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# Generalized PI observer design for descriptor linear system

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A kind of generalized proportional-integral(GPI) observer for descriptor linear systems is introduced. We first propose two complete parametric solutions to generalized Sylvester matrix equation corresponding to the left eigenvector matrices in the case of Jordan form. Then a parametric design approach for the observer is presented. The proposed method provides all parametric expression of the gain matrices and the corresponding finite left eigenvector matrix and guarantees the regularity and impulse-freeness of the expanded error system. Two numerical examples are given to explain the design procedure and illustrate the effectiveness of the proposed method.

**Key words:** generalized proportional-integral observer, Sylvester matrix equation, regularity, impulse-freeness

## 1. Introduction

In many control systems and applications, the system states aren't always able to measure on account of the measured error and the infeasibility of measured approach. For this situation, the observer can be designed to estimate the unmeasured state. In accordance with the control theory, the integral actions has important effect on increasing the steady-state accuracy of control system. In virtue of the duality of controllability and observability, the integral actions can be injected to the observer. There are a lot of wally character for this type of observer since it not only use the current information of system, but also utilize the departed information of system. The observer that contains propor-

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tional and integral terms is entitled as proportional-integral (PI) observer. The first PI observer was presented by Wojciechowski [1] for SISO linear normal systems. After that, the thought was extended to the multivariable linear system. Since this type of observers can provide more degrees of design freedom than the conventional proportional ones, many scholars have been appealed to study it and the relative problems. The problem of disturbance and fault detection using PI observer was studied by Shafai et al. [2]. PI observer for attenuating noise was investigated by Krishna and Pousga [3]. Recent years, Duan et al. [4] proposed a parametric design method to PI observer for conventional continuous-time linear systems. The method proposed in [4] provided all the degrees of design freedom, therefore great convenience could be obtained to further system design.

Relative to normal system observer, the descriptor system observer is widely used in engineering application, for example, power system, electrical networks, social economic systems, and so on. The research with regard to descriptor system observer are relatively abundant. For instance, the descriptor system observer can be divided into regular system observer [5] and nonregular system observer [6] according to the system regularity; Luenberger observer [7] and descriptor observer [8] on the basis of the form of observer; certain system observer [9] and uncertain system observer [10] considering of disturbance; continuous-time system observer [11] and discrete-time system observer [12] for the system continuity.

Similar to conventional linear system theory, the issue of designing PI observer for descriptor linear system has also been studied by many researches. Wu and Duan [13, 14] studied the PI observer for continuous-time and discrete-time descriptor linear system. In [15, 16], the Luenberger-type normal full-order and reduced-order PI observers are introduced for non-square descriptor linear systems with unknown inputs. The Luenberger-type full order PI state observer for square descriptor linear systems is investigated in [17]. Moreover, some other kinds of PI observers for descriptor linear systems are introduced by [18, 19].

In this paper, we design a generalized proportional-integral(GPI) observer for descriptor linear system taking advantage of the parametric design method [20–22]. We firstly get the corresponding generalized Sylvester matrix equation for the expanded error system. Then the equation is solved by taking Smith form reduction or the right coprime decomposition for the matrices. Finally, we solve gain matrices of the observer based on the above solution and some parameters which represent the degrees of freedom and satisfy certain constrains. This approach not only guarantees the regularity of the expanded error system, but also eliminates its impulsive responses. Furthermore, it provides all the degrees of freedom in the problem and which can be utilized to achieve various desired system specifications and performances.

## 2. Problem formulation

Consider the following time-invariant descriptor linear system

$$\begin{cases} E\dot{x} = Ax + Bu, \\ y = C_1\dot{x} + C_0 x, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$ and $y \in \mathbb{R}^m$ are, respectively, state vector, input vector, and output vector, $E,\ A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C_0,\ C_1 \in \mathbb{R}^{m\times n}$ are known real matrices and satisfy the following assumptions,

Assumption A1: $\operatorname{rank}\begin{bmatrix} C_0 & C_1 \end{bmatrix} = m \leqslant n$.

Assumption A2: $\operatorname{rank}\begin{bmatrix} sE - A \\ sC_1 + C_0 \end{bmatrix} = n$.

For system (1), we introduce an observer in the following form:

$$\begin{cases} E\dot{\hat{x}} = A\hat{x} + Bu + L\left(y - C_1\dot{\hat{x}} - C_0\hat{x}\right) + Fw, \\ \dot{w} = K\left(y - C_1\dot{\hat{x}} - C_0\hat{x}\right), \end{cases} \tag{2}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimated state vector, $w \in \mathbb{R}^p$ is a vector representing the integral of the weighted output estimation error, and $L \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{n\times p}$, $K \in \mathbb{R}^{p\times m}$ are the observer gains.

**Definition 1** *The system (2) is called a GPI observer for system (1) if, for any admissible initial conditions $x(0)$, $\hat{x}(0)$ and $w(0)$ and any input $u(t)$, the following relations hold:*

$$\lim_{t\to\infty} (x(t) - \hat{x}(t)) = 0, \qquad \lim_{t\to\infty} w(t) = 0.$$

Let $e(t) = x(t) - \hat{x}(t)$, then combining (1) and (2) gives the following expanded error system:

$$E_0 \begin{bmatrix} \dot{e} \\ \dot{w} \end{bmatrix} = A_0 \begin{bmatrix} e \\ w \end{bmatrix} \tag{3}$$

with

$$E_0 = \begin{bmatrix} E + LC_1 & 0 \\ -KC_1 & I_p \end{bmatrix}, \qquad A_0 = \begin{bmatrix} A - LC_0 & -F \\ KC_0 & 0 \end{bmatrix}. \tag{4}$$

Thus, system (2) is a GPI observer for system (1) if the matrix pair in (4) is Hurwitz stable. Furthermore, let $n_0 = \operatorname{rank}\begin{bmatrix} E \\ C_1 \end{bmatrix}$, we demand that the matrix pair $(E_0, A_0)$ has $(p + n_0)$ finite eigenvalues, which ensures the elimination of impulse for system (3) with $\operatorname{rank}(E + LC_1) = n_0$. Based on the above discussion, we state the GPI observer design problem for descriptor linear system (1) as follows.

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**Problem 1 (GPIO)** *Given matrices* $E$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $C_0$, $C_1 \in \mathbb{R}^{m\times n}$ *satisfying Assumptions A1 and A2, find the complete parametric expression for the matrices* $L$, $F$, $K$, *such that the matrix pair* $(E_0, A_0)$ *in (4) satisfies the following conditions:*

1. *It is Hurwitz stable;*
2. *It is regular;*
3. *It is impulse-free, that is, it possesses* $(p + n_0)$ *finite eigenvalues and* $\mathrm{rank}(E + LC_1) = n_0$.

## 3. Solution to Problem GPI

### 3.1. Basic relations

Suppose matrix pair $(E_0, A_0)$ has the following Jordan form:

$$\Lambda = \mathrm{diag}\left(J_1, J_2, \ldots, J_q\right) \in \mathbb{C}^{(n_0+p)\times(n_0+p)}, \tag{5}$$

$$J_i = \begin{bmatrix} s_i & 1 & & \\ & s_i & \ddots & \\ & & \ddots & 1 \\ & & & s_i \end{bmatrix},$$

where $s_i, i = 1, 2, \ldots, q$, are not necessarily distinct. Obviously, $s_i$, $i = 1, 2, \ldots, q$, are the finite eigenvalues of $(E_0, A_0)$. Thus, in order to ensure $(E_0, A_0)$ is Hurwitz stable, the following constraint must be satisfied.

**Constraint C1** $s_i$, $i = 1, 2, \ldots, q$, are self-conjugate and $\mathrm{Re}(s_i) < 0$, $i = 1, 2, \ldots, q$.

Denote the left eigenvector chain of $(E_0, A_0)$ associated with finite eigenvalue $s_i$ by $u_{ij} \in \mathbb{C}^{n+p}$, $j = 1, 2, \ldots, p_i$, $i = 1, 2, \ldots, q$. Constructing the left eigenvector matrix as

$$\begin{cases} U = \begin{bmatrix} U_1 & U_2 & \cdots & U_q \end{bmatrix} \in \mathbb{C}^{(n+p)\times(n_0+p)}, \\ U_i = \begin{bmatrix} u_{i1} & u_{i2} & \cdots & u_{ip_i} \end{bmatrix} \in \mathbb{C}^{(n+p)\times p_i}, \end{cases}$$

then by definition we demand that $\mathrm{rank}U = n_0 + p$, and

$$U^{\mathrm{T}} A_0 = \Lambda U^{\mathrm{T}} E_0\,. \tag{6}$$

Partitioning $U$ into the following form:

$$U = \begin{bmatrix} T \\ V \end{bmatrix}, \qquad T \in \mathbb{C}^{n\times(n_0+p)}, \quad V \in \mathbb{C}^{p\times(n_0+p)}. \tag{7}$$

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Substituting (7) and (4) into the (6), we have

$$T^{\mathrm{T}}(A - LC_0) + V^{\mathrm{T}}KC_0 = \Lambda T^{\mathrm{T}}(E + LC_1) - \Lambda V^{\mathrm{T}}KC_1 \tag{8}$$

and

$$-F^{\mathrm{T}}T = V\Lambda^{\mathrm{T}}. \tag{9}$$

Let

$$Z^{\mathrm{T}} = V^{\mathrm{T}}K - T^{\mathrm{T}}L, \tag{10}$$

then (8) can be equivalently written as

$$\Lambda T^{\mathrm{T}}E - T^{\mathrm{T}}A = \Lambda Z^{\mathrm{T}}C_1 + Z^{\mathrm{T}}C_0, \tag{11}$$

### 3.2. Solution to the matrices $T$ and $Z$

Taking transpose to (11) gives

$$E^{\mathrm{T}}T\Lambda^{\mathrm{T}} - A^{\mathrm{T}}T = C_1^{\mathrm{T}}Z\Lambda^{\mathrm{T}} + C_0^{\mathrm{T}}Z. \tag{12}$$

**Theorem 1** *Given the descriptor linear system (1) satisfying Assumptions A1 and A2, and $\Lambda$ defined by (5), then all the solution to matrix equation (12) are given by*

$$\begin{gathered}\begin{bmatrix} t_{ij} \\ z_{ij} \end{bmatrix} = Q(s_i)\begin{bmatrix} g_{ij} \\ P(s_i)(E^{\mathrm{T}}t_{i(j+1)} - C_1^{\mathrm{T}}z_{i(j+1)}) \end{bmatrix} \\ t_{i(p_i+1)} = 0, \qquad z_{i(p_i+1)} = 0, \qquad j = 1, 2, \ldots, \qquad p_i, i = 1, 2 \ldots, q, \end{gathered} \tag{13}$$

*where $g_{ij} \in \mathbb{C}^m$, $j = 1, 2 \ldots, p_i$, $i = 1, 2, \ldots, q$, is a set of parametric vectors, unimodular matrices $P(s) \in \mathbb{R}^{n\times n}[s]$ and $Q(s) \in \mathbb{R}^{(n+m)\times(n+m)}[s]$ satisfy the following Smith form reduction:*

$$P(s)\begin{bmatrix} A^{\mathrm{T}} - sE^{\mathrm{T}} & sC_1^{\mathrm{T}} + C_0^{\mathrm{T}} \end{bmatrix} Q(s) = \begin{bmatrix} 0 & I \end{bmatrix}. \tag{14}$$

**Proof.** The matrices $T$ and $Z$ can be divided as follows, according to the construction of Jordan matrix $\Lambda$

$$\begin{aligned} T &= \begin{bmatrix} T_1 & T_2 & \cdots & T_q \end{bmatrix}, & Z &= \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_q \end{bmatrix}, \\ T_i &= \begin{bmatrix} t_{i1} & t_{i2} & \cdots & t_{ip_i} \end{bmatrix}, & Z_i &= \begin{bmatrix} z_{i1} & z_{i2} & \cdots & z_{ip_i} \end{bmatrix}. \end{aligned} \tag{15}$$

Equation (12) can be equivalently written into the following vector form:

$$\begin{gathered}\begin{bmatrix} A^{\mathrm{T}} - s_iE^{\mathrm{T}} & s_iC_1^{\mathrm{T}} + C_0^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} t_{ij} \\ z_{ij} \end{bmatrix} = \begin{bmatrix} E^{\mathrm{T}} & -C_1^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} t_{i(j+1)} \\ z_{i(j+1)} \end{bmatrix}, \\ j = 1, 2, \ldots, p_i, \qquad t_{i(p_i+1)} = 0, \qquad i = 1, 2, \ldots, q. \end{gathered} \tag{16}$$

Thus, we need to prove that the set of vectors given by (13) are all the solution of the matrix equation of (16). First of all, we show that vectors given by (13) satisfy (16).

Using (13) and (14), we have

$$\begin{aligned}
&\begin{bmatrix} A^{\mathrm{T}}-s_iE^{\mathrm{T}} & s_iC_1^{\mathrm{T}}+C_0^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} t_{ij} \\ z_{ij} \end{bmatrix} \\
&\quad = \begin{bmatrix} A^{\mathrm{T}}-s_iE^{\mathrm{T}} & s_iC_1^{\mathrm{T}}+C_0^{\mathrm{T}} \end{bmatrix} Q(s_i) \begin{bmatrix} g_{ij} \\ P(s_i)\left(E^{\mathrm{T}}t_{i(j+1)} - C_1^{\mathrm{T}}Z_{i(j+1)}\right) \end{bmatrix} \\
&\quad = P^{-1}(s_i)\begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} g_{ij} \\ P(s_i)(E^{\mathrm{T}}t_{i(j+1)} - C_1^{\mathrm{T}}Z_{i(j+1)}) \end{bmatrix} \\
&\quad = E^{\mathrm{T}}t_{i(j+1)} - C_1^{\mathrm{T}}Z_{i(j+1)} \\
&\qquad\quad j = 1, 2, \ldots, p_i, \qquad i = 1, 2, \ldots, q.
\end{aligned}$$

Therefore the vectors given by (13) satisfy (16).

Then, we need to show that all the solution of the equation (16) can be expressed by (13).

Post-multiplying by $P(s_i)$ on both sides of equation (16), we get

$$P(s_i)\begin{bmatrix} A^{\mathrm{T}} - s_iE^{\mathrm{T}} & s_iC_1^{\mathrm{T}} + C_0^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} t_{ij} \\ z_{ij} \end{bmatrix} = P(s_i)\begin{bmatrix} E^{\mathrm{T}} & -C_1^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} t_{i(j+1)} \\ z_{i(j+1)} \end{bmatrix}. \tag{17}$$

Let

$$\begin{bmatrix} g_{ij} \\ l_{ij} \end{bmatrix} = Q^{-1}\begin{bmatrix} t_{ij} \\ z_{ij} \end{bmatrix} \tag{18}$$

according to (14) and (17), we get

$$\begin{gathered}
\begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} g_{ij} \\ l_{ij} \end{bmatrix} = P(s_i)\begin{bmatrix} E^{\mathrm{T}} & -C_1^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} t_{i(j+1)} \\ z_{i(j+1)} \end{bmatrix}, \\
j = 1, 2, \ldots, p_i, \qquad i = 1, 2, \ldots, q
\end{gathered}$$

from which we derive

$$\begin{gathered}
l_{ij} = P(s_i)\left(E^{\mathrm{T}}t_{i(j+1)} - C_1^{\mathrm{T}}z_{i(j+1)}\right) \\
j = 1, 2, \ldots, p_i, \qquad i = 1, 2, \ldots, q.
\end{gathered} \tag{19}$$

Substituting (19) into (18), and left multiplying $Q(s_i)$ on both sides of obtained equation, yields (13). Therefore, (13) represents all the solution of matrix equation (12). □

**Corollary 3** *Given the descriptor linear system(1) satisfying Assumptions A1 and A2, and* $\Lambda = \operatorname{diag}\left(s_1, s_2, \ldots, s_{n_0+p}\right) \in \mathbb{C}^{(n_0+p)\times(n_0+p)}$*, then all the solution to matrix equation (12) are given by*

$$\begin{bmatrix} t_i \\ z_i \end{bmatrix} = Q(s_i)\begin{bmatrix} g_i \\ 0 \end{bmatrix}, \qquad i = 1, 2, \ldots, n_0 + p,$$

*where* $g_i \in \mathbb{C}^m$, $i = 1, 2, \ldots, n_0 + p$*, is set of parametric vectors, unimodular matrix* $Q(s) \in \mathbb{R}^{(n+m)\times(n+m)}[s]$ *satisfying the Smith form reduction (14).*

**Theorem 2** *Suppose the descriptor linear system (1) satisfying Assumptions A1 and A2, and* $\Lambda$ *defined by (5), then all the solution to matrix equation (12) are given by*

$$\begin{bmatrix} t_{ij} \\ z_{ij} \end{bmatrix} = \sum_{k=0}^{p_i-j} \frac{1}{k!}\begin{bmatrix} N(s_i) \\ D(s_i) \end{bmatrix} g_{i(j+k)}, \qquad j = 1, 2, \ldots, p_i, \quad i = 1, 2, \ldots, q, \tag{20}$$

*where* $N(s) \in \mathbb{R}^{n\times m}[s]$ *and* $D(s) \in \mathbb{R}^{m\times m}[s]$ *are right coprime polynomial matrices satisfying*

$$\left(sE^{\mathrm{T}} - A^{\mathrm{T}}\right)^{-1}\left(sC_1^{\mathrm{T}} + C_0^{\mathrm{T}}\right) = N(s)D^{-1}(s) \tag{21}$$

*and* $g_{ij} \in \mathbb{C}^m$, $j = 1, 2, \ldots, p_i$, $i = 1, 2, \ldots, q$*, are arbitrarily chosen parametric vectors.*

**Proof.** Noting that the number of free parameters contained in (13) and (20) are equal, we need only to prove that vectors given by (20) satisfy (16).

Converting (21) into the following form:

$$\left(A^{\mathrm{T}} - sE^{\mathrm{T}}\right)N(s) + \left(sC_1^{\mathrm{T}} + C_0^{\mathrm{T}}\right)D(s) = 0$$

and taking $k$-th differential in both sides of the above equation, we obtain

$$\begin{aligned} \left(A^{\mathrm{T}} - sE^{\mathrm{T}}\right)\frac{\mathrm{d}^k}{\mathrm{d}s^k}N(s) + \left(sC_1^{\mathrm{T}} + C_0^{\mathrm{T}}\right)\frac{\mathrm{d}^k}{\mathrm{d}s^k}D(s) &= \\ = kE^{\mathrm{T}}\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}}N(s) - kC_1^{\mathrm{T}}\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}}D(s)&. \end{aligned} \tag{22}$$

Substituting $s$ by $s_i$ and post-multiplying by vector $(1/k!)g_{i(j+k)}$ on both sides of (22), gives

$$\begin{aligned} &\left(A^{\mathrm{T}} - s_iE^{\mathrm{T}}\right)\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}s^k}N(s_i)g_{i(j+k)} + \left(s_iC_1^{\mathrm{T}} + C_0^{\mathrm{T}}\right)\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}s^k}D(s_i)g_{i(j+k)} \\ &\quad = E^{\mathrm{T}}\frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}}N(s_i)g_{i(j+k)} - C_1^{\mathrm{T}}\frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k-1}}D(s_i)g_{i(j+k)}, \\ &\quad k = 0, 1, 2, \ldots, p_i - j, \quad j = 1, 2, \ldots, p_i, \quad i = 1, 2, \ldots, q. \end{aligned} \tag{23}$$

Summing up all the equations in (23) for $k = 0, 1, 2, \ldots, p_i - j$, and using (20), we obtain (16). □

**Corollary 4** *Given the descriptor linear system (1) satisfying Assumptions A1 and A2, and* $\Lambda = \mathrm{diag}\left(s_1, s_2, \ldots, s_{n_0+p}\right) \in \mathbb{C}^{(n_0+p)\times(n_0+p)}$, *then all the solution of matrix equation (12) are given by*

$$\begin{bmatrix} t_i \\ z_i \end{bmatrix} = \begin{bmatrix} N(s_i) \\ D(s_i) \end{bmatrix} g_i, \qquad i = 1, 2, \ldots, n_0 + p,$$

*where* $g_{ij} \in \mathbb{C}^m$, $i = 1, 2, \ldots, n_0 + p$, *are arbitrarily chosen parametric vectors, right coprime polynomial matrices* $N(s) \in \mathbb{R}^{n\times m}[s]$ *and* $D(s) \in \mathbb{R}^{m\times m}[s]$ *satisfy equation (21).*

### 3.3. Solution to the matrices $V$ and $F$

Owing to Constraint C1, it follows from (9) that the matrix $V$ can be expressed as

$$V = -F^{\mathrm{T}} T \Lambda^{-1}, \tag{24}$$

where $F$ can be regarded as a parametric matrix. Further, the general parametric form for the left eigenvector matrix $U$ can be obtained according to (7) using (15) and (24).

Since $U$ is the left eigenvector matrix, the following constraint on the parametric matrix $F$ as well as the other design parameters $s_i, g_{ij}$, $j = 1, 2, \ldots, p_i$, $i = 1, 2, \ldots, q$, must be satisfied.

**Constraint C2**: $\mathrm{rank}\, U = n_0 + p$.

### 3.4. Solution to the matrices $K$ and $L$

Equation (10) can be written as

$$Z^T = \begin{bmatrix} -T^{\mathrm{T}} & V^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} L \\ K \end{bmatrix}. \tag{25}$$

To ensure the existence of solution to the above equation, the following condition must be met:

$$\mathrm{rank} \begin{bmatrix} T^{\mathrm{T}} & V^{\mathrm{T}} \end{bmatrix} = \mathrm{rank} \begin{bmatrix} T^{\mathrm{T}} & V^{\mathrm{T}} & Z^{\mathrm{T}} \end{bmatrix} = n_0 + p.$$

Applying singular value decomposition for matrix $\begin{bmatrix} -T^{\mathrm{T}} & V^{\mathrm{T}} \end{bmatrix}$, we get two unitary matrices $P_1 \in \mathbb{C}^{(n_0+p)\times(n_0+p)}$, $Q_1 \in \mathbb{C}^{(n+p)\times(n+p)}$ satisfying

$$P_1 \begin{bmatrix} -T^{\mathrm{T}} & V^{\mathrm{T}} \end{bmatrix} Q_1 = \begin{bmatrix} \Sigma & 0 \end{bmatrix}.$$

Thus, (25) can be written as

$$Z^{\mathrm{T}} = \begin{bmatrix} P_1^{\mathrm{T}}\Sigma & 0 \end{bmatrix} Q_1^{\mathrm{T}} \begin{bmatrix} L \\ K \end{bmatrix}.$$

From the above relation, a solution for the matrices $L$ and $K$ is easily obtained as

$$\begin{bmatrix} L \\ K \end{bmatrix} = Q_1 \begin{bmatrix} \Sigma^{-1} P_1 Z^{\mathrm{T}} \\ K' \end{bmatrix}, \tag{26}$$

where $K' \in \mathbb{R}^{p\times m}$ is an arbitrary real parametric matrix. To ensure that matrices $L$ and $K$ are both real, the following constraint must be met.

**Constraint C3:** $g_{il} = \bar{g}_{jl}$, $l = 1, 2, \ldots, p_i$, if $s_i = \bar{s}_j$, $i, j = 1, 2, \ldots, q$.

According to the above results, we get the following theorem about solution to Problem 1.

**Theorem 3** *Given descriptor linear system (1) satisfying Assumptions A1, A2. Problem 1 have solution if there exist parameters $s_i$, $g_{ij}$, $j = 1, 2, \ldots, p_i$, $i = 1, 2, \ldots, q$, and parametric matrices $F$, $K'$ satisfying Constraints C1–C3. On this case, the gain matrices are given by (26).*

According to the above deduction, we can list the procedures to solve Problem 1 as follows.

**Algorithm 1** *(For Problem 1)*

**Step 1.** *Obtain $P(s)$ and $Q(s)$ satisfying equation (14) by taking Smith form reduction for $\begin{bmatrix} sE^{\mathrm{T}} - A^{\mathrm{T}} & sC_1^{\mathrm{T}} + C_0^{\mathrm{T}} \end{bmatrix}$. Partition the matrix $Q(s)$ into blocks as $Q(s) = \begin{bmatrix} N(s) & * \\ D(s) & * \end{bmatrix}$, where $N(s) \in \mathbb{R}^{n\times m}[s]$ and $D(s) \in \mathbb{R}^{m\times m}[s]$ satisfy right coprime factorization (21).*

**Step 2.** *Seek parameters $s_i$, $i = 1, 2, \ldots, q$, matrix $F$ and vectors $g_{ij}$, $j = 1, 2, \ldots, p_i$, $i = 1, 2, \ldots, q$, satisfying Constraints C1, C2 and C3. If such parameters do not exist, Problem 1 does not have solution.*

**Step 3.** *Calculate matrices $T$ and $Z$ according to formulas (15) and (20) (or (13)) on account to the parameters $s_i$, $g_{ij}$, $j = 1, 2, \ldots, p_i$, $i = 1, 2, \ldots, q$.*

**Step 4** *Compute $V$ by equation (24), then $U$ is obtained by formula (7). The gain matrices $L$ and $K$ can be obtained by formula (26).*

## 4. Examples

**Example 1.** Consider a descriptor linear system in the form of (1) with the following parameters

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

where $n = 5$, $m = 2$ and $n_0 = 4$. It is easy to verify that Assumptions A1–A2 are satisfied. According to the first step of Algorithm 1, we obtain

$$N(s) = \begin{bmatrix} 2s^2 + s & -2s^3 - s \\ s + 1 & -s^2 - 1 \\ s & 0 \\ -2s - 1 & 2s^2 + 2 \\ -2s^2 - s & s + s(2s^2 + 1) \end{bmatrix}, \qquad D(s) = \begin{bmatrix} s & -s^2 - 1 \\ -s - 1 & 2s^2 + 1 \end{bmatrix}.$$

In the following, we design a GPI observer in the form of (2) with $p = 1$. Moreover, we can assign five finite eigenvalues for the matrix pair $(E_0, A_0)$. For simplicity, we restrict the assigned eigenvalues $s_i$, $i = 1, \ldots, 5$, to be negative real numbers, thus Constrain C1 is met. In this case, the parametric vectors $g_i, i = 1, \ldots, 5$, are also restricted to be real, so Constraint C3 is satisfied.

Denote

$$g_i = \begin{bmatrix} g_{i1} \\ g_{i2} \end{bmatrix}, \qquad i = 1, \ldots, 5$$

$$F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \end{bmatrix}^{\mathrm{T}},$$

then, in view of Constraint C2, we demand,

$$\operatorname{rank} U = \operatorname{rank} \begin{bmatrix} T \\ V \end{bmatrix} = 5,$$

where

$$T_i = \begin{bmatrix} g_{i1}(2s_i^2 + s_i) + g_{i2}(2s_i^3 + s_i) \\ g_{i1}(s_i + 1) - g_{i2}(s_i^2 + 1) \\ s_i g_{i1} \\ g_{i2}(2s_{i2}^2 + 2) - g_{i1}(2s_i + 1) \\ g_{i2}(s_i + s_i(2s_i^2 + 1)) - g_{i1}(2s_i^2 + s_i) \end{bmatrix},$$

$$V_i = -s_i^{-1} F^{\mathrm{T}} T_i, \qquad i = 1, 2, \ldots, 5.$$

Specially, we choose the parameters satisfying the preceding constraint as follows:

$$s_1 = -1, \quad s_2 = -1.7, \quad s_3 = -0.5, \quad s_4 = -0.4, \quad s_5 = -2.9$$

$$g_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad g_4 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad g_5 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 10 & 20 & 1 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Based on the foregoing parameters and Theorem 2, we have

$$T = \begin{bmatrix} 2.0000 & 11.5260 & 2.2500 & 1.5040 & 93.4380 \\ -2.0000 & -3.8900 & -2.7500 & -2.8800 & -15.1100 \\ 1.0000 & 0 & -1.0000 & -0.4000 & -8.7000 \\ 3.0000 & 7.7800 & 7.5000 & 6.7600 & 33.2200 \\ -3.0000 & -13.2260 & -3.7500 & -2.7040 & -96.3380 \end{bmatrix},$$

$$Z = \begin{bmatrix} -1.0000 & -3.8900 & -4.7500 & -3.8800 & -18.1100 \\ 3.0000 & 6.7800 & 3.5000 & 3.3600 & 23.5200 \end{bmatrix}.$$

According to (26), we obtain the observer gain matrices as

$$L = \begin{bmatrix} 13.7092 & -5.0673 \\ -1.2570 & -4.2050 \\ -6.6242 & -2.6464 \\ 8.8651 & -4.8165 \\ 16.3012 & -5.2735 \end{bmatrix}, \quad K = \begin{bmatrix} -0.4747 & -0.1147 \end{bmatrix}.$$

The simulation of the expanded error system $e$ and integral term $w$ are showed by Figures 1 and 2.

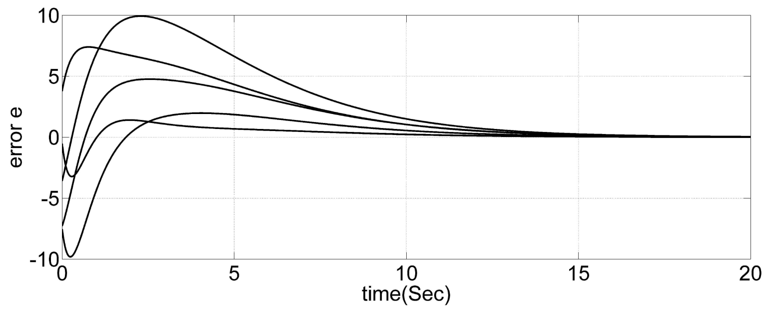


Figure 1: Simulation of errors of Example 4

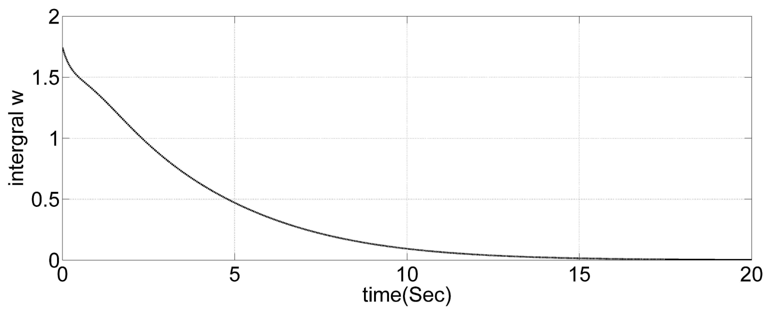


Figure 2: Simulation of integral term of Example 4

**Example 2.** Consider a descriptor linear system in the form of (1) with the following parameters

$$E=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1&0&0&0&1\\1&0&0&0&0\end{bmatrix},\qquad A=\begin{bmatrix}0&0&1&0&0\\1&0&0&1&0\\0&1&0&0&0\\0&0&0&1&0\\1&0&0&0&1\end{bmatrix},$$

$$C_0=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\end{bmatrix},\qquad C_1=\begin{bmatrix}0&0&0&1&0\\1&0&0&0&0\end{bmatrix},$$

where $n=5$, $m=2$ and $n_0=5$. It is easy to verify that Assumptions A1–A2 are satisfied. According to the first step of Algorithm 1, we obtain

$$N(s)=\begin{bmatrix}s^3+s^2+1 & -\dfrac{s^2}{2}+\dfrac{s}{2}\\ 2s & 0\\ 2s^2 & 1\\ -2s & s\\ -2s^2 & s^2\end{bmatrix},$$

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$$D(s) = \begin{bmatrix} 0 & -1 \\ s^3 - s^2 - 1 & \dfrac{s^2}{2} + \dfrac{s}{2} \end{bmatrix}.$$

In the following, we design a GPI observer in the form of (2) with $p = 1$. Moreover, we assign six finite eigenvalues for the matrix pair $(E_0, A_0)$. For simplicity, we restrict the assigned eigenvalues $s_i$, $i = 1, \ldots, 6$, to be negative real numbers, thus Constrain C1 is met. In this case, the parametric vectors $g_i$, $i = 1, \ldots, 6$, are also restricted to be real, so we need not consider Constraint C3.

Denote

$$g_i = \begin{bmatrix} g_{i1} \\ g_{i2} \end{bmatrix}, \qquad i = 1, \ldots, 6,$$

$$F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \end{bmatrix}^{\mathrm{T}},$$

then based on Constraint C2, we demand

$$\operatorname{rank} U = \operatorname{rank} \begin{bmatrix} T \\ V \end{bmatrix} = 6,$$

$$T_i = \begin{bmatrix} g_{i1}\left(s_i^3 + s_i^2 + 1\right) + g_{i2}\left(-\dfrac{s_i^2}{2} + \dfrac{s_i}{2}\right) \\ 2s_i g_{i1} \\ 2g_{i1}s_i^2 + g_{i2} \\ -2s_i g_{i1} + s_i g_{i2} \\ -2s_i^2 g_{i1} + s_i^2 g_{i2} \end{bmatrix},$$

$$V_i = -s_i^{-1} F^{\mathrm{T}} T_i, \qquad i = 1, 2, \ldots, 6.$$

Specially, we choose a set of parameters satisfying the foregoing constraint as follows:

$$s_1 = -1, \quad s_2 = -2, \quad s_3 - 1.3, \quad s_4 = -1.4, \quad s_5 = -1.5, \quad s_6 = -0.6$$

$$g_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

$$g_4 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad g_5 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad g_6 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 10 & 20 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

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Based on the preceding parameters and Theorem 2, we have

$$T = \begin{bmatrix} -2.0000 & -3.0000 & -3.4990 & -4.8240 & -2.2500 & 1.1440 \\ 2.0000 & 0 & -5.2000 & -2.8000 & -9.0000 & -1.2000 \\ -1.0000 & 1.0000 & 9.7600 & 6.9200 & 14.5000 & 0.7200 \\ -3.0000 & -2.0000 & 1.3000 & -1.4000 & 7.5000 & 1.2000 \\ 3.0000 & 4.0000 & -1.6900 & 1.9600 & -11.2500 & -0.7200 \end{bmatrix},$$

$$Z = \begin{bmatrix} -1.0000 & -1.0000 & -3.0000 & -3.0000 & -1.0000 & 0 \\ 3.0000 & 1.0000 & -9.1890 & -4.8640 & -19.5000 & -1.5760 \end{bmatrix}.$$

According to (26), we obtain the observer gain matrices as

$$L = \begin{bmatrix} 0.6059 & 1.4607 \\ 0.6050 & 3.6870 \\ 0.9703 & 4.7736 \\ -0.3789 & 1.7876 \\ 0.2784 & 0.8946 \end{bmatrix}, \qquad K = \begin{bmatrix} 0.0022 & 0.1268 \end{bmatrix}.$$

The simulation of the expanded error system $e$ and integral term $w$ are showed by Figures 3 and 4.

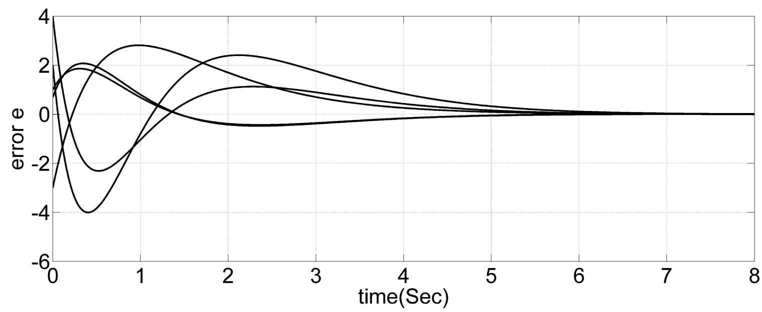


Figure 3: Simulation of errors of Example 4

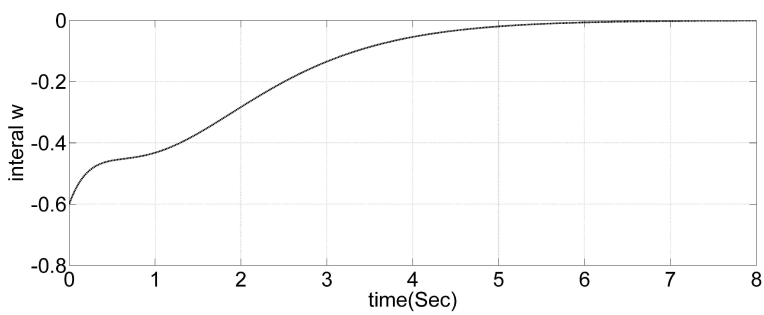


Figure 4: Simulation of integral term of Example 4

## 5. Conclusion

A type of GPI observer for descriptor linear system and a parametric method is proposed. Based on a general solution to a type of generalized Sylvester matrix equations, a parametric approach for the GPI observer is presented. In terms of four groups of parameters, the parametric expressions of all observer gain matrices are given. The proposed method offers all the degrees of design freedom which can be utilized to achieve various system specifications. The method also guarantees the regularity of the expanded error system and can eliminate the impulsive responses.

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