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CIVIL ENGINEERING

Proposal of the criterion for transmission line lumped parameters analysis

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Abstract. The paper concerns the problem of choosing a criterion for transmission line when lumped parameters analysis is required. First, the formal introduction of transmission line scheme is presented. Secondly, a new criterion for lumped-parameter analysis of transmission line is proposed. The criterion has clear physical meaning and simple mathematical form. The proposed criterion takes into account not only wave length, but also the dissipation of transmission line. The criterion can be easily adjusted to some requirements, such as needed level of no-load output voltage change.

Key words: transmission line, lumped parameters analysis, no-load output voltage change, adjustable criterion.

1. Introduction

Transmission line is a system which should be treated as a distributed system described by partial differential equations system, generally. Transmission line model is often willingly simplified into lumped-parameter circuit. For many engineering applications, such a simplification is required but, on the other hand, it often leads to errors. The responsible decision concerning transmission line model simplification should be taken after verifying a criterion. The more rigorous the proposed criterion, the lower error level admitted. Moreover, the criterion should have a simple mathematical form and clear physical interpretation. The paper presents mathematical and physical backgrounds of criterion for transmission line model simplification (allowing for lumped parameters analysis). First, the formal introduction of transmission line model is presented. Secondly, new criterion for lumped-parameter analysis of transmission line is introduced. Thirdly, the adaptive modification way of the criterion is discussed.

2. Transmission line equations

Transmission line state equations (the so-called Heaviside equations) result from Maxwell electromagnetic field equations. Those equations allow for introducing the model transmission line. Works [1–5, 11] introduce transmission line equations for arbitrary chosen model scheme (so-called equivalent circuit). Based on assumed transmission line scheme, the main equations are derived and the difference equations are rearranged into differential state equations.

This paragraph introduces transmission line scheme by means of electromagnetic field equation analysis. For cylindrical co-ordinate system, the model (two-port scheme) is introduced strictly for transmission line segment. Subsequently, electromagnetic field equations, i.e. Maxwell equations, are applied for transmission line mathematical description (not the assumed circuit diagram).

The analysis is based on Maxwell equations for one phase linear transmission line. First, a single linear conductor in considered. The symmetry of the electromagnetic field points out that the most appropriate is the cylindrical co-ordinate system (Fig. 1). The z axis coincides with symmetry axis of cylindrical conductor, which is one of two conductors of one phase transmission line of infinite length placed in isotropic region indicated in Fig. 1.

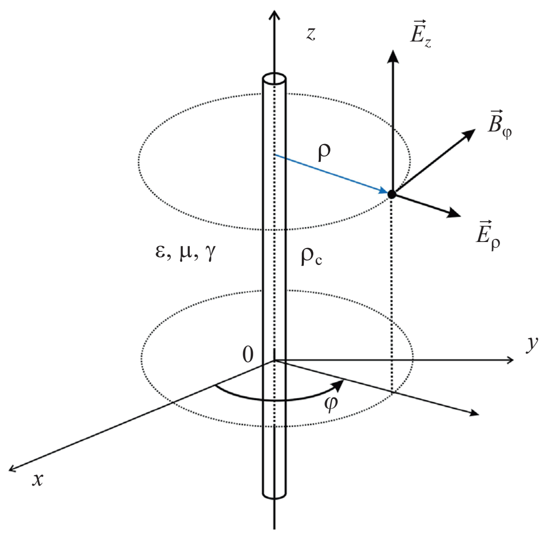

Fig. 1. Electromagnetic field components of single conductor in cylindrical co-ordinate system

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The electromagnetic field is described by Maxwell equations [1–6] are usually written as follows

$$\mathrm{rot}\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \tag{1}$$

and

$$\mathrm{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{2}$$

Due to axial symmetry the electric field strength vector tangential component equals to zero, thus

$$\vec{E} = E_\rho \vec{i}_\rho + E_z \vec{i}_z, \tag{3}$$

and magnetic flux density has only one component

$$\vec{B} = B\vec{i}_\varphi. \tag{4}$$

Hence, the Maxwell equations for this problem can be written as follows

$$-\frac{1}{\rho}\frac{\partial(\rho H_\varphi)}{\partial z} = \gamma E_\rho + \frac{\partial D_\rho}{\partial t}. \tag{5}$$

and

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = -\frac{\partial B_\varphi}{\partial t}. \tag{6}$$

After integrating over φ (multiplied by ρ) one obtains

$$-\frac{\partial}{\partial z}\int_0^{2\pi} H_\varphi \rho \,\mathrm{d}\varphi = \int_0^{2\pi} \gamma E_\rho \rho \,\mathrm{d}\varphi + \frac{\partial}{\partial t}\int_0^{2\pi} D_\rho \rho \,\mathrm{d}\varphi. \tag{7}$$

The three integrals represent subsequently: conductor current

$$\int_0^{2\pi} H_\varphi \rho \,\mathrm{d}\varphi = i, \tag{8}$$

stray (parasitic) current i_u (over dielectric surrounding the conductor) per length l

$$\int_0^{2\pi} \gamma E_\rho \rho \,\mathrm{d}\varphi = \frac{i_u}{l} = G_1 V, \tag{9}$$

and electric charge on conductor per length

$$\int_0^{2\pi} D_\rho \rho \,\mathrm{d}\varphi = \frac{q}{l} = C_1 V, \tag{10}$$

where $V = V(t, z)$ means local potential along conductor. Hence, the first (current) partial difference equation for transmission line conductor is as follows

$$-\frac{\partial i}{\partial z} = G_1 V + C_1 \frac{\partial V}{\partial t}. \tag{11}$$

Subsequently, after integrating (6) over ρ one obtains

$$\frac{\partial}{\partial z}\int_0^{\infty} E_\rho \,\mathrm{d}\rho = \int_0^{\infty} \frac{\partial E_z}{\partial \rho}\mathrm{d}\rho - \frac{\partial}{\partial t}\int_0^{\infty} B_\varphi \,\mathrm{d}\rho. \tag{12}$$

The three integrals in (12) represent subsequently: local potential

$$\int_0^{\infty} E_\rho \,\mathrm{d}\rho = V(0) - V(\infty) = V(0) = V, \tag{13}$$

voltage drop per length

$$\begin{aligned}\int_0^{\infty} \frac{\partial E_z}{\partial \rho}\mathrm{d}\rho &= E_z(\infty) - E_z(0) = -E_z(0) = \\ &= -\rho_c J_z = -R_1 i\end{aligned} \tag{14}$$

(where ρ_c conductor resistivity, R_1 resistance per length), and magnetic flux per length

$$\int_0^{\infty} B_\varphi \,\mathrm{d}\rho = \frac{\Phi}{l} = L_1 i, \tag{15}$$

where L_1 means inductance per length. Thus, the second (voltage) partial differential equation for transmission line conductor takes the form of

$$\frac{\partial V}{\partial z} = -R_1 i - L_1 \frac{\partial i}{\partial t}. \tag{16}$$

Let us consider one phase transmission line of two conductors is depicted in Fig. 2.

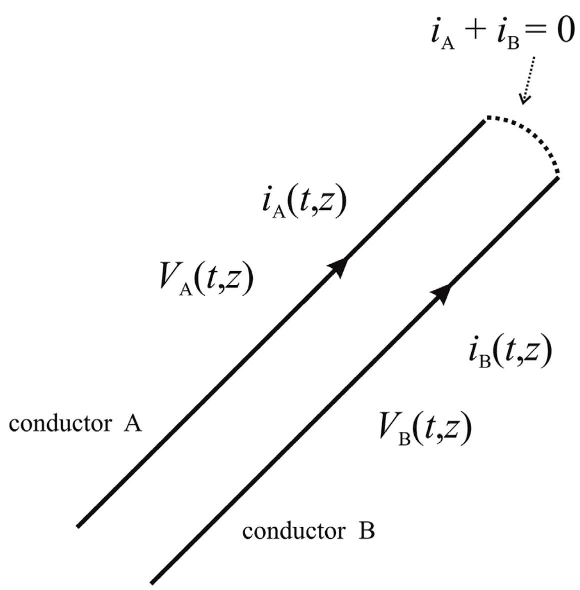

Fig. 2. One phase – two conductors transmission line

The current and voltage equations for conductor A take the forms

$$-\frac{\partial i_A}{\partial z} = G_1 V_A + C_1 \frac{\partial V_A}{\partial t}, \tag{17}$$

and

$$-\frac{\partial V_A}{\partial z} = R_1 i_A + L_1 \frac{\partial i_A}{\partial t}. \tag{18}$$

Analogously, the equations for conductor B are as follows

$$-\frac{\partial i_B}{\partial z} = G_1 V_B + C_1 \frac{\partial V_B}{\partial t}, \tag{19}$$

and

$$-\frac{\partial V_B}{\partial z} = R_1 i_B + L_1 \frac{\partial i_B}{\partial t}. \tag{20}$$

Subtractions (17) minus (19) and subsequently (18) minus (20) lead to equations

$$-\frac{\partial (i_A - i_B)}{\partial z} = G_1 (V_A - V_B) + C_1 \frac{\partial (V_A - V_B)}{\partial t}, \tag{21}$$

and

$$-\frac{\partial (V_A - V_B)}{\partial z} = R_1 (i_A - i_B) + L_1 \frac{\partial (i_A - i_B)}{\partial t}. \tag{22}$$

Finally, after regarding constraint $i_B = -i_A = -i$, and definition $u = V_A - V_B$ the transmission line equations are derived as follows

$$-\frac{\partial i}{\partial z} = G_0 u + C_0 \frac{\partial u}{\partial t}, \tag{23}$$

and

$$-\frac{\partial u}{\partial z} = R_0 i + L_0 \frac{\partial i}{\partial t}, \tag{24}$$

where transmission line per length parameters are defined as follows $G_0 = \frac{1}{2} G_1$, $C_0 = \frac{1}{2} C_1$, $R_0 = 2R_1$, $L_0 = 2L_1$. Moreover, the presented way of introduction transmission line equations enables for calculations of transmission line parameters.

The proved transmission line equations (23) and (24), based only on Maxwell electromagnetic field equations, introduce strictly the correct equivalent scheme. In Fig. 3 element of infinitesimal line length Δx is shown ($x = -z$ is measured from the transmission line end).

The transmission line equations (23) and (24) lead for sine steady-state of work (Fig. 4) to the following relations [6, 8, 10]

$$U_1 = U_2 \cosh(\Gamma l) + Z_C I_2 \sinh(\Gamma l), \tag{25}$$

and

$$I_1 = \frac{U_2}{Z_C} \sinh(\Gamma l) + I_2 \cosh(\Gamma l), \tag{26}$$

where propagation constant

$$\Gamma = \sqrt{(R_0 + j\omega L_0)(G_0 + j\omega C_0)} = \alpha + j\beta. \tag{27}$$

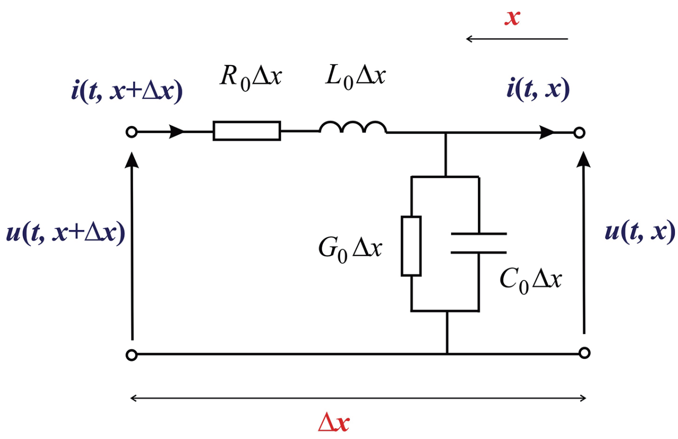

Fig. 3. Transmission line – equivalent circuit scheme

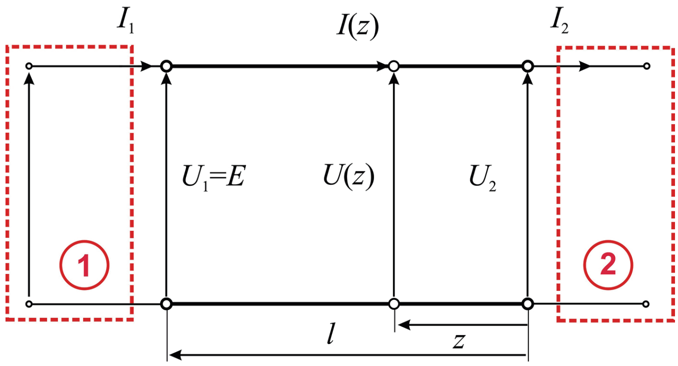

Fig. 4. Transmission line at no-load state of work

3. Criterion background for lumped-parameters analysis of transmission line

The proposed criterion for any approximation should be as simple as possible from mathematical point of view and has to have a clear physical interpretation. The no-load state of work is chosen, which guarantees that influence of load changes on output voltage change is omitted. The criterion should base on transmission line parameters not on line state of work. Let us consider a one phase transmission line supply by sin time varying source $U(l) = E$ at no–load state of work $I_2 = 0$ (Fig. 4).

At no-load state of work, the output voltage is changed in comparison to supply voltage due to transmission line phenomena pointed out in paragraph 1 (stray and capacitance currents, voltage drop). These phenomena result from distributed feature of transmission line. The phenomena leads to partial differential equations (23) and (24). For many engineering purposes, approximated lumped parameters are required. The question is: when could such an analysis be admitted? Let us consider the output voltage at no-load state of work as a background for proposing a new criterion of lumped parameter analysis. Mathematically, the output voltage change depends on transmission line parameters G_0, C_0, R_0, L_0 [4, 11, 12]. Relative voltage change at no-load state of work is defined as follows

$$\delta U_0(l) \overset{\text{def}}{=} \frac{U_2 - E}{E}, \tag{28}$$

where according to (25) is satisfied

$$E = U_1 = U_2 \cosh(\Gamma l). \tag{29}$$

Theoretically, the above implicates that only for line length $l \to 0$ the no-load output voltage U_2 is equal to supply voltage E. For lossless transmission line $\left(G_0 = 0,\ R_0 = 0,\ \alpha = 0,\ \Gamma = \mathrm{j}\omega\sqrt{L_0 C_0} = \mathrm{j}\beta,\ \lambda = 2\pi/\beta\right)$, the relative voltage change (29) is presented in Fig. 5.

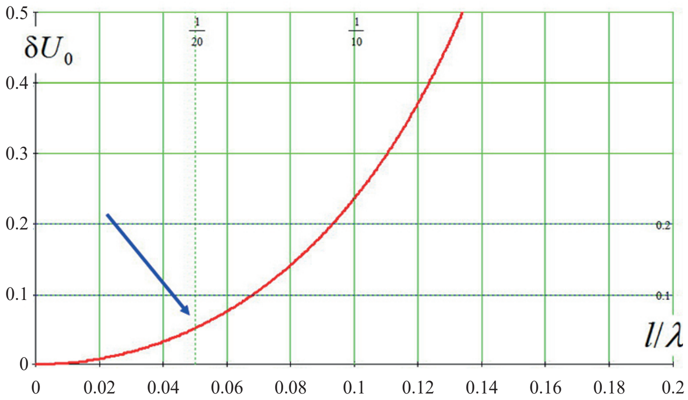

Fig. 5. Relative voltage change δU_0 module vs. relative line length l/λ for lossless transmission line

The curve yields

$$\left|\delta U_0(\lambda/20)\right| = 0.0515\ \ldots \cong 5\%, \tag{30}$$

which means if the length equals

$$l = \lambda/20, \tag{31}$$

the relative voltage change is lower than 5%.

Hence, a criterion could be formulated for lossless transmission line which demands satisfying condition (31) giving the guarantee that relative voltage is bounded by about 5%. This criterion has the clear physical interpretation (determines relative no-load voltage change) and simple mathematical form (can be easily changed for more either strict or soft criterion). Additionally, it should be remarked that the length of $\lambda/20$ is lower than the first anti-node placement of $\lambda/4$.

For example, if 10–percentage relative voltage change level can be accepted then based on the curve shown in Fig. 5 (or directly from equation (29)), the limit is as follows

$$\left|\delta U_0(\lambda/15)\right| = 0.0946\ \ldots \cong 10\%. \tag{32}$$

It is nteresting, from technical point of view, to consider transmission line when dissipation α cannot be omitted.

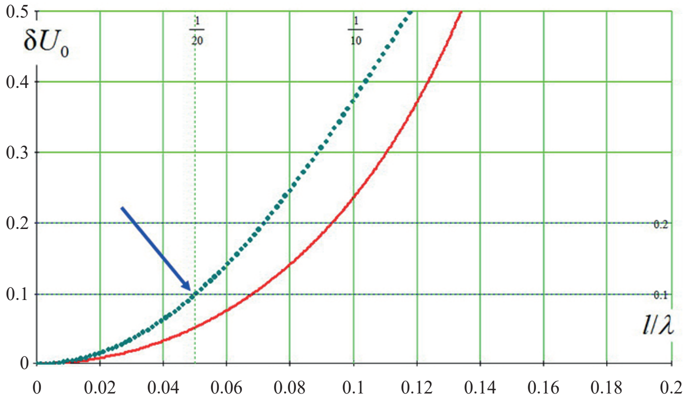

Fig. 6. Relative voltage change δU_0 moduli vs. relative line length l/λ for lossless $\alpha = 0$ (continuous line) and dissipative $\alpha = \beta$ (points) transmission lines

Figure 6 confirms that the relative voltage change rises with dissipation of transmission line, e.g. for $\alpha = \beta$

$$\left|\delta U_0(\lambda/20)\right| = 0.0984\ \ldots \cong 10\%. \tag{33}$$

Moreover, from the theoretical point of view it is interesting to consider transmission line for lower frequency (wave length λ is significantly great) when the dissipation α is dominant i.e. $\alpha >> \beta$.

Form the Fig. 7 is evident that dissipation α must be incorporated to the proposed criterion for lumped parameters analysis of transmission line.

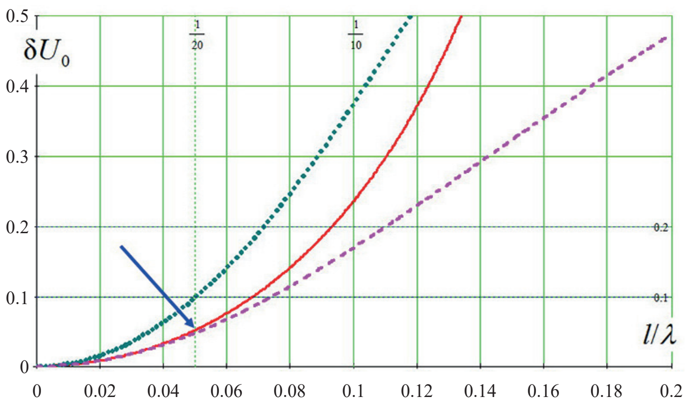

Fig. 7. Relative voltage change δU_0 moduli vs. relative line length $l/\lambda = l\beta/2\pi$ (or $l\alpha/2\pi$ for line for strong dissipative case) for lossless $\alpha = 0$ (continuous line), dissipative $\alpha = \beta$ (points) and strong dissipative (dashed line) transmission lines

Generally, the criterion should satisfy the inequality

$$\left|\delta U_0(l)\right| \le k, \tag{34}$$

which implicates condition in complex domain for transmission line length

$$\left|\delta U_0(l)\right| = \left|\cosh(\Gamma l)^{-1} - 1)\right| \le k. \tag{35}$$

Condition (35) is equivalent to conjunction of two inequalities as follows

$$\frac{l}{\lambda} \le \frac{1}{2\pi}\left|\operatorname{Im}\left\{\operatorname{acosh}\left(\frac{1}{1+k\mathrm{e}^{\mathrm{j}\varphi}}\right)\right\}\right|, \tag{36}$$

and

$$\alpha l \le \operatorname{Re}\left\{\operatorname{acosh}\left(\frac{1}{1+k\mathrm{e}^{\mathrm{j}\varphi}}\right)\right\}, \tag{37}$$

where φ is argument of $\cosh(\Gamma l) - 1$ [7–10].

The conjunction of two inequalities (36) and (37) constitute the necessary and sufficient conditions for (34). Inequalities (36) and (37) are interpreted graphically in Fig. 8. The criterion regards the influence of dissipation phenomena. In Fig. 9. the graphical algorithm of the criterion is explained. Namely, for given α the value of $\alpha l/2\pi$ is calculated, and a proper point is founded on real part curve. It is asked whether or not l/λ is lower than adequate value on curve of imaginary part of complex relative voltage change.

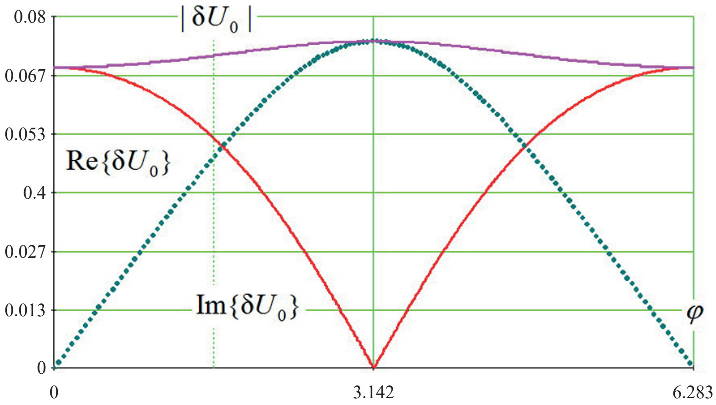

Fig. 8. Real (points), imaginary (line) parts and module (upper line) of complex relative voltage change δU_0 vs. argument φ

4. Adaptive criterion formulas allowing for lumped parameters analysis

The conjunction of (36) and (37) defines necessary and sufficient conditions for limiting relative voltage change. However, fulfillment of conjunction of (36) and (37) is not a simple algorithm for engineers. Hence, it is required more convenient formulation of the proposed criterion. Let us introduce a certain sufficient condition which develops a more convenient criterion form. Namely, the conjunction of two inequalities (36) and (37) is satisfied if the module (upper line in Fig. 8) is limited. The most rigorous (strict) condition is accepted if the minimal value of all moduli (for any φ) is chosen as a critical value for both real and imaginary parts. Thus, the criterion is more rigorous and independent from argument φ (argument φ has not to be evaluated). Hence, the new criterion can be easily applied by engineers and could be also adjusted due to some particular requirements. Exemplary, for few moduli of relative output voltage change (1st column in Table 1) the different-level criteria are established.

Table1
Exemplary criteria for transmission line

Relative voltage change module $\lvert\delta U_0\rvert$ lower than k	Limit value (for given k) $\left\lvert\frac{\Gamma l}{2\pi}\right\rvert = \left\lvert\frac{\alpha l}{2\pi} + \mathrm{j}\frac{l}{\lambda}\right\rvert$	Proposed criterion $\left\lvert\frac{\alpha l}{2\pi} + \mathrm{j}\frac{l}{\lambda}\right\rvert$
5%	$= 0.0493\ldots \approx 1/20$	$< 1/20$
10%	$= 0.0684\ldots \approx 1/15$	$< 1/15$
15%	$= 0.0822\ldots \approx 1/12$	$< 1/12$
20%	$= 0.0932\ldots \approx 1/11$	$< 1/11$

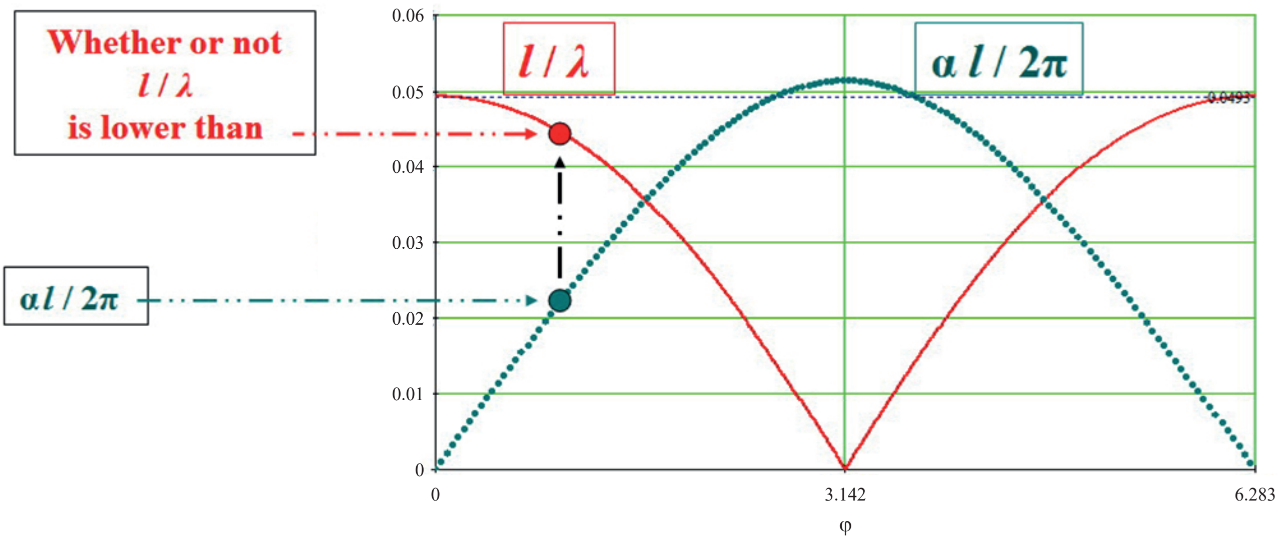

Fig. 9. Graphical interpretation for proposed criterion for dissipative transmission line

The most reasonable seems to admit the criterion which guarantees level $k = 0.05$ for given transmission line, thus

$$\left|\frac{\Gamma l}{2\pi}\right| = \left|\frac{\alpha l}{2\pi} + \mathrm{j}\frac{l}{\lambda}\right| < \frac{1}{20}. \tag{38}$$

The above written criterion for lossless transmission line ($\alpha = 0$) takes the form of

$$\frac{l}{\lambda} < \frac{1}{20}, \tag{39}$$

and is more strict than the one often presented in bibliography [1, 2, 4, 5] of the form $l < \lambda/10$.

5. Conclusion

The presented discussion indicates that criterion for transmission line cannot be arbitrarily chosen (e.g. $l < \lambda/10$), but it has to be well-motivated and strictly proved. The criterion has to have the well-motivated physical interpretation and a mathematical form as simple as possible. Such a way of introducing criterion enables engineers for its simple practical interpretation and adaptation. The criterion allows for simplifying electromagnetic field phenomena and performing lumped parameters analysis.

The proposed criterion (38) is based on output voltage change at no-load steady state of work for sine input voltage. This criterion takes into account dissipation i.e. G_0, R_0 of transmission line, not only the wave length λ. Moreover, the criterion's physical motivation indicates the technical consequences bounded with its application.

The criterion in the form of $l < \lambda/10$ (e.g. [1, 4, 5]) cannot be accepted due to great relative voltage change over 23% and failing to take into account the dissipation of transmission line (Fig. 5).

The proposed criterion (38) allows for accepting the lumped parameters analysis for transmission line and limiting the relative output voltage change at no-load state of work on level 5%.

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