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SPECIAL SECTION

Center of circles intersection, a new defuzzification method for fuzzy numbers

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Abstract. The article introduces a new proposal of a defuzzification method, which can be implemented in fuzzy controllers. The first chapter refers to the origin of fuzzy sets. Next, a modern development based on this theory is presented in the form of ordered fuzzy numbers (OFN). The most important characteristics of ordered fuzzy numbers are also presented. In the following chapter, details about the defuzzification process are given as part of the fuzzy controller model. Then a new method of defuzzification is presented. The method is named center of circles intersection (CCI). The authors compare this method with a similar geometric solution: triangular expanding (TE) and geometric mean (GM). Also, the results are compared with other methods such as center of gravity (COG), first of maxima (FOM) and last of maxima (LOM). The analysis shows that the proposed solution works correctly and provides results for traditional fuzzy numbers as well as directed fuzzy numbers. The last chapter contains a summary, in which more detailed conclusions are provided and further directions of research are indicated.

Key words: defuzzification, fuzzy logic, ordered fuzzy numbers.

1. Introduction

Precise or clear form of expression often poses several difficulties in the communication between humans and computer programs. One of many possible solutions is the use of fuzzy logic as a tool for describing inaccurate phenomena. The above statement is confirmed in studies such as [1] and [2–5]. The theory [6] formulated by Zadeh in mid-sixties has become an alternative to existing mathematical models. The next step in improving the solutions of fuzzy logic is the proposition [7] of Dubois and Prade, of introducing a limited class function belonging to two L and R shape functions. Naturally, also in this case a certain type of imperfection is described as: rounding errors, expansion of the carrier or inconsistency with the arithmetic of real numbers. The above inconvenience has become the next step in the development of fuzzy logic. The forerunner of the new theory is Professor W. Kosinski together with his PhD student P. Prokopowicz and colleague D. Ślęzak. In 2001–2002 they proposed a new concept of fuzzy number [8, 9] as ordered fuzzy number (OFN), where the main advantage used at that time was the ability to solve equation (1).

$$A + B = C, \tag{1}$$

where A, B are any directed (ordered) numbers for which equality should result.

$$C - B = A \tag{2}$$

In the case of the number proposed by Dubois and Prade, equation (2) will be incorrect due to the increased imprecision of the fuzzy number. The use of OFN numbers enables us to freely perform arithmetic operations on OFNs and on real numbers.

2. Preliminaries

Referring to Zadeh's article "Fuzzy sets" published in the Information and Control magazine in 1965, we call a set of ordered pairs with a fuzzy set [6].

$$A = \{(x, \mu_A(x)) : x \in X\} \;\; thus \;\; \mu_A(x) : X \to [0, 1], \tag{3}$$

where the μ function is called the membership function of a fuzzy set A.

Thus, we will identify ordered fuzzy number with the ordered (directed) pair of functions as given in the following definition (1).

Definition 1. Ordered fuzzy number A is an ordered pair of functions

$$A = (f_A, g_A) \tag{4}$$

where: $f_A, g_A : [0, 1] \to R$ are continuous functions.

The introduced definition of ordered fuzzy number shows that it is an extension of classic fuzzy numbers elsewhere called convex ones. Classic fuzzy numbers and their applications

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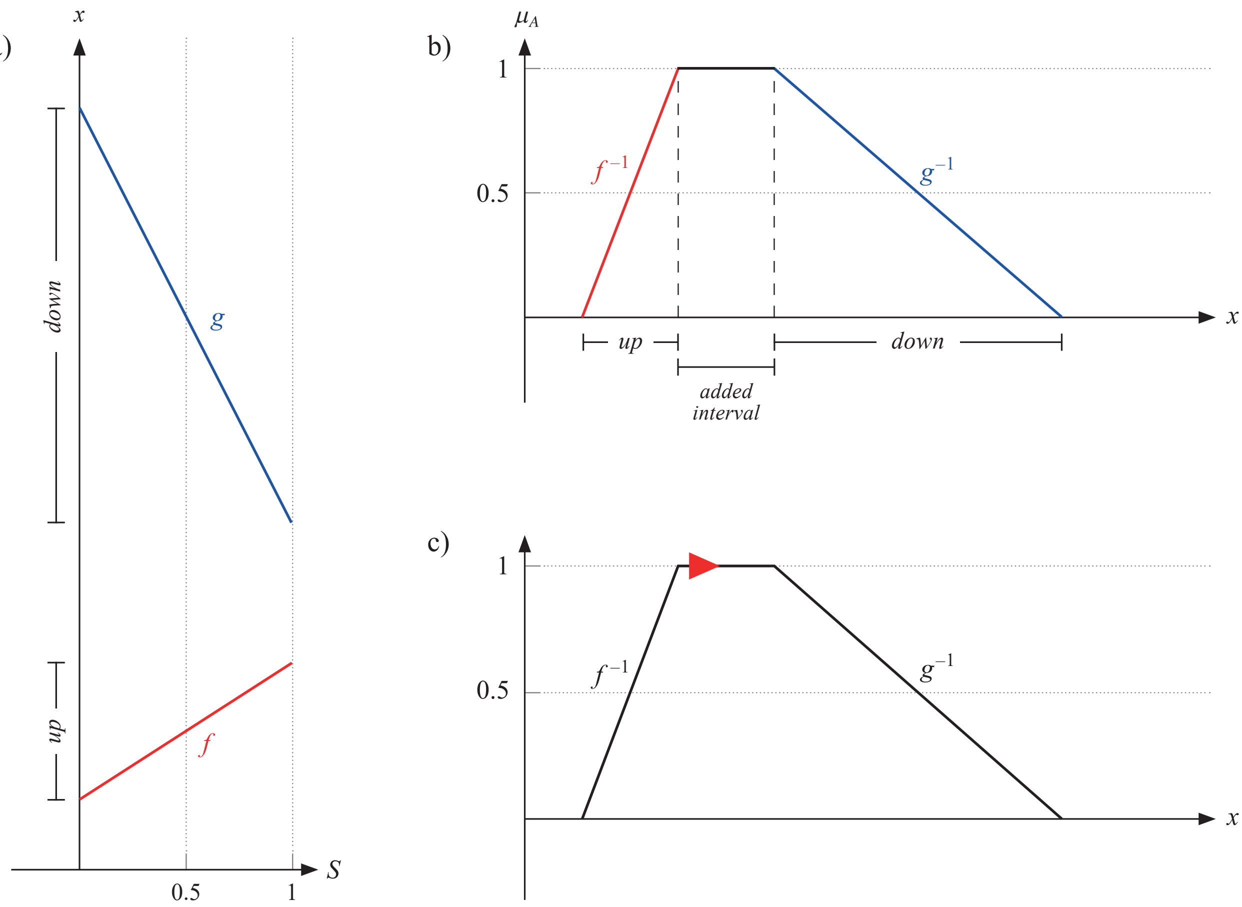

Fig. 1. Example (a) of ordered fuzzy number and interpretation (b) and (c) referring to convex fuzzy number

are widely described and available in numerous publications [10–13]. Graphic interpretation of the ordered fuzzy number and its reference to convex numbers is presented in Fig. 1.

In essence, the solution being discussed is a new feature that introduces new possibilities of describing reality. While classic solutions of fuzzy logic support a number of activities, some information about a given process is skipped. Thus, the introduction of a direction in fuzzy logic is an important element. Therefore, the order (direction) is called the property resulting from the definition of the ordered fuzzy number, where the order of the couples of functions f and g is important. Consequently, number (f, g) is not the same as number (f, g). And so we can distinguish between orientation. Positive orientation, for which the direction is consistent with the *OX* axis and, a negative one, the direction of which is opposite to the *OX* axis. The form of directing depends on the background and the specification of the problem. In many cases, order (direction) is the same as the trend of a given phenomenon [14, 15]. Further appropriate example discussing the order (direction) will be the interpretation taken from publication [16]. The author,

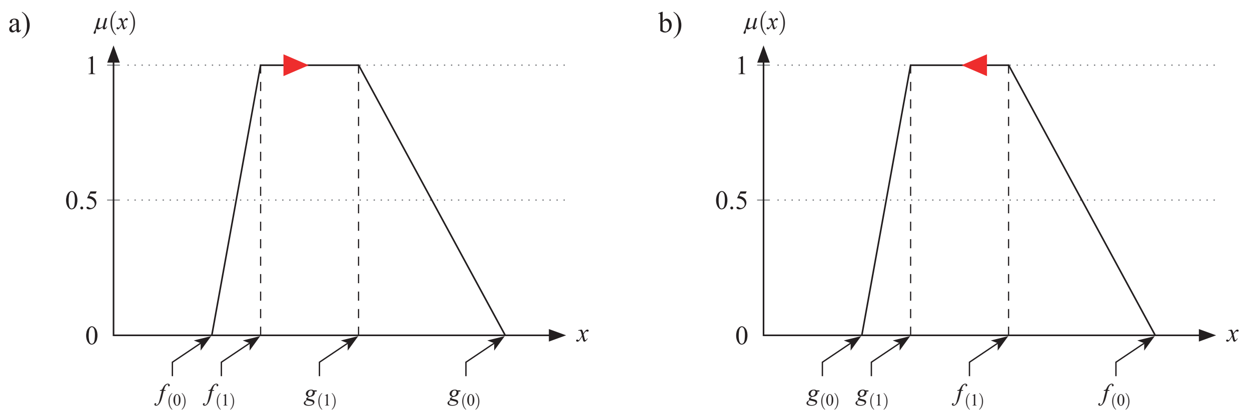

Fig. 2. Ordered fuzzy number in two variants of orientation along with characteristic points: a) positive orientation, b) negative orientation

basing on the case of train speed assessment, explains the situation in which the mere finding of speed is insufficient to fully assess the situation. The speed of the object "about 20 km/h" in the classical notation says nothing about the direction of this change. In particular, whether the train is coming or going away from the observer. Thus, using the ordered fuzzy number, we obtain specific information, e.g. whether the distance from the station will increase or decrease. In this case, we have information in which direction the movement proceeds.

It is worth noting that OFN finds wider and wider applications in practice [17, 18], also among the achievements of the main authors of this article, especially in the area of control devices and tools [19–22] and in finance [14, 15, 23, 24].

3. Characteristics of defuzzification methods

The mathematical foundations of the fuzzy controller can be found in the literature [13, 25–27]. The characteristics of this topic are related to concepts such as: a linguistic variable which assumes the role of the description of entry, the output of the state we intend to express, evaluation using a linguistic description; with the linguistic value directly concerning verbal evaluation of the linguistic dimension [28]. For example, the linguistic variable of "voltage" adopts the linguistic values of "small, medium and high". The fuzzy control process is shown in Fig. 3, which includes operations such as: fuzzification, inference, defuzzification. In the first stage there is a fuzzification operation connected with the calculation of the degree of belonging to individual fuzzy sets. In the inference stage from the input membership levels, one can calculate the resultant membership function. The ending operation of the system is the defuzzification block, which will be discussed in more detail later in the article.

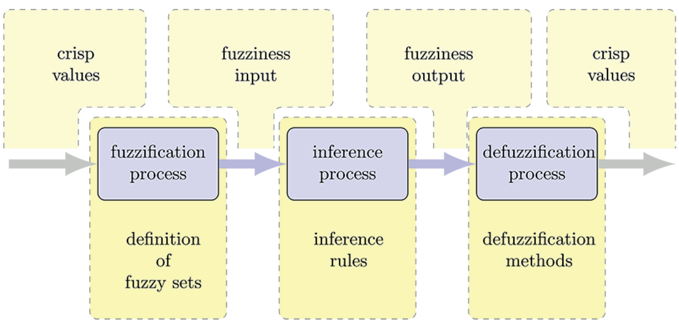

Fig. 3. Schematic diagram of the fuzzy controller

3.1. Defuzzification operator. The actual value of the resultant fuzzy set, as the conclusion of the fuzzy system, enables us to control the given process. This activity is carried out on the basis of the defuzzification component, having the appropriate method at its disposal. Therefore, the defuzzification apparatus can be defined as:

$$W: \{f : X \to [0, 1]\} \to X \tag{5}$$

where: W – defuzzification operator, f – membership function and X – set of the universe on which membership functions are defined.

3.2. Discussing methods. Due to the fact that fuzzy logic is used in many areas of activity, we have access to various methods of defuzzification such as those discussed in [29, 30]. This part of the study will present succinctly important methods that are detailed in publications such as [31–33].

The variety of forms of obtaining crisp (real) value is concentrated in three main categories, i.e. maxima methods: random choice of maxima (RCOM), first of maxima (FOM), last of maxima (LOM), middle of maxima (MOM) – in which selection of crisp values is made from the fuzzy set kernel; distribution methods: center of gravity (COG), mean of maxima (MOM), basic defuzzifcation distribution (BADD), generalized level set defuzzifcation (GLSD), indexed centre of gravity (ICOG), fuzzy mean (FM) – transformation of membership function to probability distribution, and area methods: center of area (COA) – division of the area under the curve into parts.

4. New method of defuzzifcation

The proposal of a new method of defuzzification for traditional and ordered fuzzy numbers is the center of circles intersection (CCI) method that extends the techniques from [20, 32]. A graphical interpretation of this method is presented

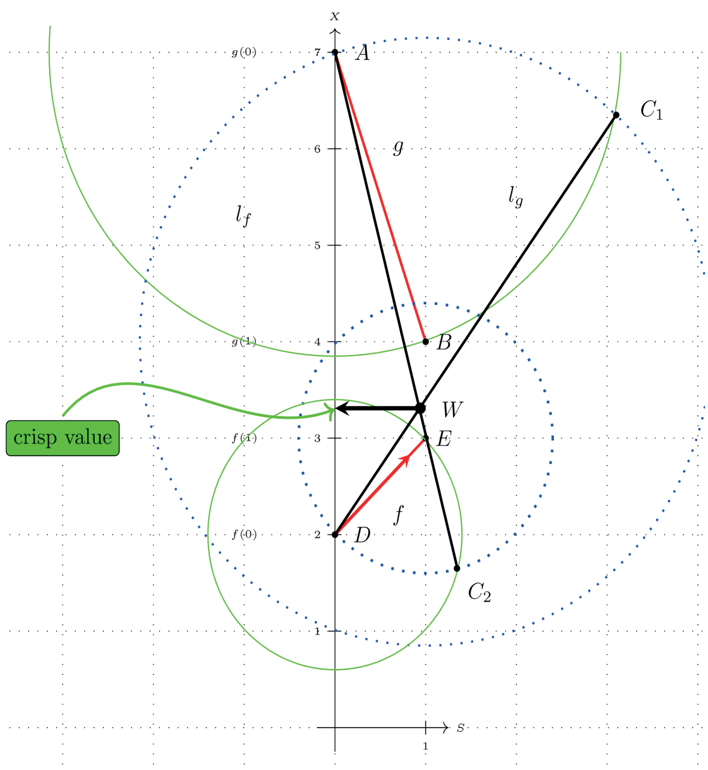

Fig. 4. Graphical interpretation of the center of circles intersection method

in Fig. 4. CCI builds on each side (leg) of fuzzy number two circles. C_1 and C_2 are the points of intersection of these circles over the OX axis. Then one must connect these 2 points diagonally with D and A. The intersection of lines C_1D and C_2A will indicate the result after defuzzification. In this method, crisp value W is consistent with the COG and GM methods for a fuzzy number with symmetrical sides (legs) of equal length. However, if one of the sides is significantly longer than the other, inversely to COG and GM, the CCI method will tend to move the W value closer to the shorter side of the fuzzy number. This interesting property can be useful for many fuzzy controllers.

More specifically, points $A = (0, g(0))$ and $B = (1, g(1))$ form a line AB of the length expressed as $|AB|$, and this value equals radius r_1. The radius is used to build two circles at the ends of the AB section, a circle with center A and radius $r1$ and a circle with center B and the same radius r_1. We will mark them respectively as $KA(A, r_1)$ and $KB(B, r_1)$. Circles KA and KB intersect in two places. The intersection of circles for the positive (higher) values of the membership function is defined as C_1. The choice of point C_1 should be sought to exclude a situation where point C_1 will be located under the axis of the abscissa or under the alternative point of intersection. For the two found intersection points $C_{1U} = (x_{1U}; \mu_{1U})$ and $C_{1T} = (x_{1T}; \mu_{1T})$, the following condition should be met:

$$
\begin{aligned}
C_1 = C_{1U} \Leftrightarrow x_{1U} \in R \quad \wedge \quad \mu_{1U} > \mu_{1T} \\
or \\
C_1 = C_{1T} \Leftrightarrow x_{1T} \in R \quad \wedge \quad \mu_{1T} > \mu_{1U}
\end{aligned}
\tag{6}
$$

where: x_{1U} – coordinate of the point on the x axis; μ_{1U} – coordinate of the point on the abscissa (membership function); U, T – distinguishes between alternative intersection points under or above the x axis.

Analogously, points $D = (0, f(0))$ and $E = (1, f(1))$ form a DE segment with length $|DE|$ which is equal to radius r_2. Using radius r_2, one creates two circles: $KD(D, r_2)$ and $KE(E, r_2)$. The intersection of circles in the positive values of the membership function is referred to as C_2. For the two found intersection points $C_{2U} = (x_{2U}, \mu_{2U})$ and $C_{2T} = (x_{2T}, \mu_{2T})$, the following condition should be met:

$$
\begin{aligned}
C_2 = C_{2U} \Leftrightarrow x_{2U} \in R \quad \wedge \quad \mu_{2U} > \mu_{2T} \\
or \\
C_2 = C_{2T} \Leftrightarrow X_{2T} \in R \quad \wedge \quad \mu_{2T} > \mu_{2U}.
\end{aligned}
\tag{7}
$$

For the points C_1 and C_2 obtained in this way, the lines C_1D and C_2A should be created. The intersection of segments C_1D and C_2A at point W will indicate the result of defuzzification (crisp value) read from the OX axis. W is the final result of the center of circles intersection method. It should be noted that if there is no point of intersection of lines C_1D and C_2A, defuzzification by this method is impossible. Such a case (e.g. parallel lines) may occur for a very unusual shape of a fuzzy number.

5. Comparison of methods

This chapter compares the proposed center of circles intersection (CCI) method with other selected methods of defuzzification including COG, GM, TE, FOM and LOM. The COG (centre of gravity) method is the first method chosen for comparison. This method is characterized by relatively good results for both ordered (directed) and convex fuzzy numbers. The software developed by Ł. Lewiński and M. Szymański was used to determine the value for this method. For other methods, the crisp value was determined using the GeoGebra – Dynamic Mathematics for EveryoneFF tool.

A detailed description of the GM mean method is available in the paper by D. Wilczyńska [16]. The value of defuzzification is obtained by finding the intersection point of two lines placed in the poles of the OFN.

The TE method was described in detail by the authors in paper [31, 32]. It is similar to CCI, using a one-sided intersection of circles based on the leg (lateral side) of OFN.

The last for comparison are two methods from the group of maxims. In the FOM (first of maxima) method, the value of defuzzification is equal to the value of x for the first element of the kernel. Where the kernel forms part of the domain for which the membership function has a maximum value of one. Similarly, the LOM (last of maxima) method is a method in which the value of defuzzification equals the last element of the kernel. In the experiment, two OFNs, the first $H = [4, 6, 8, 14]$ and the second oppositely directed $Z = [14, 8, 6, 4]$ were used. The graphical interpretation of the numbers H and Z is demonstrated in the figure below. Above, two OFNs are presented. They are stretched to the same values, but they differ in their directing. The numbers H and Z were used as the base for defuzzification using different methods. The crisp values determined can be

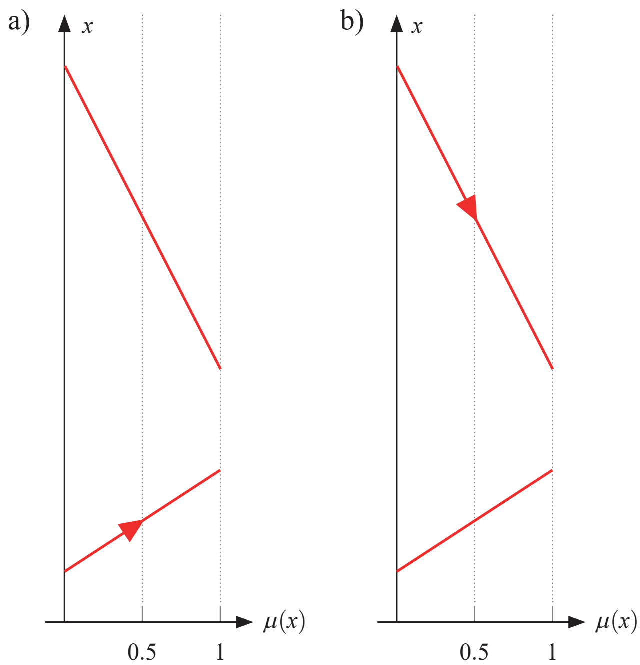

Fig. 5. Ordered fuzzy number. a) $H = [4, 6, 8, 14]$, b) $Z = [14, 8, 6, 4]$

Table 1
Measurements of defuzzification for two OFNs

	Defuzzification method					
Ordered fuzzy number	CCI	COG	GM	TE	FOM	LOM
$H = [4, 6, 8, 14]$	6.7	8.2	7.3	6.6	6	8
$Z = [14, 8, 6, 4]$	6.7	8.2	7.3	7.3	8	6

found in Table 1. GM and COG methods are one of those for which direction does not affect the value of defuzzification. On the other hand, if we use methods such as FOM or LOM, then the defuzzification results (crisp values) depend on the OFN direction, and the boundary values of the kernel region are returned. The results referring to the method of center of circles intersection confirm that this method is not sensitive to directing. Both for the number H and for the number Z, the value of defuzzification is 6.7. This means that the CCI method can be applied to traditional fuzzy numbers as well as to OFN. Methods sensitive to directing are TE, FOM and LOM are methods sensitive to directing. The CCI method can return the same results as GM. In particular if both sides (legs) of the fuzzy number are symmetrical, i.e. they intersect the x axis at the same angle.

6. Concluding remarks

Control devices that are based on directed fuzzy numbers are increasingly used in practice as inference or control tools [20]. The huge advantage of OFN in in comparison with traditional fuzzy numbers is the possibility of using additional arithmetic operations, thus extending the functionality of controllers. An important element of the controllers is the defuzzification block containing the defuzzification algorithm.

The article proposes a new method of defuzzification – center of circles intersection (CCI). This method is conceptually related to the TR and extended TR methods. However, in practice, this is an alternative to existing solutions that are insensitive to directing, in particular the GM (geometric mean) or COG (centre of gravity). The results indicate that for the given simple examples GM, COG and CCI return similar results. The created method can be applied to traditional as well as ordered fuzzy numbers.

The authors intend to undertake further research in order to find the best applications for the method. In another separate study, a description of the method will be prepared with detailed derivation of all CCI formulas. It is also important to compare the new solution with further methods of defuzzification such as COA, and to examine the range of results for fuzzy numbers with unusual shape.

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