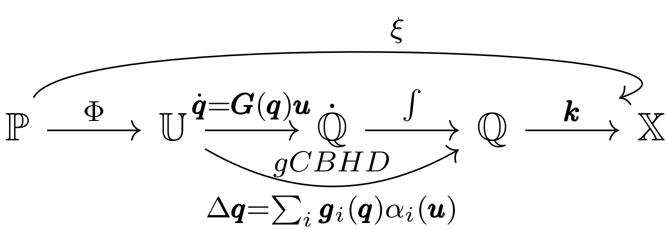

Figure 1: Spaces and transformations among them.

in a configuration space to avoid an infinite dimensional space of controls is to select a fixed basis (polynomials or harmonic functions are the most popular) $\phi_i(t)$ in the space of controls, and select a finite representation of controls within this basis

$$u_i(t) = \sum_{j=1}^{K_i} p_{i,j}\phi_j(t), \quad i = 1, \ldots, m \tag{18}$$

Coefficients $p_{i,j}$ are determined based on a required direction of motion at a given configuration. In order to reduce computational complexity of this task, values of K_i should be relatively small, but large enough not to lose controllability [6]. It is known that the higher layer is required to satisfy condition (6), the more numerous representation of controls should be. From the gCBHD formula [7] it follows that vector fields that belong to the same layer are qualitatively the same as controls generating them are uniform with respect to degree of vector field generated. In practice of motion planning with Lie algebraic methods, it means that even a vector field is not required to a satisfy controllability condition, still it should be taken into account while generating a desired motion, if only any other vector field with the same degree was actively involved in checking the Q-STLC condition. The meaning of previously defined maximal/minimal X-layer relies on their relationship with controls required to generate a motion of system (1). Transferring actions of controls from the configuration space into the (lower dimensional, $r < n$) task-space gives a theoretical chance to decrease the number of coefficients to describe controls, cf. Eq. (18) comparing to the number required to steer a system within its configuration space $\mathbb{Q}$.

4. Computational examples

Let us consider two input driftless nonholonomic system (1), $m = 2$, with harmonic controls represented as

$$\begin{aligned} u_1(t) &= a_1 + a_2\sin(\omega t) + a_3\cos(\omega t), \\ u_2(t) &= b_1 + b_2\sin(\omega t) + b_3\cos(\omega t), \end{aligned} \tag{19}$$

where T is a small time of motion and $\omega = 2\pi/T$ represents the basis frequency. The vector of coefficients parameterizing control is equal to $\boldsymbol{p} = (a_1, a_2, a_3, b_1, b_2, b_3)^T$. Applying the gCBHD formula, one gets

$$\Delta\boldsymbol{q} \simeq \left[\boldsymbol{g}_1\ \boldsymbol{g}_2\ [\boldsymbol{g}_1, \boldsymbol{g}_2]\ [\boldsymbol{g}_1, [\boldsymbol{g}_1, \boldsymbol{g}_2]]\ [\boldsymbol{g}_2, [\boldsymbol{g}_1, \boldsymbol{g}_2]] \ldots\right] \left[\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5 \ldots\right]^T \tag{20}$$

and for controls (19) dominant (for small T) coefficients $\boldsymbol{\alpha}$ are the following

$$\begin{aligned}
&\alpha_1 = a_1, \qquad \alpha_2 = b_1,\\
&\alpha_3 = (2a_2b_1 - 2a_1b_2 + a_3b_2 - a_2b_3)/(4\pi),\\
&\alpha_4 = \{+a_2b_2(-3a_1+2a_3)+a_2^2(3b_1-2b_3)+(4a_1-a_3)(a_1b_3-a_3b_1)\}/(16\pi^2),\\
&\alpha_5 = \{-a_2b_2(-3b_1+2b_3)-b_2^2(3a_1-2a_3)+(4b_1-b_3)(a_1b_3-a_3b_1)\}/(16\pi^2).
\end{aligned} \tag{21}$$

From a practical point of view, an interesting question to answer is how to select controls in the parametric form (19). It is known that their number should be equal to n at least, and the mapping $\boldsymbol{p} \rightarrow \Delta\boldsymbol{q}$, based on (20), (21) should be surjective. Under those circumstances there is a possibility to generate any $\Delta\boldsymbol{q}$ with appropriate values of $\boldsymbol{p}$. Thus, the case when representation of $u_1(t)$ is numerous (more than n coefficients) while the other control is equal to zero should be excluded.

From a numerical point of view, the vector $\boldsymbol{p}$ should be relatively short but long enough not to lose controllability. The procedure of selecting a vector parameterizing controls is performed in off-line mode and it is possible to evaluate many sets of parameters starting from those less numerous. Later on, only minimal sets of parameters required to get X-STLC for some output functions are of interest.

In the first example the simplest wheeled mobile robot – the unicycle, Fig. 2a, is considered with the configuration $\boldsymbol{q} = (q_1, q_2, q_3)^T = (x, y, \theta)^T$ and generators equal to

$$\boldsymbol{g}_1 = (\cos(q_3),\ \sin(q_3),\ 0)^T, \qquad \boldsymbol{g}_2 = (0,\ 0,\ 1)^T.$$

The vector fields $\boldsymbol{g}_1, \boldsymbol{g}_2$ supplemented with $[\boldsymbol{g}_1, \boldsymbol{g}_2] = (\sin(q_3), -\cos(q_3), 0)^T$ satisfy the Q-STLC condition and nonholonomy of the system. For a trivial output function $\boldsymbol{k}(\boldsymbol{q}) = q_3$ the first layer of vector fields is enough to satisfy X-STLC, thus $\hat{p} = 1$. Moreover, a minimal set of controls generating motion in any direction in the X-space is $\boldsymbol{p} = (b_1)^T$. For the output function $\boldsymbol{k}(\boldsymbol{q}) = (q_1, q_2)^T$, more vector fields are required. In this case $\boldsymbol{g}_1, \boldsymbol{g}_2, [\boldsymbol{g}_1, \boldsymbol{g}_2]$ satisfy X-STLC for

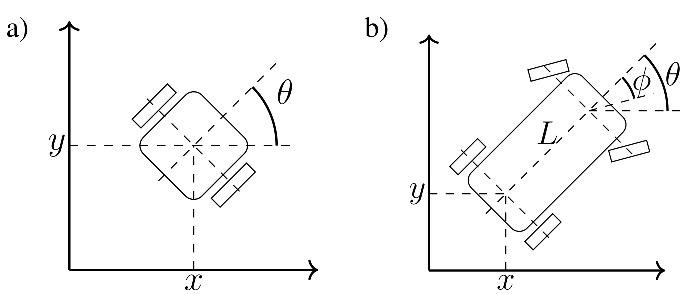

Figure 2: Unicycle (a) kinematic car (b) with coordinates defined.

any $\boldsymbol{q} \in \mathbb{Q}$, thus $\hat{p} = 2$. Vectors of coefficients of controls for motions in different directions follow

$$\begin{array}{lll} \boldsymbol{p} = (a_1, b_1, b_2)^T & \text{when} & a_1 \neq 0, \\ \boldsymbol{p} = (a_1, b_1, a_2)^T & \text{when} & b_1 \neq 0, \\ \boldsymbol{p} = (a_2, b_3)^T \text{ or } \boldsymbol{p} = (a_3, b_2)^T & \text{when} & b_1 = a_1 = 0. \end{array} \tag{22}$$

It appears (22) that there exist three minimal sets of parameters generating locally a motion in any direction. In regions of the configuration space requiring $a_1 \simeq b_1 \simeq 0$, the first two sets in (22) are admissible, but amplitudes of controls are likely to be very high and practically unrealizable. Therefore, a small redundancy is advised in selection of parameters of controls. For the output function $\boldsymbol{k}(\boldsymbol{q}) = (q_1, q_3)^T$, vector fields $\boldsymbol{g}_1, \boldsymbol{g}_2$ are enough to satisfy X-STLC for $q_2 \neq \pi/2 + k\pi$, and inside the region one more vector field $[\boldsymbol{g}_1, \boldsymbol{g}_2]$ is required as $\hat{p}_{\min} = 1$ and $\hat{p}_{\max} = 2$. When the output function is in the form $k(\boldsymbol{q}) = (q_1 \cos(q_3) + q_2 \sin(q_3), q_3)^T$ (with a non-singular Jacobian matrix everywhere), then $\hat{p}_{min} = \hat{p}_{max} = \hat{p} = 1$ and a single layer of vector fields is enough to satisfy X-STLC. From the aforementioned examples, it can be concluded that there is no single minimal set of parameters of controls to satisfy X-STLC at any configuration $\boldsymbol{q}$.

The second example of system (1) is a kinematic car, Fig. 2b, with configuration $\boldsymbol{q} = (q_1, q_2, q_3, q_4)^T = (x, y, \theta, \phi)^T$ and generators

$$\boldsymbol{g}_1(\boldsymbol{q}) = (L\, c_3 c_4, L\, s_3 c_4, s_4, 0)^T, \quad \boldsymbol{g}_2(\boldsymbol{q}) = (0, 0, 0, 1)^T,$$

where $c_i = \cos(q_i)$, $s_i = \sin(q_i)$, and L denotes the distance between front and rear axle of the car; it is assumed that $L = 1$. After computing necessary Lie brackets (the first and the last column are present just to improve readability) matrix $\boldsymbol{F}_3^Q(\boldsymbol{q})$ is obtained

$$\boldsymbol{F}_3^Q(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{g}_1 & \boldsymbol{g}_2 & [\boldsymbol{g}_1, \boldsymbol{g}_2] & [\boldsymbol{g}_1, [\boldsymbol{g}_1, \boldsymbol{g}_2]] & [\boldsymbol{g}_2, [\boldsymbol{g}_1, \boldsymbol{g}_2]] & \\ c_3 c_4 & 0 & c_3 s_4 & -s_3 & c_3 c_4 & x \\ s_3 c_4 & 0 & s_3 s_4 & c_3 & s_3 c_4 & y \\ s_4 & 0 & -c_4 & 0 & s_4 & \theta \\ 0 & 1 & 0 & 0 & 0 & \phi \end{bmatrix}. \tag{23}$$

The determinant of the sub-matrix composed of the first four columns is constant thus vector fields $\boldsymbol{g}_1, \boldsymbol{g}_2, [\boldsymbol{g}_1, \boldsymbol{g}_2], [\boldsymbol{g}_1, [\boldsymbol{g}_1, \boldsymbol{g}_2]]$ satisfy the Q-STLC condition and the vector field $[\boldsymbol{g}_2, [\boldsymbol{g}_1, \boldsymbol{g}_2]] = \boldsymbol{g}_1$ depends on the other everywhere so it is useless. Any triple of coordinates selected from among $\boldsymbol{q}$ as an output function does not satisfy X-STLC, when associated with the first three vector fields (i.e. $\boldsymbol{F}_2^Q(\boldsymbol{q})$). Consequently, in order to satisfy X-STLC the vector field $[\boldsymbol{g}_1, [\boldsymbol{g}_1, \boldsymbol{g}_2]]$

should be also used and the number of parameters of controls to satisfy Q-STLC and X-STLC is exactly the same. When the output function belongs to the family $k(\boldsymbol{q}) = (\xi(q_3, q_4)\{q_1\cos(q_3) + q_2\sin(q_3)\}, q_3, q_3, q_4)^T$ with $\xi \in \mathbb{C}^1$ and $\forall q_3, q_4\ \xi \neq 0$, then $\hat{p} = 2$ an two layers of vector fields are enough to satisfy X-STLC but they do not satisfy Q-STLC.

5. Comparison with the endogenous configuration space approach

An endogenous configuration space method (ECSM) [14] is applicable to a wider class of system (1) (with an uncontrolled drift $\boldsymbol{g}_0(\boldsymbol{q})$ added to the right hand side of the equation) than Lie-algebraic method (LAM). All further considerations are taken on a fixed time interval $[0, T]$ [14]. For given (initially assumed) controls $\boldsymbol{u}(\cdot)$ and a given initial $\boldsymbol{q}_0$, a linear approximation along the trajectory initialized at $\boldsymbol{q}_0$ and corresponding to controls $\boldsymbol{u}(\cdot)$ is performed. Thus Eq. (1) is transformed into [14]

$$\dot{\boldsymbol{\xi}} = \boldsymbol{A}(t)\boldsymbol{\xi} + \boldsymbol{B}(t)\boldsymbol{v}, \quad \boldsymbol{\eta} = \boldsymbol{C}(t)\boldsymbol{\xi}, \tag{24}$$

where $\boldsymbol{v}(\cdot)$ is a small variation of controls, $\boldsymbol{\eta}$ is a small displacement variation in the output space and the time-dependent matrices are the following

$$\boldsymbol{A}(t) = \frac{\partial \boldsymbol{G}(\boldsymbol{q}(t))\boldsymbol{u}(t)}{\partial \boldsymbol{q}}, \;\; \boldsymbol{B}(t) = \frac{\partial \boldsymbol{G}(\boldsymbol{q}(t))\boldsymbol{u}(t)}{\partial \boldsymbol{u}} = \boldsymbol{G}(\boldsymbol{q}), \;\; \boldsymbol{C}(t) = \frac{\partial \boldsymbol{k}(\boldsymbol{q}(t))}{\partial \boldsymbol{q}}, \tag{25}$$

where the fundamental matrix $\boldsymbol{\Phi}(t, s)$ is the solution of differential equation with the identity initial condition [8]

$$\frac{\partial \boldsymbol{\Phi}(t, s)}{\partial t} = \boldsymbol{A}(t)\boldsymbol{\Phi}(t, s), \qquad \boldsymbol{\Phi}(s, s) = \boldsymbol{I}_n\,. \tag{26}$$

Here one can see the first but the most important difference between LAM and ECSM as names and interpretations of spaces differ. In the former: a configuration space is $\mathbb{Q}$ and the input space $\mathbb{U}$ while in the latter: a configuration space is $\mathbb{U}$ and $\mathbb{Q}$ serves as a space where actions of controls are primarily observed. As a consequence, the configuration space is finite dimensional ($= n$) in the former case, and infinite dimensional in the latter one.

Controls $\boldsymbol{u}(\cdot)$ applied on time horizon $[0, T]$ to system (1) initialized at $\boldsymbol{q}_0$ attain a point $\boldsymbol{q}(T)$. Around that point, a displacement in the Q-space due to controls $\boldsymbol{v}(\cdot)$ is equal to [14]

$$\boldsymbol{\eta}(T) = \boldsymbol{C}(T)\int_{s=0}^{T} \boldsymbol{\Phi}(T, s)\boldsymbol{B}(s)\boldsymbol{v}(s)ds, \tag{27}$$

and singular configurations are determined based on the mobility matrix

$$\boldsymbol{M}^U(T) = \int_{s=0}^{T} \boldsymbol{\Phi}(T,s)\boldsymbol{B}(s)\boldsymbol{B}^T(s)\boldsymbol{\Phi}^T(T,s)ds \tag{28}$$

mapped into the output space

$$\boldsymbol{M}^X(T) = \boldsymbol{C}(T)\boldsymbol{M}^U(T)\boldsymbol{C}^T(T). \tag{29}$$

Singular configurations (i.e. controls) appear when the rank of the matrix $\boldsymbol{M}^X(T)$ drops below the maximal available value equal to r. Now another difference is visible: in LAM singularity is considered around $\boldsymbol{q}_0$ while in ECSM around $\boldsymbol{q}(T)$. However, in both cases a singularity means a rank deficit and inability to move along some directions. Moreover, both methods take advantage from the theory developed for manipulators, cf. Eq. (13), (17), (28), (29). In ECSM controls-configurations $\boldsymbol{u}(\cdot) = \boldsymbol{0}$ are always singular and other singular configurations can be computed numerically only as $\boldsymbol{\Phi}(T,s)$ can not be computed analytically for almost any practical system. In LAM singular configurations are given by analytic expressions. There are also other differences between ECSM and LAM, mainly due to implementation issues, that are collected in Table 1. From a practical point of view, both methods, LAM and ECSM, are useful in motion planning of driftless nonholonomic systems.

Table 1: Lie-algebraic method vs. the endogenous configuration space method – the comparison.

Factor	Methods	
	LAM	ECSM
input space	$\boldsymbol{u}$	–
configuration space	$\boldsymbol{q}$	$\boldsymbol{u}$
transfer space	–	$\boldsymbol{q}$
output space	$\boldsymbol{x}$	$\boldsymbol{x}$
time of motion $T > 0$	small	any
singularities	analytic	numeric
range of planning	local	global
controls usually	parameterized	parameterized
continuity of controls	no	yes
motion controlled around	$\boldsymbol{q}_0$	$\boldsymbol{q}(T)$
computational complexity	low	heavy
controlling localization of a trajectory	easy	difficult

6. Conclusions

In this paper some objects and terms related with a small time local controllability, known from the Lie algebraic method of motion planning applied in a configuration space, have been transferred into a task-space via output function and its Jacobian. The output function can introduce some singularities whose nature is quite different than singularities of stationary manipulators. Nevertheless, singularities denote that an infinitesimal motion in some directions is impossible to realize. It appears that the only source of singularities in this setting is the output function when the rank of its Jacobian matrix is smaller than maximal possible. A practical aspect of introducing the task space with a smaller dimension than the configuration space is the possibility to decrease the number of parameters of controls required to satisfy small time local controllability condition. An important observation made is that there is no one universal minimal parameter setting for controls to satisfy the condition at any point. Therefore, while using Lie-algebraic methods of motion planning it is advised to switch among a few parameterizations depending on a current configuration. It appears that the presented Lie-algebraic construction has got a lot in common with the endogenous configuration space method but interpretations of some space differ substantially the methods.

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