10.24425/acs.2021.136887

Archives of Control Sciences
Volume 31(LXVII), 2021
No. 1*, pages 213–247*

Interval-valued dual hesitant fuzzy prioritized aggregation operators based on Archimedean t-conorm and t-norm and their applications to multi-criteria decision making

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Multi-criteria decision making (MCDM) technique and approach have been a trending topic in decision making and systems engineering to choosing the probable optimal options. The primary purpose of this article is to develop prioritized operators to multi-criteria decision making (MCDM) based on Archimedean t-conorm and t-norms (At-CN&t-Ns) under interval-valued dual hesitant fuzzy (IVDHF) environment. A new score function is defined for finding the rank of alternatives in MCDM problems with IVDHF information based on priority levels of criteria imposed by the decision maker. This paper introduces two aggregation operators: At-CN&t-N-based IVDHF prioritized weighted averaging (AIVDHFPWA), and weighted geometric (AIVDHFPWG) aggregation operators. Some of their desirable properties are also investigated in details. A methodology for prioritization-based MCDM is derived under IVDHF information. An illustrative example concerning MCDM problem about a Chinese university for appointing outstanding oversea teachers to strengthen academic education is considered. The method is also applicable for solving other real-life MCDM problems having IVDHF information.

Key words: multi-criteria decision-making, interval-valued dual hesitant fuzzy elements, Archimedean t-conorm and t-norm, prioritized weighted averaging operator, prioritized weighted geometric operator

1. Introduction

The ambiguity of information is becoming an unalterable situation due to the rising complexity of our lifestyle rapidly. Multi-criteria decision making (MCDM) methods are a handy tool to grip this type of situation. Therefore,

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Received 20.07.2020.

MCDM has been an inexorable process to assess an object precisely. Besides the prior several decades, various methods have been proposed for solving different MCDM problems. Decision-maker (DM) can give their opinion by hesitant fuzzy (HF) set (HFS) [1,2] to defeat any hesitations. Generally, aggregation operators are essential tools for dealing with such MCDM problems. Xia and Xu [3] proposed a series of weighted averaging (WA) and weighted geometric (WG) aggregation operators based on HF environment viz., HF WA, HF ordered WA and their geometric operators. Based on Einstein operation, Zhou and Li [4] defined HF Einstein WG, and HF Einstein ordered WG operators and established the connections between the proposed operators. Zhang [5] proposed a method for deriving the weights of DMs and solved a multi-criteria group decision making (MCGDM) problem under HF information. Based on Hamacher t-conorm (t-CN) and t-norms (t-Ns), Son et al. [6] introduced some new HF power aggregation operators. Inspired by the concept of intuitionistic fuzzy (IF) set (IFS) and HFS, Zhu et al. [7] introduced dual HF (DHF) set (DHFS) by considering possible membership degrees and non-membership degrees with the condition that sum of maximum membership and non-membership degrees is less or equal to one. Under the DHF context, Wang et al. [8] defined some WA and WG aggregation operators: DHF WA, DHF WG, DHF ordered WA and DHF ordered WG operators. With Hamacher operations, Ju et al. [9] developed several aggregation operators viz., DHF Hamacher WA, DHF Hamacher WG, DHF Hamacher ordered WA, DHF Hamacher ordered WG operators, etc. Yu et al. [10] introduced the aggregation operators for aggregating DHF elements (DHFEs) and described these operators' properties. Zhao et al. [11] proposed some arithmetic operations of DHFEs based on Einstein t-CN and t-N, and some DHF aggregation operators are also introduced. Tang et al. [12] proposed the generalized rules of DHFS based on Frank t-CN and t-N, and used to construct the aggregation operators on DHF assessments in the context of MCDM.

However, in several real-life MCDM models, due to insufficiency in available information, DM are unable to exert their opinion with a crisp number but are comfortable to putting the decision values by interval numbers within $[0, 1]$. To address this situation, Ju et al. [13] introduced the concept of interval-valued DHF (IVDHF) sets (IVDHFSs), which takes the hesitant membership and non-membership degrees in the form of interval-valued fuzzy numbers. It should be noted that when both the membership degree and non-membership degree of each element to a given set have single interval value, the IVDHFS reduces to the interval-valued IFS [14] and when the upper and lower limits of interval values are identical, IVDHFS becomes DHFS [7]. Thus, it is clear that IVDHFS is a more generalized form than other extensions of fuzzy sets. To aggregate the IVDHF elements (IVDHFEs), Ju et al. [13] developed IVDHF WA aggregation operator. Further, Zhang et al. [15] imposed Einstein t-CN and t-N on IVDHF environment to develop IVDHF Einstein WA and IVDHF Einstein WG operators.

During the aggregation process, the selection of appropriate operational laws is a crucial phase. The Archimedean t-CN and t-N (At-CN&t-N) provides a general rule of operational laws and more choices for DM. Different classes of t-CNs and t-Ns can be derived from At-CN&t-N [16, 17], such as t-CNs and t-Ns of the Algebraic, Einstein, Hamacher, Frank, and so on. Based on At-CN&t-N, Xia et al. [18] introduced At-CN&t-N-based IF WA and WG operators. Zhang and Wu [19] developed several At-CN&t-N-based interval-valued HF (IVHF) WA and WG aggregation operators. On DHF environment, Yu [20] proposed DHF WA and WG aggregation operators based on At-CN&t-N operations. Recently, Sarkar and Biswas [21] introduced At-CN&t-N operations on Pythagorean HF sets and defined a class of At-CN&t-N-based Pythagorean HF WA and WG operators. Again Sarkar and Biswas [22] applied At-CN&t-N on the IVDHF information and introduced a class of aggregation operators.

The above methods are all used under the premise that all criteria are in the same priority level. Most applications involve selecting or ordering of a group of alternatives based upon their satisfaction to a collection of criteria. To deal with this issue, Yager [23] developed prioritized average (PA) operators by modelling the criteria priority on the weights associated with criteria, which are dependent on the satisfaction of higher priority criteria. Yager [24] further focused on PA operators and proposed two methods for formulating this type of aggregation process. It is well known that the PA operator has many advantages over other operators. On HF environment, Yu [25] developed a family of aggregation operators based on Einstein t-CN and t-N, such as HF Einstein prioritized WA, WG and power WA operators. Wei [26] developed two prioritized aggregation operators for aggregating HFEs: HF prioritized WA (HFPWA), and HF prioritized WG (HFPWG) operators. Chen [27] introduced interval-valued IF prioritized aggregation operator and illustrated the proposed methodology by solving the watershed site selection problem. Liang et al. [28] derived generalized intuitionistic trapezoidal fuzzy prioritized WA and WG operators, also construct an approach for MCGDM under intuitionistic trapezoidal fuzzy environment. Under the IVHF context, Ye [29] proposed IVHF prioritized WA and WG operators and presented some properties of the proposed aggregation operators. Jin et al. [30] introduced Einstein operational laws on IVHF sets, and also developed two prioritized aggregation operators: IVHF Einstein prioritized WA (IVHFEPWA) and IVHF Einstein prioritized WG (IVHFEPWG) operators. Ren and Wei [31] proposed a prioritized multi-attribute decision-making method to solve decision problems under DHF environment. Recently, Biswas and Sarkar [32] introduced Einstein operations-based DHF prioritized WA (DHFPWA), and WG (DHFPWG) operators and constructed an approach for MCGDM. However, prioritized aggregation operators are applied in various contexts viz., IF, HF, IVHF, DHF for MCDM. But many prioritized-based MCDM problems can not be solved which are designed on IVDHF environment. And to overcome such situation, a methodology is proposed for IVDHF prioritized MCDM, which is the main motivation of this article. To do this at first define two prioritized aggregation operators based on At-CN&t-Ns under IVDHF information.

This article is organized as follows. Some preliminary concepts on DHFS, IVDHFS, At-CN&t-Ns and At-CN&t-Ns-based operations on IVDHFEs are studied in Section 2. A new score function of IVDHFE is defined in Section 3. In Section 4, At-CN&t-Ns-based IVDHF prioritized WA (AIVDHFPWA), and WG (AIVDHFPWG) aggregation operators are proposed to aggregate the IVDHFEs. After that classification of the proposed operators is made for different types of decreasing functions. Some desired properties and special cases of the proposed operators are also investigated. Section 5 gives an approach to MCDM under IVDHF environment. In Section 6, an illustrative example is solved using the proposed method, and sensitivity analysis is performed by varying the parameter. Finally, conclusion and scope for future studies have been described in Section 7.

2. Preliminaries

This section briefly reviews some basic concepts of DHFS, IVDHFS, At-CN&t-Ns and prioritized aggregation operators.

2.1. DHFS

Definition 1 [7] *The concept of DHFS was presented by Zhu et al. [7]. Let X be a fixed set. Then a DHFS is defined as*

$$P = \{\langle x, h_P(x), g_P(x)\rangle \mid x \in X\}, \tag{1}$$

where $\{\mu \mid \mu \in h_P(x)\}$ and $\{\nu \mid \nu \in g_P(x)\}$ denote the set of possible membership and non-membership degrees, respectively, of the element $x \in X$ to the set P, satisfying the conditions:

$0 \leqslant \mu, \nu \leqslant 1,\ \ 0 \leqslant \max\{\mu\} + \max\{\nu\} \leqslant 0$, *for all $x \in X$. For convenience, $\langle h_P(x), g_P(x)\rangle$ is called the DHF element (DHFE) and denoted by $p = \langle h, g\rangle$.*

To compare among the DHFEs, Zhu et al. [7] derived the following comparison formula.

Definition 2 [7] *Let $p = \langle h, g\rangle$ be a DHFE. Then the score function $S(p)$ and accuracy function $A(p)$ of p is defined by*

$$S(p) = \hat{h} - \hat{g} \quad \textit{and} \quad A(p) = \hat{h} + \hat{g},$$

where $\hat{h} = \dfrac{1}{\#h}\sum_{\mu\in h}\mu$ and $\hat{g} = \dfrac{1}{\#g}\sum_{\nu\in g}\nu$, and $\#h$ and $\#g$ denote the number of elements in h and g, respectively.

For any two DHFEs p_1 and p_2, if $S(p_1) > S(p_2)$ then $p_1 \succ p_2$.

To compute DMs' preference values by an interval number within [0, 1] instead of crisp numbers, Ju et al. [13] defined the concept of IVDHFSs.

Definition 3 [13] *Let X be a given set, then an IVDHFS $\widetilde{\mathrm{A}}$ on X is described as:*

$$\widetilde{\mathrm{A}} = \left\{ \left\langle x, \widetilde{h}_{\widetilde{\mathrm{A}}}(x), \widetilde{g}_{\widetilde{\mathrm{A}}}(x) \right\rangle \mid x \in X \right\}, \tag{2}$$

in which $\widetilde{h}_{\widetilde{\mathrm{A}}}(x) = \bigcup_{[\gamma^l,\gamma^u]\in\widetilde{h}(x)} \left\{ \left[\gamma^l, \gamma^u\right] \right\}$ and $\widetilde{g}_{\widetilde{\mathrm{A}}}(x) = \bigcup_{[\eta^l,\eta^u]\in\widetilde{g}(x)} \left\{ \left[\eta^l, \eta^u\right] \right\}$ are two sets of interval values in [0, 1]*, representing the possible membership degree and non-membership degree of the element $x \in X$ to the set $\widetilde{\mathrm{A}}$, respectively, with $\left[\gamma^l, \gamma^u\right] \left[\eta^l, \eta^u\right] \subset [0, 1]$ and $0 \leqslant \max\{\gamma^u\} + \max\{\eta^u\} \leqslant 1$, for all $x \in X$. For convenience, Ju et al. [13] called the pair $\widetilde{\alpha}(x) = \left(\widetilde{h}(x), \widetilde{g}(x)\right)$ an IVDHF element (IVDHFE) and denoted by $\widetilde{\alpha} = \left(\widetilde{h}, \widetilde{g}\right)$.*

To compare the IVDHFEs, Ju et al. [13] defined the score function and accuracy function in the following manner.

Definition 4 [13] *Score function of IVDHFE $\widetilde{\alpha} = \left(\widetilde{h}, \widetilde{g}\right)$ is defined as*

$$H(\widetilde{\alpha}) = \frac{1}{2}\left(\frac{1}{\Delta\widetilde{h}} \sum_{[\gamma^l,\gamma^u]\in\widetilde{h}} \left(\gamma^l + \gamma^u\right) - \frac{1}{\Delta\widetilde{g}} \sum_{[\eta^l,\eta^u]\in\widetilde{g}} \left(\eta^l + \eta^u\right) \right), \tag{3}$$

and accuracy function of IVDHFE $\widetilde{\alpha} = \left(\widetilde{h}, \widetilde{g}\right)$ is defined as

$$A(\widetilde{\alpha}) = \frac{1}{2}\left(\frac{1}{\Delta\widetilde{h}} \sum_{[\gamma^l,\gamma^u]\in\widetilde{h}} \left(\gamma^l + \gamma^u\right) + \frac{1}{\Delta\widetilde{g}} \sum_{[\eta^l,\eta^u]\in\widetilde{g}} \left(\eta^l + \eta^u\right) \right), \tag{4}$$

where $\Delta\widetilde{h}$ and $\Delta\widetilde{g}$ is the number of intervals in $\widetilde{h}$ and $\widetilde{g}$ respectively.

Definition 5 *Let $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ be any two IVDHFEs,*

(i) If $H(\widetilde{\alpha}_1) > H(\widetilde{\alpha}_2)$ then $\widetilde{\alpha}_1 > \widetilde{\alpha}_2$;

(ii) If $H(\widetilde{\alpha}_1) = H(\widetilde{\alpha}_2)$ then if $A(\widetilde{\alpha}_1) > A(\widetilde{\alpha}_2)$ then $\widetilde{\alpha}_1 > \widetilde{\alpha}_2$; if $A(\widetilde{\alpha}_1) = A(\widetilde{\alpha}_2)$ then $\widetilde{\alpha}_1 = \widetilde{\alpha}_2$.

2.2. A t-CN&t-Ns

In this section, the definition of At-CN&t-Ns is displayed.

Definition 6 [16, 17] *A function $U : [0,1] \times [0,1] \to [0,1]$ is called a t-CN if it satisfies associativity, symmetricity, non-decreasing and $U(x,0) = x$ for all $x \in [0,1]$. If a binary operation $I : [0,1] \times [0,1] \to [0,1]$ satisfies associativity, symmetricity, non-decreasing and $I(x,1) = x$ for all $x \in [0,1]$ then I is known as a t-N.*

Archimedean t-CN (At-CN) and Archimedean t-N (At-N) operations are expressed as follows:

Definition 7 *[33] An At-CN U is formulated using increasing function g as*

$$U(x,y) = g^{(-1)}\left(g(x) + g(y)\right), \tag{5}$$

similarly, using decreasing function f, an At-N I is represented as

$$I(x,y) = f^{(-1)}\left(f(x) + f(y)\right) \quad \text{with } g(t) = f(1-t) \text{ for all } x,y,t \in [0,\ 1]. \tag{6}$$

Several t-CNs and t-Ns are derived by Klement and Mesiar [32] using different forms of increasing and decreasing functions; and using these functions Sarkar and Biswas [22] defined some operational rules for IVDHFEs based on algebraic, Einstein, Hamacher, and Frank classes of t-CN and t-Ns.

Definition 8 [22] *Let $\widetilde{\alpha}_i = \left(\widetilde{h}_i, \widetilde{g}_i\right)$ $(i = 1,2)$ and $\widetilde{\alpha} = \left(\widetilde{h}, \widetilde{g}\right)$ be any three IVDHFEs, $\lambda > 0$ be any scalar. At-CN&t-Ns-based operational laws for the IVDHFEs are presented bellow.*

(1) $\widetilde{\alpha}_1 \oplus_A \widetilde{\alpha}_2 =$

$$\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i\\ ,i=1,2}} \left\{\left[U\left(\gamma_1^l,\gamma_2^l\right), U\left(\gamma_1^u,\gamma_2^u\right)\right]\right\}, \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2}} \left\{\left[I\left(\eta_1^l,\eta_2^l\right), I\left(\eta_1^u,\eta_2^u\right)\right]\right\}\right)$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2}} \left\{\left[g^{-1}\left(g\left(\gamma_1^l\right) + g\left(\gamma_2^l\right)\right), g^{-1}\left(g\left(\gamma_1^u\right) + g\left(\gamma_2^u\right)\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2}} \left\{\left[f^{-1}\left(f\left(\eta_1^l\right) + f\left(\eta_2^l\right)\right), f^{-1}\left(f\left(\eta_1^u\right) + f\left(\eta_2^u\right)\right)\right]\right\}\right);$$

(2) $\widetilde{\alpha}_1 \otimes_A \widetilde{\alpha}_2 =$

$$\left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ i=1,2}} \left\{ \left[I\left(\gamma_1^l, \gamma_2^l\right), I\left(\gamma_1^u, \gamma_2^u\right) \right] \right\}, \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, \\ i=1,2}} \left\{ \left[U\left(\eta_1^l, \eta_2^l\right), U\left(, \eta_1^u, \eta_2^u\right) \right] \right\} \right)$$

$$= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ i=1,2}} \left\{ \left[f^{-1}\left(f\left(\gamma_1^l\right) + f\left(\gamma_2^l\right)\right), f^{-1}\left(f\left(\gamma_1^u\right) + f\left(\gamma_2^u\right)\right) \right] \right\}, \right.$$

$$\left. \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, \\ i=1,2}} \left\{ \left[g^{-1}\left(g\left(\eta_1^l\right) + g\left(\eta_2^l\right)\right), g^{-1}\left(g\left(\eta_1^u\right) + g\left(\eta_2^u\right)\right) \right] \right\} \right);$$

(3) $\lambda \widetilde{\alpha} =$

$$\left(\bigcup_{[\gamma^l, \gamma^u] \in \widetilde{h}} \left\{ \left[g^{-1}\left(\lambda g\left(\gamma^l\right)\right), g^{-1}\left(\lambda g\left(\gamma^u\right)\right) \right] \right\}, \right.$$

$$\left. \bigcup_{[\eta^l, \eta^u] \in \widetilde{g}} \left\{ \left[f^{-1}\left(\lambda f\left(\eta^l\right)\right), f^{-1}\left(\lambda f\left(\eta^u\right)\right) \right] \right\} \right);$$

(4) $\widetilde{\alpha}^{\lambda} =$

$$\left(\bigcup_{[\gamma^l, \gamma^u] \in \widetilde{h}} \left\{ \left[f^{-1}\left(\lambda f\left(\gamma^l\right)\right), f^{-1}\left(\lambda f\left(\gamma^u\right)\right) \right] \right\}, \right.$$

$$\left. \bigcup_{[\eta^l, \eta^u] \in \widetilde{g}} \left\{ \left[g^{-1}\left(\lambda g\left(\eta^l\right)\right), g^{-1}\left(\lambda g\left(\eta^u\right)\right) \right] \right\} \right).$$

2.3. PA Operator

PA operator for MCDM problems was introduced by Yager [23], which is defined in the following manner:

Definition 9 *[23] Let $\{C_i\}$ $(i = 1, 2, \ldots, n)$ be a collection of criteria, and their priority is expressed by the linear ordering $C_1 \succ C_2 \succ \ldots \succ C_n$. This ordering indicates criteria C_j has a higher priority than C_k if $j < k$. The value $C_j(z)$ is the performance of any alternative z under criteria C_j, and satisfies $C_j(z) \in [0, 1]$.*

$$\text{If } PA\left(C_j(z)\right) = \sum_{j=1}^{n} w_j C_j(z), \text{ where } w_j = \frac{T_j}{\sum_{j=1}^{n} T_j},\ T_j = \prod_{k=1}^{j-1} C_k(z)$$

$(j = 2, \ldots, n)$, $T_1 = 1$. Then PA is called the PA operator.

In the following section, a new score function of IVDHFEs is introduced. The drawback of score function defined by Ju et al. [9] is that the score value may be negative.

3. Score value of IVDHFE

Definition 10 *Score function of IVDHFE $\widetilde{\alpha} = \left(\widetilde{h}, \widetilde{g}\right)$ is defined as*

$$S\left(\widetilde{\alpha}\right) = \left(\left(\frac{1}{2}\left(\frac{1}{\Delta\widetilde{h}}\left(\sum_{[\gamma^l,\gamma^u]\in h}\left(\gamma^l + \gamma^u\right)\right) - \frac{1}{\Delta\widetilde{g}}\left(\sum_{[\eta^l,\eta^u]\in g}\left(\eta^l + \eta^u\right)\right)\right)\right) + 1\right)\Big/2, \quad (7)$$

where $\Delta\widetilde{h}$ and $\Delta\widetilde{g}$ denote the number of intervals in $\widetilde{h}$ and $\widetilde{g}$, respectively.

To compare among the IVDHFEs, a comparative rule is presented as follows:

Definition 11 *Let $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ be any two IVDHFEs, then*
If $S\left(\widetilde{\alpha}_1\right) > S\left(\widetilde{\alpha}_2\right)$ then $\widetilde{\alpha}_1 > \widetilde{\alpha}_2$.

4. Development of At-CN&t-Ns-based IVDHF prioritized weighted aggregation operators

In this section, the IVDFEs are fused with PA operator based on At-CN&t-Ns and proposed the AIVDHFPWA and AIVDHFPWG operators.

Definition 12 *Let $\widetilde{\alpha}_i = \left(\widetilde{h}_i, \widetilde{g}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IVDHFEs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the weight vectors of $\widetilde{\alpha}_i$ with $\omega_i \in [0, 1]$, where $w_i = \frac{T_i}{\sum_{i=1}^{n} T_i}$, $T_i = \prod_{k=1}^{i-1} S\left(\widetilde{\alpha}_k\right)$ $(i = 2, \ldots, n))$, $T_1 = 1$ and $S\left(\widetilde{\alpha}_i\right)$ is the score of $\widetilde{\alpha}_i$.*

Then, AIVDHFPWA operator is a mapping $\widetilde{\Omega}^n \to \widetilde{\Omega}$, where

$$AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) = \bigoplus_{i=1}^{n}{}_A \left(\frac{T_i}{\sum\limits_{i=1}^{n} T_i} \widetilde{\alpha}_i \right).$$

$\bigoplus_A$ *conveys the meaning as described in Definition 8.*

Theorem 1 *Let $\widetilde{\alpha}_i = \left(\widetilde{h}_i, \widetilde{g}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IVDHFEs, then the aggregated value by using AIVDHFPWA operator is also an IVDHFE and*

$$\begin{aligned} AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) &= \bigoplus_{i=1}^{n}{}_A \left(\frac{T_i}{\sum\limits_{i=1}^{n} T_i} \widetilde{\alpha}_i \right) \\ &= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ i=1,2,\ldots,n}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma_i^l\right) \right), g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma_i^u\right) \right) \right] \right\}, \right. \\ &\quad \left. \bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, \\ i=1,2,\ldots,n}} \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta_i^l\right) \right), f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta_i^u\right) \right) \right] \right\} \right). \end{aligned} \tag{8}$$

Proof. The theorem will be proved using the mathematical induction method.

The theorem is obvious for $n = 1$.

Assume that theorem is valid for $n = p$, it will prove that it is also valid for $n = p + 1$.

when $n = p$,

$$\begin{aligned} AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p\right) &= \\ = \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ i=1,2,\ldots,p}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{p} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma_i^l\right) \right), g^{-1}\left(\sum_{i=1}^{p} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma_i^u\right) \right) \right] \right\}, \right. \end{aligned}$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,p}} \left\{\left[f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\eta_i^l\right)\right), f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\eta_i^u\right)\right)\right]\right\}\Bigg).$$

Now when $n = p + 1$,

$$AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p, \widetilde{\alpha}_{p+1}\right) =$$

$$= AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p\right) \bigoplus_A \left(\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}\widetilde{\alpha}_{p+1}\right),$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,p}} \left\{\left[g^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum_{i=1}^{n}T_i}g\left(\gamma_i^l\right)\right), g^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum_{i=1}^{n}T_i}g\left(\gamma_i^u\right)\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,p}} \left\{\left[f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\eta_i^l\right)\right), f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\eta_i^u\right)\right)\right]\right\}\right) \bigoplus_A$$

$$\left(\bigcup_{[\gamma_{p+1}^l,\gamma_{p+1}^u]\in\widetilde{h}_{p+1}} \left\{\left[g^{-1}\left(\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}g\left(\gamma_{p+1}^l\right)\right), g^{-1}\left(\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}g\left(\gamma_{p+1}^u\right)\right)\right]\right\},\right.$$

$$\left.\bigcup_{[\eta_{p+1}^l,\eta_{p+1}^u]\in\widetilde{g}_{p+1}} \left\{\left[f^{-1}\left(\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}f\left(\eta_{p+1}^l\right)\right), f^{-1}\left(\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}f\left(\eta_{p+1}^u\right)\right)\right]\right\}\right),$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,p,p+1}} \left\{\left[g^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum_{i=1}^{n}T_i}g\left(\gamma_i^l\right) + fracT_{p+1}\sum_{i=1}^{n}T_i g\left(\gamma_{p+1}^l\right)\right),\right.\right.\right.$$

$$\left.\left. g^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum_{i=1}^{n}T_i}g\left(\gamma_i^u\right) + \frac{T_{p+1}}{\sum_{i=1}^{n}T_i}g\left(\gamma_{p+1}^u\right)\right)\right]\right\}$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,p,p+1}} \left\{\left[f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)+\frac{T_{p+1}}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_{p+1}^l\right)\right),\right.\right.$$

$$\left.\left.\left. f^{-1}\left(\sum_{i=1}^{p}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)+\frac{T_{p+1}}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_{p+1}^l\right)\right)\right]\right\}\right),$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,p,p+1}} \left\{\left[g^{-1}\left(\sum_{i=1}^{p+1}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^l\right)\right),g^{-1}\left(\sum_{i=1}^{p+1}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^u\right)\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,p,p+1}} \left\{\left[f^{-1}\left(\sum_{i=1}^{p+1}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)\right),f^{-1}\left(\sum_{i=1}^{p+1}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^u\right)\right)\right]\right\}\right),$$

$$=\bigoplus_{i=1}^{p+1}{}_{A}\left(\frac{T_i}{\sum\limits_{i=1}^{n}T_i}\widetilde{\alpha}_i\right)=AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_p,\widetilde{\alpha}_{p+1}\right).$$

Hence the theorem is proved for $p+1$ and thus true for all n.

Hence $AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)$ is an IVDHFE.

This completes the proof.

Theorem 2 *(Boundary) Let* $\widetilde{\alpha}_i=\left(\widetilde{h}_i,\widetilde{g}_i\right)$ *(*$i=1,2,\ldots,n$*) be a collection of IVDHFEs, and let for all* $i=1,2,\ldots,n$.

$$\gamma_{\min}^l=\min\left\{\min_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i}\left\{\gamma_i^l\right\}\right\},\qquad \gamma_{\min}^u=\min\left\{\min_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i}\left\{\gamma_i^u\right\}\right\},$$

$$\gamma_{\max}^l=\max\left\{\max_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i}\left\{\gamma_i^l\right\}\right\},\qquad \gamma_{\max}^u=\max\left\{\max_{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i}\left\{\gamma_i^u\right\}\right\},$$

$$\eta_{\min}^l=\min\left\{\min_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i}\left\{\eta_i^l\right\}\right\},\qquad \eta_{\min}^u=\min\left\{\min_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i}\left\{\eta_i^u\right\}\right\},$$

$$\eta_{\max}^l=\max\left\{\max_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i}\left\{\eta_i^l\right\}\right\},\qquad \eta_{\max}^u=\max\left\{\max_{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i}\left\{\eta_i^u\right\}\right\}.$$

Then if $\widetilde{\alpha}_{-} = \left(\left[\gamma^l_{\min}, \gamma^u_{\min}\right], \left[\eta^l_{\max}, \eta^u_{\max}\right]\right)$ *and* $\widetilde{\alpha}_{+} = \left(\left[\gamma^l_{\max}, \gamma^u_{\max}\right], \left[\eta^l_{\min}, \eta^u_{\min}\right]\right)$,

$$\widetilde{\alpha}_{-} \leqslant AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) \leqslant \widetilde{\alpha}_{+}\,. \tag{9}$$

Proof. . For any $i = 1, 2, \ldots, n$, it is clear that $\gamma^l_{\min} \leqslant \gamma^l_i \leqslant \gamma^l_{\max}$ and $\gamma^u_{\min} \leqslant \gamma^u_i \leqslant \gamma^u_{\max}$. Since $g(t)$ $(t \in [0, 1])$ is a monotonic increasing function,

$$g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma^l_{\min}\right)\right) \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma^l_i\right)\right) \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma^l_{\max}\right)\right),$$

which implies that

$$\gamma^l_{\min} \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma^l_i\right)\right) \leqslant \gamma^l_{\max}\,. \tag{10}$$

Similarly, find that

$$\gamma^u_{\min} \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} g\left(\gamma^u_i\right)\right) \leqslant \gamma^u_{\max}\,, \tag{11}$$

for any $i = 1, 2, \ldots, n$, $\eta^l_{\min} \leqslant \eta^l_i \leqslant \eta^l_{\max}$.

Since $f(t)$ $(t \in [0, 1])$ is a decreasing function,

$$f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta^l_{\max}\right)\right) \leqslant f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta^l_i\right)\right) \leqslant f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta^l_{\min}\right)\right),$$

which implies that

$$\eta^l_{\max} \leqslant f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta^l_i\right)\right) \leqslant \eta^l_{\min}\,. \tag{12}$$

Similarly,

$$\eta^u_{\max} \leqslant f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum\limits_{i=1}^{n} T_i} f\left(\eta^u_i\right)\right) \leqslant \eta^u_{\min}\,. \tag{13}$$

From (10) and (12), it is obtained that

$$\gamma^l_{\min} - \eta^l_{\min} \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma^l_i\right)\right) - f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta^l_i\right)\right) \leqslant \gamma^l_{\max} - \eta^l_{\max}\,.$$

Also, from (11) and (13), it is found that

$$\gamma^u_{\min} - \eta^u_{\min} \leqslant g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma^u_i\right)\right) - f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta^u_i\right)\right) \leqslant \gamma^u_{\max} - \eta^u_{\max}\,,$$

i.e., $S\left(\widetilde{\alpha}_-\right) \leqslant S\left(AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right)\right) \leqslant S\left(\widetilde{\alpha}_+\right)$.

Therefore, $\widetilde{\alpha}_- \leqslant AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) \leqslant \widetilde{\alpha}_+$.

Theorem 3 *Let $\widetilde{\alpha}_i$ $(i = 1, 2, \ldots, n$ be a collection of IVDHFEs, $\omega_i = \dfrac{T_i}{\sum_{i=1}^{n} T_i}$ $(i = 1, 2, \ldots, n)$ be their corresponding weight vectors, if $\widetilde{\alpha}$ be an IVDHFE, then*

$$AIVDHFPWA\left(\widetilde{\alpha}_1 \oplus_A \widetilde{\alpha}, \widetilde{\alpha}_2 \oplus_A \widetilde{\alpha}, \ldots, \widetilde{\alpha}_n \oplus_A \widetilde{\alpha}\right) = \\ = AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) \oplus_A \widetilde{\alpha}.$$

Proof.

$$\widetilde{\alpha}_i \oplus_A \widetilde{\alpha} = \left(\bigcup_{\substack{[\gamma^l_i, \gamma^u_i] \in \widetilde{h}_i, \\ [\gamma^l, \gamma^u] \in \widetilde{h} \\ (i=1,2,\ldots,n)}} \left\{\left[g^{-1}\left(g\left(\gamma^l_i\right) + g\left(\gamma^l\right)\right), g^{-1}\left(g\left(\gamma^u_i\right) + g\left(\gamma^u\right)\right)\right]\right\}, \right.$$

$$\left. \bigcup_{\substack{[\eta^l_i, \eta^u_i] \in \widetilde{g}_i, \\ [\eta^l, \eta^u] \in \widetilde{g} \\ (i=1,2,\ldots,n)}} \left\{\left[f^{-1}\left(f\left(\eta^l_i\right) + f\left(\eta^l\right)\right), f^{-1}\left(f\left(\eta^u_i\right) + f\left(\eta^u\right)\right)\right]\right\} \right).$$

So,

$$AIVDHFPWA\left(\widetilde{\alpha}_1 \oplus_A \widetilde{\alpha}, \widetilde{\alpha}_2 \oplus_A \widetilde{\alpha}, \ldots, \widetilde{\alpha}_n \oplus_A \widetilde{\alpha}\right)$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ [\gamma^l,\gamma^u]\in\widetilde{h}\\ (i=1,2,\ldots,n)}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(g^{-1}\left(g\left(\gamma_i^l\right)+g\left(\gamma^l\right)\right)\right) \right), g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(g^{-1}\left(g\left(\gamma_i^u\right)+g\left(\gamma^u\right)\right)\right) \right) \right] \right\} \right.$$

$$\left. \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ [\eta^l,\eta^u]\in\widetilde{g}\\ (i=1,2,\ldots,n)}} \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(f^{-1}\left(f\left(\eta_i^l\right)+f\left(\eta^l\right)\right)\right) \right), f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(f^{-1}\left(f\left(\eta_i^l\right)+f\left(\eta^l\right)\right)\right) \right) \right] \right\} \right),$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ [\gamma^l,\gamma^u]\in\widetilde{h}\\ (i=1,2,\ldots,n)}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^l\right)+g\left(\gamma^l\right) \right), g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^u\right)+g\left(\gamma^u\right) \right) \right] \right\} \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ [\eta^l,\eta^u]\in\widetilde{g}\\ (i=1,2,\ldots,n)}} \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^l\right)+f\left(\eta^l\right) \right), f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^u\right)+f\left(\eta^u\right) \right) \right] \right\} \right).$$

Now,

$$AIVDHFPWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) \oplus_A \widetilde{\alpha} =$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i\\(i=1,2,\ldots,n)}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^l\right) \right), g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^u\right) \right) \right] \right\}, \right.$$

$$\left. \bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i\\(i=1,2,\ldots,n)}} \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^l\right) \right), f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^u\right) \right) \right] \right\} \right) \oplus_A \left(\left\{ \left[\gamma^l, \gamma^u \right] \right\}, \left\{ \left[\eta^l, \eta^u \right] \right\} \right)$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ [\gamma^l,\gamma^u]\in\widetilde{h}\\(i=1,2,\ldots,n)}} \left\{ \left[g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^l\right) \right) \right) + g\left(\gamma^l\right) \right), \right. \right. \right.$$

$$\left. \left. g^{-1}\left(g\left(g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^u\right) \right) \right) + g\left(\gamma^u\right) \right) \right] \right\},$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ [\eta^l,\eta^u]\in\widetilde{g}\\(i=1,2,\ldots,n)}} \left\{ \left[f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^l\right) \right) \right) + f\left(\eta^l\right) \right), \right. \right.$$

$$\left. \left. \left. f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} f\left(\eta_i^u\right) \right) \right) + f\left(\eta^u\right) \right) \right] \right\} \right),$$

$$= \left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ [\gamma^l,\gamma^u]\in\widetilde{h}\\(i=1,2,\ldots,n)}} \left\{ \left[g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^l\right) + g\left(\gamma^l\right) \right), \; g^{-1}\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{i=1}^{n} T_i} g\left(\gamma_i^u\right) + g\left(\gamma^u\right) \right) \right] \right\} \right.$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ [\eta^l,\eta^u]\in\widetilde{g}\\ (i=1,2,\ldots,n)}} \left\{\left[f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)+f\left(\eta^l\right)\right),\ f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^u\right)+f\left(\eta^u\right)\right)\right]\right\}\Bigg).$$

Therefore,

$$AIVDHFPWA\left(\widetilde{\alpha}_1\oplus_A\widetilde{\alpha},\widetilde{\alpha}_2\oplus_A\widetilde{\alpha},\ldots,\widetilde{\alpha}_n\oplus_A\widetilde{\alpha}\right)= \\ =AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)\oplus_A\widetilde{\alpha}.$$

Hence the theorem is proved.

Theorem 4 *(Idempotency) If all $\widetilde{\alpha}_i$ $(i=1,2,\ldots,n)$ are equal and let $\widetilde{\alpha}_i=\left(\left\{\left[\gamma^l,\ \gamma^u\right]\right\},\left\{\left[\eta^l,\ \eta^u\right]\right\}\right)$ for all $(i=1,2,\ldots,n)$, then*

$$AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=\left(\left\{\left[\gamma^l,\ \gamma^u\right]\right\},\left\{\left[\eta^l,\ \eta^u\right]\right\}\right).$$

Proof.

$$AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)= \\ =\left(\bigcup_{\substack{[\gamma^l,\gamma^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^l\right)\right),g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}g\left(\gamma_i^u\right)\right)\right]\right\},\right. \\ \left.\bigcup_{\substack{[\eta^l,\eta^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^l\right)\right),f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}f\left(\eta_i^u\right)\right)\right]\right\}\right).$$

Now, since $\widetilde{\alpha}_i=\left(\left\{\left[\gamma^l,\ \gamma^u\right]\right\},\left\{\left[\eta^l,\ \eta^u\right]\right\}\right)$ for all $(i=1,2,\ldots,n)$, $\gamma_i^l=\gamma^l$, $\gamma_i^u=\gamma^u$, $\eta_i^l=\eta^l$ and $\eta_i^u=\eta^u$ for all $(i=1,2,\ldots,n)$.

Therefore,

$$AIVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)= \\ =\left(\bigcup_{\substack{[\gamma^l,\gamma^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[g^{-1}\left(g\left(\gamma^l\right)\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}\right),g^{-1}\left(g\left(\gamma^u\right)\sum_{i=1}^{n}\frac{T_i}{\sum\limits_{i=1}^{n}T_i}\right)\right]\right\},\right.$$

$$\bigcup_{\substack{[\eta^l,\eta^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}} \left\{\left[f^{-1}\left(f\left(\eta^l\right)\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}\right), f^{-1}\left(f\left(\eta^u\right)\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}\right)\right]\right\}\Bigg)$$

$$=\left(\bigcup_{\substack{[\gamma^l,\gamma^u]\in\widetilde{h}_i\\ i=1,2,\ldots,n}}\left\{\left[\gamma^l,\gamma^u\right]\right\}, \bigcup_{\substack{[\eta^l,\eta^u]\in\widetilde{g}_i\\ i=1,2,\ldots,n}}\left\{\left[\eta^l,\eta^u\right]\right\}\right),$$

$$=\left(\left\{\left[\gamma^l,\,\gamma^u\right]\right\},\left\{\left[\eta^l,\,\eta^u\right]\right\}\right).$$

Hence the theorem is proved.

At-CN&t-N-based IVDHF prioritized WG (AIVDHFPWG) operator is defined as follows.

Definition 13 *Let $\widetilde{\alpha}_i=\left(\widetilde{h}_i,\widetilde{g}_i\right)$ $(i=1,2,\ldots,n)$ be a collection of IVDHFEs and $\frac{T_i}{\sum_{i=1}^{n}T_i}$ indicates preference degree of $\widetilde{\alpha}_i$, where $T_i=\prod_{k=1}^{i-1}S\left(\widetilde{\alpha}_k\right)$ $(i=2,\ldots,n)$, $T_1=1$ and $S\left(\widetilde{\alpha}_i\right)$ is the score value of $\widetilde{\alpha}_i$. If*

$$AIVDHFPWG\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=\bigotimes_{i=1}^{n}{}_A\left(\frac{T_i}{\sum_{i=1}^{n}T_i}\widetilde{\alpha}_i\right),$$

then $AIVDHFPWG$ is called the IVDHF prioritized WG (AIVDHFPWG) operator.

$\otimes_A$ conveys the meaning as described in Definition 8.

Theorem 5 *Let $\widetilde{\alpha}_i=\left(\widetilde{h}_i,\widetilde{g}_i\right)$ $(i=1,2,\ldots,n)$ be a collection of IVDHFEs, then the aggregated value using AIVDHFPWG operator is also an IVDHFE and*

$$AIVDHFPWG\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=$$

$$\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\gamma_i^l\right)\right), f^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}f\left(\gamma_i^u\right)\right)\right]\right\},\right.$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}g\left(\eta_i^l\right)\right),g^{-1}\left(\sum_{i=1}^{n}\frac{T_i}{\sum_{i=1}^{n}T_i}g\left(\eta_i^u\right)\right)\right]\right\}\Bigg). \tag{14}$$

Proof. The proof is similar to Theorem 1.

The proposed AIVDHFPWA and AIVDHFPWG operators provide a general expression with the generators $f(x)$ and $g(x)$. Some particular cases of the proposed PA operators are presented as follows:

Case 1 If $f(x) = -\log x$ is considered, then the AIVDHFPWA and AIVDHFPWG operators reduced to the IVDHF prioritized WA (IVDHFPWA) and WG (IVDHFPWG) operators, respectively, which are shown as follows:

$$IVDHFPWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right) =$$
$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[1-\prod_{i=1}^{n}\left(1-\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}},1-\prod_{i=1}^{n}\left(1-\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\right]\right\},\right.$$
$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[\prod_{i=1}^{n}\left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}},\prod_{i=1}^{n}\left(\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\right]\right\}\right), \tag{15}$$

and

$$IVDHFPWG\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right) =$$
$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}},\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\right]\right\},\right.$$
$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[1-\prod_{i=1}^{n}\left(1-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}},1-\prod_{i=1}^{n}\left(1-\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\right]\right\}\right). \tag{16}$$

Case 2 For adopting the Einstein operations, the AIVDHFPWA operator turned to the IVDHF prioritized Einstein WA (IVDHFPEWA), and IVDHF prioritized Einstein WG (IVDHFPEWG) operators, sequentially, defined as:

$$IVDHFPEWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=$$

$$=\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[\frac{\prod_{i=1}^{n}\left(1+\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(1-\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(1-\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\right.\right.\right.$$

$$\left.\left.\frac{\prod_{i=1}^{n}\left(1+\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(1-\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(1-\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right]\right\},$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[\frac{2\prod_{i=1}^{n}\left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(2-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\frac{2\prod_{i=1}^{n}\left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(2-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right]\right\}\right),\tag{17}$$

and

$$IVDHFEPWG\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=$$

$$\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,2,\ldots,n}}\left\{\left[\frac{2\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(2-\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\frac{2\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(2-\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right]\right\},\right.$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[\frac{\prod_{i=1}^{n}\left(1+\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(1-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(1-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\right.\right.$$

$$\left.\left.\left.\frac{\prod_{i=1}^{n}\left(1+\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(1-\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(1-\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right]\right\}\right).\tag{18}$$

Case 3 When putting $f(x) = \log\left(\dfrac{\sigma + (1-\sigma)x}{x}\right)$, $\sigma > 0$, i.e., for consideration of Hamacher operations, the AIVDHFPWA and AIVDHFPWA operators converted, respectively, to the IVDHF prioritized Hamacher WA (IVDHFPHWA), and IVDHF prioritized Hamacher WG (IVDHFPHWG) operators, which are described as:

$$IVDHFPHWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) =$$

$$= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i,\\ i=1,2,\ldots,n}} \left\{ \left[\frac{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\,\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} \left(1 - \gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\,\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\sigma - 1) \prod_{i=1}^{n} \left(1 - \gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right.\right.\right.$$

$$\left.\left. \frac{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\,\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} \left(1 - \gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\,\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\sigma - 1) \prod_{i=1}^{n} \left(1 - \gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right] \right\},$$

$$\bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i,\\ i=1,2,\ldots,n}} \left\{ \left[\frac{\sigma \prod_{i=1}^{n} \left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\left(1 - \eta_i^l\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\sigma - 1) \prod_{i=1}^{n} \left(\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right.\right.$$

$$\left.\left.\left. \frac{\sigma \prod_{i=1}^{n} \left(\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\left(1 - \eta_i^u\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\sigma - 1) \prod_{i=1}^{n} \left(\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right] \right\} \right), \tag{19}$$

and,

$$IVDHFHPWG\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) =$$

$$= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i,\\ i=1,2,\ldots,n}} \left\{ \left[\frac{\sigma \prod_{i=1}^{n} \left(\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(1 + (\sigma - 1)\left(1 - \gamma_i^l\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\sigma - 1) \prod_{i=1}^{n} \left(\gamma_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right.\right.\right.$$

$$\left.\left.\frac{\sigma\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+(\sigma-1)\left(1-\gamma_i^u\right)\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\sigma-1)\prod_{i=1}^{n}\left(\gamma_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right]\right\}\right)$$

$$\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,2,\ldots,n}}\left\{\left[\frac{\prod_{i=1}^{n}\left(1+(\sigma-1)\,\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(1-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+(\sigma-1)\,\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\sigma-1)\prod_{i=1}^{n}\left(1-\eta_i^l\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\right.\right.$$

$$\left.\left.\left.\frac{\prod_{i=1}^{n}\left(1+(\sigma-1)\,\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(1-\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(1+(\sigma-1)\,\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\sigma-1)\prod_{i=1}^{n}\left(1-\eta_i^u\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right]\right\}\right). \tag{20}$$

Case 4 The AIVDHFPWA and AIVDHFPWG operators switched to the IVDHF prioritized Frank WA (IVDHFPFWA), and IVDHF prioritized Frank WG (IVDHFPFWG) operators for calculating with the Frank t-CN and t-N, $f(x)=\log\left(\frac{\tau-1}{\tau^x-1}\right)$, $\tau>1$, respectively, which are expressed as:

$$IVDHFPFWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=$$

$$\left(\bigcup_{\substack{[\gamma_i^l,\gamma_i^u]\in\widetilde{h}_i,\\ i=1,\ldots,n}}\left\{\left[1-\log_\tau\left(1+\frac{\prod_{i=1}^{n}\left(\tau^{1-\gamma_i^l}-1\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\tau-1}\right),1-\log_\tau\left(1+\frac{\prod_{i=1}^{n}\left(\tau^{1-\gamma_i^u}-1\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\tau-1}\right)\right]\right\},\right.$$

$$\left.\bigcup_{\substack{[\eta_i^l,\eta_i^u]\in\widetilde{g}_i,\\ i=1,\ldots,n}}\left\{\left[\log_\tau\left(1+\frac{\prod_{i=1}^{n}\left(\tau^{\eta_i^l}-1\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\tau-1}\right),\log_\tau\left(1+\frac{\prod_{i=1}^{n}\left(\tau^{\eta_i^u}-1\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\tau-1}\right)\right]\right\}\right), \tag{21}$$

and

$$IVDHFFPWG\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) =$$
$$= \left(\bigcup_{\substack{[\gamma_i^l, \gamma_i^u] \in \widetilde{h}_i, \\ i=1,2,\ldots,n}} \left\{ \left[\log_\tau \left(1 + \frac{\prod_{i=1}^{n} \left(\tau^{\gamma_i^l} - 1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau - 1} \right), \log_\tau \left(1 + \frac{\prod_{i=1}^{n} \left(\tau^{\gamma_i^u} - 1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau - 1} \right) \right] \right\}, \right.$$
$$\bigcup_{\substack{[\eta_i^l, \eta_i^u] \in \widetilde{g}_i, \\ i=1,2,\ldots,n}} \left\{ \left[1 - \log_\tau \left(1 + \frac{\prod_{i=1}^{n} \left(\tau^{1-\eta_i^l} - 1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau - 1} \right), \right. \right.$$
$$\left. \left. \left. 1 - \log_\tau \left(1 + \frac{\prod_{i=1}^{n} \left(\tau^{1-\eta_i^u} - 1\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\tau - 1} \right) \right] \right\} \right). \tag{22}$$

AIVDHFPWG operator also obeys the above properties as like AIVDHFPWA operator.

In the following sections, the methodological development of the MCDM method is incorporated and are described subsequently.

5. An approach to MCDM with the prioritization under IVDHF environment

In this section, the proposed AIVDHFPWA and AIVDHFPWG operators are applied on MCDM with IVDHFEs, in which the criteria are in different priority level. Let $\{z_1, z_2, \ldots, z_m\}$ be a set of alternatives, $\{C_1, C_2, \ldots, C_n\}$ be a set of criteria, and there prioritization relationship is $C_1 \succ C_2 \succ \ldots \succ C_n$. Suppose that $\widetilde{D} = \left[\widetilde{\alpha}_{ij}\right]_{m \times n}$ be an IVDHF decision matrix (IVDHFDM), where $\widetilde{\alpha}_{ij} = \left(\widetilde{h}_{ij}, \widetilde{g}_{ij}\right)$ is provided by the DM for the alternative z_i satisfying the criteria c_j. Then, the proposed AIVDHFPWA (or AIVDHFPWG) operators are used to develop an approach for solving MCDM problems in IVDHF environment. The proposed methodology is described through the following steps:

Step 1. In general, criteria are categorized into two types: one is benefit criteria, and the other one is cost criteria. If the IVDHFDM possesses cost type criteria, the matrix $\widetilde{D} = \left[\widetilde{\alpha}_{ij}\right]_{m \times n}$ can be converted into the normalized

IVDHFDM form as $\widetilde{R} = \left(\widetilde{r}_{ij}\right)_{m\times n}$ in the following way,

$$\widetilde{r}_{ij} = \begin{cases} \widetilde{\alpha}_{ij} & \text{for benefit criteria } C_j\,, \\ \widetilde{\alpha}^c_{ij} & \text{for cost criteria } C_j\,, \end{cases} \tag{23}$$

$i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Where $\widetilde{\alpha}^c_{ij}$ is the complement $\widetilde{\alpha}_{ij}$.

Step 2. Calculate the values of T_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ based on the following equations,

$$T_{ij} = \prod_{k=1}^{j-1} S\left(\widetilde{r}_{ik}\right) \qquad (i = 1, 2, \ldots, m;\ j = 2, \ldots, n); \tag{24}$$

$$T_{i1} = 1, \qquad i = 1, 2, \ldots, m. \tag{25}$$

Step 3. Aggregate the IVDHFEs $\widetilde{r}_{ij}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ for each alternative z_i using the IVDHFPHWA (or IVDHFPHWG) or IVDHFPFWA (or IVDHFPFWG) operator as follows:

$$\widetilde{r}_i = IVDHFPHWA\left(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots\ldots, \widetilde{r}_{in}\right)$$

$$= \left(\bigcup_{[\gamma^l_{ij}, \gamma^u_{ij}] \in \widetilde{h}_{ij}} \left\{ \left[\frac{\prod_{j=1}^{n}\left(1 + (\sigma - 1)\,\gamma^l_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} - \prod_{j=1}^{n}\left(1 - \gamma^l_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n}\left(1 + (\sigma - 1)\,\gamma^l_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + (\sigma - 1)\prod_{j=1}^{n}\left(1 - \gamma^l_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}, \right.\right.\right.$$

$$\left.\left. \frac{\prod_{j=1}^{n}\left(1 + (\sigma - 1)\,\gamma^u_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} - \prod_{i=1}^{n}\left(1 - \gamma^u_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n}\left(1 + (\sigma - 1)\,\gamma^u_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + (\sigma - 1)\prod_{j=1}^{n}\left(1 - \gamma^u_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}} \right]\right\}\right),$$

$$\bigcup_{[\eta^l_{ij}, \eta^u_{ij}] \in \widetilde{g}_{ij}} \left\{ \left[\frac{\sigma \prod_{j=1}^{n}\left(\eta^l_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n}\left(1 + (\sigma - 1)\left(1 - \eta^l_{ij}\right)\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}} + (\sigma - 1)\prod_{j=1}^{n}\left(\eta^l_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n} T_{ij}}}}, \right.\right.$$

$$
\left.\left.\left.\frac{\sigma\prod_{j=1}^{n}\left(\eta_{ij}^{u}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\prod_{j=1}^{n}\left(1+(\sigma-1)\left(1-\eta_{ij}^{u}\right)\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}+(\sigma-1)\prod_{j=1}^{n}\left(\eta_{ij}^{u}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}\right]\right\}\right), \tag{26}
$$

or utilizing the proposed operator IVDHFPHWG, which is presented above by Eq. (16), aggregate the IVDHFEs elements as:

$$
\widetilde{r}_{i}=IVDHFPHWG\left(\widetilde{r}_{i1},\widetilde{r}_{i2},\ldots,\widetilde{r}_{in}\right)
$$

or

$$
\begin{aligned}
\widetilde{r}_{i} &= IVDHFPFWA\left(\widetilde{r}_{i1},\widetilde{r}_{i2},\ldots,\widetilde{r}_{in}\right)\\
&=\left(\bigcup_{[\gamma_{ij}^{l},\gamma_{ij}^{u}]\in\widetilde{h}_{ij}}\left\{\left[1-\log_{\tau}\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{1-\gamma_{i}^{l}j}-1\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\tau-1}\right),\right.\right.\right.\\
&\qquad\left.\left.1-\log_{\tau}\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{1-\gamma_{ij}^{u}}-1\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\tau-1}\right)\right]\right\},\\
&\left.\bigcup_{[\eta_{ij}^{l},\eta_{ij}^{u}]\in\widetilde{g}_{ij}}\left\{\left[\log_{\tau}\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{\eta_{ij}^{l}}-1\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\tau-1}\right),\log_{\tau}\left(1+\frac{\prod_{j=1}^{n}\left(\tau^{\eta_{ij}^{u}}-1\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\tau-1}\right)\right]\right\}\right),
\end{aligned}
$$

or aggregating the IVDHFEs using IVDHFPFWG operator already shown by Eq. (17).

Step 4. Using the proposed score function as in Definition 11, the rank of all alternatives are evaluated.

6. Illustrative example

In this section, an academic field related problem, adapted from an example previously studied by Jin et al. [30], is considered to illustrate the application of the proposed method and demonstrate its feasibility and effectiveness in a realistic scenario. For strengthening the academic environment of a Chinese university, the best alternative is to select among five alternatives, $\{A_1, A_2, A_3, A_4, A_5\}$, by considering four criteria: C_1: morality; C_2: research capability; C_3: teaching skill; and C_4: education background. The prioritization relationship for the criteria is $C_1 \succ C_2 \succ C_3 \succ C_4$. The alternatives are evaluated by the expert on the basis of the criteria under IVDHF environment, and the IVDHFDM is constructed as given in Table 1.

Table 1: IVDHFDM

	C_1	C_2
A_1	$(\{[0.3, 0.4], [0.5, 0.8]\}, \{[0.17, 0.2]\})$	$(\{[0.3, 0.4], [0.4, 0.7]\}, \{[0.2, 0.3]\})$
A_2	$(\{[0.3, 0.5]\}, \{[0.3, 0.4]\})$	$(\{[0.2, 0.3], [0.4, 0.5]\}, \{[0.3, 0.4], [0.4, 0.5]\})$
A_3	$(\{[0.3, 0.4], [0.5, 0.7]\}, \{[0.05, 0.1], [0.1, 0.2]\})$	$(\{[0.3, 0.5]\}, \{[0.1, 0.2]\})$
A_4	$(\{[0.3, 0.4], [0.4, 0.5], [0.5, 0.6]\}, \{[0.01, 0.1]\})$	$(\{[0.5, 0.7]\}, \{[0.05, 0.1], [0.1, 0.15]\})$
A_5	$[\{[0.3, 0.6], [0.7, 0.9]\}, \{[0.05, 0.1]\}]$	$(\{[0.4, 0.6]\}, \{[0.01, 0.1], [0.1, 0.15]\})$

	C_3	C_4
A_1	$(\{[0.6, 0.8]\}, \{[0.05, 0.1]\})$	$(\{[0.3, 0.4], [0.5, 0.6]\}, \{[0.2, 0.3], [0.3, 0.4]\})$
A_2	$\{[0.5, 0.6], [0.7, 0.8]\}, \{[0.15, 0.2]\}$	$(\{[0.4, 0.5]\}, \{[0.1, 0.2], [0.2, 0.3], [0.3, 0.4]\})$
A_3	$(\{[0.7, 0.8], [0.8, 0.9]\}, \{[0.02, 0.1]\})$	$(\{[0.6, 0.7]\}, \{[0.05, 0.1], [0.1, 0.2]\})$
A_4	$(\{[0.3, 0.5], [0.6, 0.8]\}, \{[0.06, 0.1], [0.1, 0.2]\})$	$(\{[0.8, 0.9]\}, \{[0.04, 0.1]\})$
A_5	$(\{[0.5, 0.7], [0.8, 0.9]\}, \{[0.01, 0.06]\})$	$(\{[0.7, 0.8]\}, \{[0.01, 0.1], [0.1, 0.2]\})$

To obtain the ranking results among the alternative(s), the developed AIVDHFPWA and AIVDHFPWG operators are used, and step by step execution of the proposed method is described below. In this context, it is to be noted here that three types of At-CN&t-N, viz., Hamacher, Dombi and Frank Classes are considered. Algebraic and Einstein classes can be derived as particular cases of Hamacher class of t-CN&t-Ns.

Step 1. Since all the criteria C_j $(j = 1, 2, 3, 4)$ are of the benefit type, then the criteria values do not need normalization and take $\left[\widetilde{r}_{ij}\right]_{m\times n} = \left[\widetilde{\alpha}_{ij}\right]_{m\times n}$.

Step 2. Calculating the values of T_{ij} ($i = 1, 2, \ldots, 5$; $j = 1, 2, 3, 4$)) based on the Eqs. (24) and (25) as follows:

$$T_{ij} = \begin{bmatrix} 1 & 0.6575 & 0.3945 & 0.3205 \\ 1 & 0.5250 & 0.2494 & 0.1839 \\ 1 & 0.6821 & 0.4263 & 0.3709 \\ 1 & 0.6975 & 0.5231 & 0.3753 \\ 1 & 0.7750 & 0.5464 & 0.4617 \end{bmatrix}.$$

Step 3. Utilizing the IVDHPFWA, IVDHPFEWA, IVDHFPHWA, and IVDHFPFWA operators, to aggregate all the preference values $\widetilde{r}_{ij}$, and get the overall preference values $\widetilde{r}_i$, which are shown in Tables 2–5.

Table 2: Overall preference values of $\widetilde{r}_i$ utilizing IVDHFPWA operator

$\widetilde{r}_1$ = ({[0.3622,0.5002], [.3905,.5268], [.3889,.5875], [.4160,.6095], [0.4465,0.6854], [0.4711,0.7022], [0.4697,0.7404], [0.4932,0.7542]}, {[0.1483,0.2107], [0.1567,0.2190]})
$\widetilde{r}_2$ = ({[0.3149,0.4681], [.3581,.5131], [.3658,.5140], [.4057,.5551]}, {[.2477,.3431], [0.2644,0.3564], [0.2747,0.3662], [0.2676,0.3643], [0.2856,0.3784], [0.2967,0.3888]})
$\widetilde{r}_3$ = ({[0.4435,0.5741], [0.4810,0.6220], [0.51410.6780], [0.5468,0.7142]}, {[0.0684,0.1600], [0.0758,0.1775], [0.0904,0.1885], [0.1003,0.2091]})
$\widetilde{r}_4$ = ({[0.4664,0.6295], [.5233,.6919], [.4972,.6546], [.5508,.7128], [0.5313,0.6831], [0.5813,0.7365]}, {[0.0656,0.1306], [0.0727,0.1502], [0.0790,0.1573], [0.0876,0.1809]})
$\widetilde{r}_5$ = ({[0.4546,0.6630], [0.5444,0.7284], [0.5977,0.7952], [0.6640,0.8350]}, {[0.0178,0.0905], [0.0261,0.1015], [0.0339,0.1013], [0.0496,0.1136]})

Table 3: Overall preference values of $\widetilde{r}_i$ utilizing IVDHFPEWA operator

$\widetilde{r}_1$ = ({[0.3569,0.4899], [.3848,.517], [.3842,.5776], [.4114,0.6013], [0.4417,0.6753], [0.4674,0.6946], [0.4668,0.7367], [0.4918,0.753]}, {[0.1494,0.2123], [0.1583,0.2214]})
$\widetilde{r}_2$ = ({[0.3114,0.4647], [.3475,.5042], [.3639,.5136], [.3985,.5506]}, {[.2497,.3457], [0.2654,0.3583], [0.2756,0.3680], [0.2708,0.3683], [0.2877,0.3815], [0.2986,0.3917]})
$\widetilde{r}_3$ = ({[0.4324,0.5653], [0.4642,0.6075], [0.5076,0.6748], [0.5366,0.7083]}, [0.0687,0.1607], [0.0762,0.1780], [0.0918,0.1905], [0.1017,0.2107]})
$\widetilde{r}_4$ = ({[0.4527,0.6168], [.5119,.6807], [.4869,.6459], [.5436,0.7059], [0.5230,0.677], [0.5768,0.7326]}, {[0.0657,0.1312], [0.0729,0.1509], [0.0792,0.1580], [0.0878,0.1814]})
$\widetilde{r}_5$ = ({[0.4470,0.6610], [0.5290,0.7227], [0.5924,0.7912], [0.6579,0.8316]}, {[0.0179,0.0905], [0.0263,0.1019], [0.0341,0.1015], [0.0500,0.1142]})

Table 4: Overall preference values of $\widetilde{r}_i$ utilizing IVDHFPHWA operator

$\widetilde{r}_1$ = ({[0.3541,0.4853], [.3818,.5126], [.3817,0.5729], [0.409,0.5975], [.4391,.6704], [0.4654,0.691], [0.4653,.735], [0.491,0.7524]}, {[0.1498,0.2129], [0.1589,0.2223]})
$\widetilde{r}_2$ = ({[0.3094,0.4629], [.3422,.5], [.3629,.5134], [.395,.5487]}, {[.2505,.3467], [0.2659,0.359], [0.276,0.3687], [0.2721,0.37], [0.2885,0.3828], [0.2993,0.3929]})
$\widetilde{r}_3$ = ({[0.4267,0.5613], [0.4560,0.6011], [0.5043,0.6734], [0.5316,0.7059]}, {[0.0689,0.1610], [0.0764,0.1782], [0.0923,0.1913], [0.1022,0.2113]})
$\widetilde{r}_4$ = ({[0.446,.6111], [.5063,0.6756], [.4821,.6422], [.5404,.7029], [0.519,0.6745], [0.5749,0.731]}, {[0.0658,0.1314], [0.073,0.1511], [0.0793,0.1583], [0.0878,0.1815]})
$\widetilde{r}_5$ = ({[0.4432,0.6602], [0.5215,0.7205], [0.5898,0.7895], [0.655,0.8302]}, {[0.0179,0.0906], [0.0264,0.102], [0.0342,0.1016], [0.0501,0.1144]})

Table 5: Overall preference values of $\widetilde{r}_i$ utilizing IVDHFPFWA operator

$\widetilde{r}_1$ = ({[0.3584,0.4919], [.3865,.5190], [.3855,0.5795], [0.4128,0.6029], [.4431,.677], [0.4685,0.6958], [0.4676,.7373], [0.4921,0.7531]}, {[0.1494,0.2122], [0.1583,0.2212]})
$\widetilde{r}_2$ = ({[0.3127,0.4657], [.3506,.5062], [.3645,.5137], [.4004,.5514]}, {[.2496,.3453], [0.2654,0.3580], [0.2755,0.3677], [0.2706,0.3677], [0.2875,0.3810], [0.2984,0.3912]})
$\widetilde{r}_3$ = ({[0.4353,0.5670], [0.4681,0.6097], [0.5093,0.6753], [0.5388,0.7090]}, {[0.0687,0.1607], [0.0763,0.1780], [0.0918,0.1905], [0.1017,0.2106]})
$\widetilde{r}_4$ = ({[0.4559,.6187], [.5145,0.6824], [.4891,.6470], [.5450,.7067], [0.5247,0.6777], [0.5776,0.7329]}, {[0.0657,0.1312], [0.0729,0.1509], [0.0792,0.158], [0.0878,0.1814]})
$\widetilde{r}_5$ = ({[0.4489,0.6613], [0.5324,0.7232], [0.5935,0.7914], [0.6590,0.8318]}, {[0.0179,0.0905], [0.0263,0.1019], [0.0341,0.1015], [0.0500,0.1142]})

Step 4. Calculating the score functions $S(\widetilde{r}_i)$ of the overall IVDHFEs.

Step 5. Rank all the candidates A_i $(i = 1, 2, \ldots, 5)$ in accordance with the score values $S(\widetilde{r}_i)$ of the overall IVDHFEs. From the Fig. 1–4, it is clear that when IVDHFPHWA, IVDHFPHWG, IVDHFPFWA and IVDHFPFWG operators are utilized, the same ordering of the candidates is obtained, and the most desirable candidate is A_5.

The overall IVDHF values $\widetilde{r}_i$ $(i = 1, 2, \ldots, 5$ of the candidates A_i are derived by aggregating IVDHFEs $\widetilde{r}_{ij}$ $(j = 1, 2, \ldots, 5))$ for all i with prioritized aggregation operator IVDHFPWA, and is presented in Table 2, whereas Table 3 represents the aggregating values of each candidate A_i using Einstein-based aggregation operator IVDHFPEWA instead of IVDHFPWA.

Subsequently, Hamacher $(\sigma = 3)$ and Frank $(\tau = 3)$ based aggregation operators IVDHFPHWA and IVDHFPFWA are utilized to aggregate the performance values of the alternatives A_i and is demonstrated in Tables 4 and 5, respectively.

The score values and the ranking results by varying parameters, σ and τ, in the IVDHFPHWA, IVDHFPHWG, IVDHFPFWA and IVDHFPFWG operators, are shown in Tables 6–9, respectively.

Table 6: Ranking results for different parameters of the IVDHFPHWA operator

Parameter	$S(z_1)$	$S(z_2)$	$S(z_3)$	$S(z_4)$	$S(z_5)$	Ordering
$\sigma = 1$	0.6752	0.5587	0.7190	0.7447	0.7968	$A_5 > A_4 > A_3 > A_1 > A_1$
$\sigma = 2$	0.6714	0.5550	0.7136	0.7401	0.7935	$A_5 > A_4 > A_3 > A_1 > A_1$
$\sigma = 3$	0.6696	0.5533	0.7112	0.7381	0.7920	$A_5 > A_4 > A_3 > A_1 > A_1$

Table 7: Ranking results for different parameters of the IVDHFPHWA operator

Parameter	$S(z_1)$	$S(z_2)$	$S(z_3)$	$S(z_4)$	$S(z_5)$	Ordering
$\sigma = 1$	0.6462	0.5370	0.6812	0.7039	0.7687	$A_5 > A_4 > A_3 > A_1 > A_1$
$\sigma = 2$	0.6502	0.5398	0.6858	0.7090	0.7724	$A_5 > A_4 > A_3 > A_1 > A_1$
$\sigma = 3$	0.6526	0.5414	0.6886	0.7121	0.7747	$A_5 > A_4 > A_3 > A_1 > A_1$

Table 8: Ranking results for different parameters of the IVDHFPFWA operator

Parameter	$S(z_1)$	$S(z_2)$	$S(z_3)$	$S(z_4)$	$S(z_5)$	Ordering
$\tau = 2$	0.6732	0.5568	0.7162	0.7422	0.7950	$A_5 > A_4 > A_3 > A_1 > A_1$
$\tau = 3$	0.6721	0.5558	0.7146	0.7409	0.7941	$A_5 > A_4 > A_3 > A_1 > A_1$
$\tau = 4$	0.6714	0.5552	0.7136	0.7400	0.7934	$A_5 > A_4 > A_3 > A_1 > A_1$

Table 9: RRanking results for different parameters of the IVDHFPFWG operator

Parameter	$S(z_1)$	$S(z_2)$	$S(z_3)$	$S(z_4)$	$S(z_5)$	Ordering
$\tau = 2$	0.6479	0.5383	0.6833	0.7151	0.7703	$A_5 > A_4 > A_3 > A_1 > A_1$
$\tau = 3$	0.6489	0.5390	0.6844	0.7160	0.7712	$A_5 > A_4 > A_3 > A_1 > A_1$
$\tau = 4$	0.6495	0.5394	0.6851	0.7167	0.7717	$A_5 > A_4 > A_3 > A_1 > A_1$

Now, based on the DMs' preferences, the parameter can take different values. Based on the Hamacher (or Frank) parameter σ (or τ) between 0 to 20 (or 1 to 20), the score values and ranking of the five alternatives are shown in Fig. 1–4.

From Fig. 1, when the given problem is solved with IVDHFPHWA operator, it is perceived that the ordering of the alternatives does not change. Still, with varying the Hamacher parameter σ the score value of the alternatives decreases monotonically.

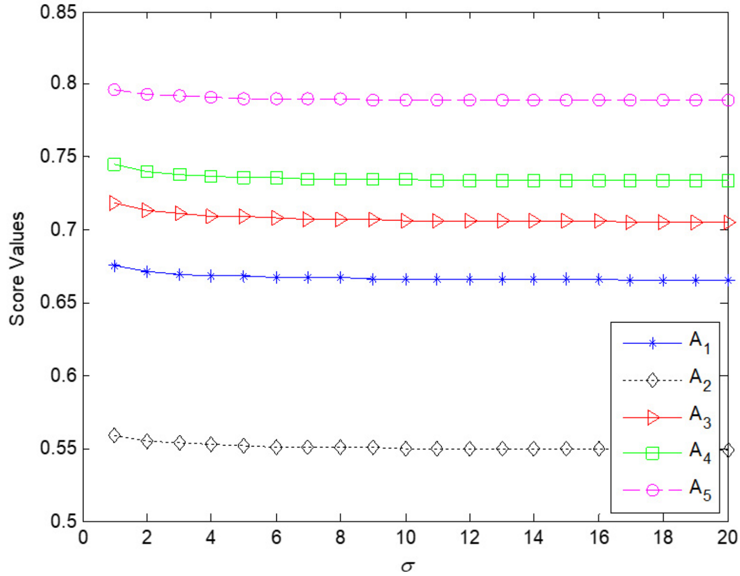

Figure 1: Changes of the score values applying

Similarly, if IVPHFHWG operator is used, the score value of alternatives are computed by varying the Hamacher parameter $\sigma \in [0, 20]$, the obtained results are advertised in the following Fig. 2. It is to be noted here that the ranking of

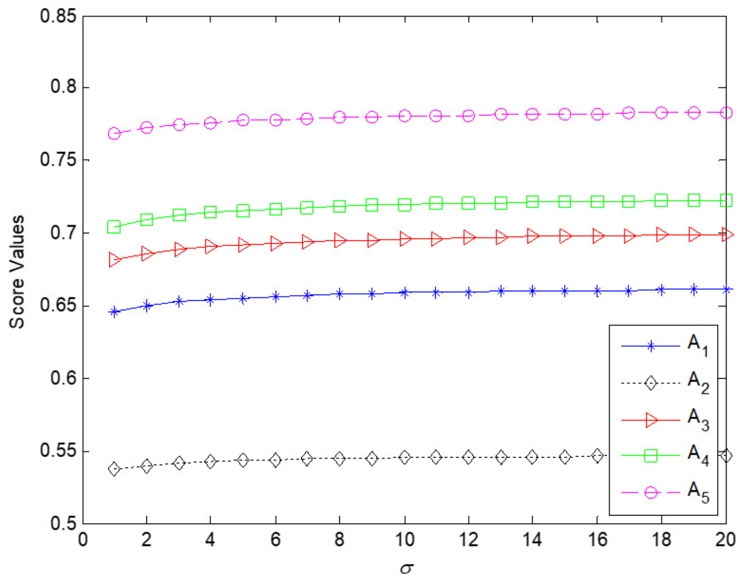

Figure 2: Changes of the score values applying IVDHFPHWG

alternatives does not modify as like using IVDHFPHWA operator. But the score value of the alternatives increases monotonically.

If IVDHFPFWA and IVDHFPFWG operators are used for the Frank parameter τ between 1 to 20, individually, the score values are presented in Fig. 3 and Fig. 4, respectively. As like the above cases, the equivalent observations are seen corresponding to averaging and geometric operators.

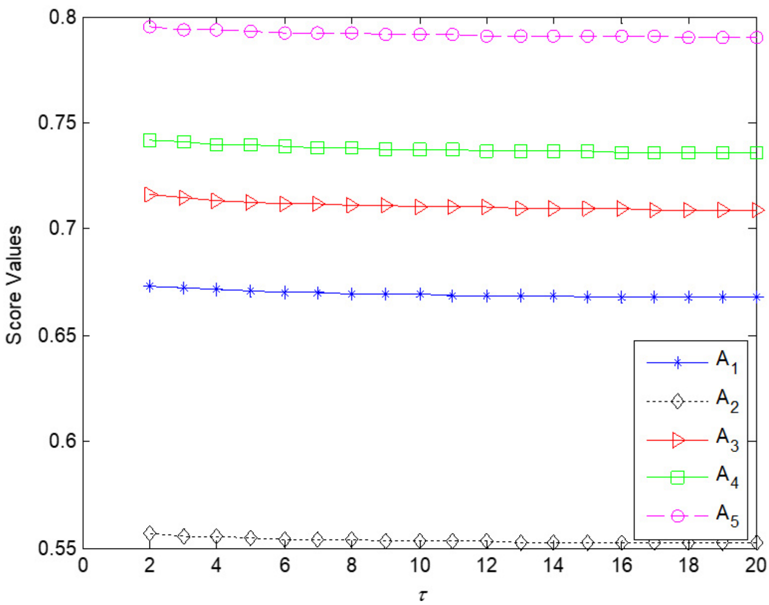

Figure 3: Changes of the score values applying IVDHFPFWA

It is decent to mention here that no changes in the ranking of the alternatives A_i ($i = 1, 2, \ldots, 5$) are found while making the decision using different PA operators. Thus it persists that the suggested methodology has a durable consistency.

It is worthy to mention here that, same ranking results of alternatives are found using the proposed method and which also covers the result of Jin et al. [30]. The technique developed by Jin et al. [30] is based on Einstein operation under DHF environment, whereas the proposed approach is based on At-CN&t-Ns under IVDHF information. Because IFS, and DHFS are the particular cases of IVDHFS and also At-CN&t-Ns contains an adjustable parameter. So it is claimed that the approach of Jin et al. [30] is a special case of the proposed method. Thus, the proposed methodology is more consistent than the technique developed by Jin et al. [30].

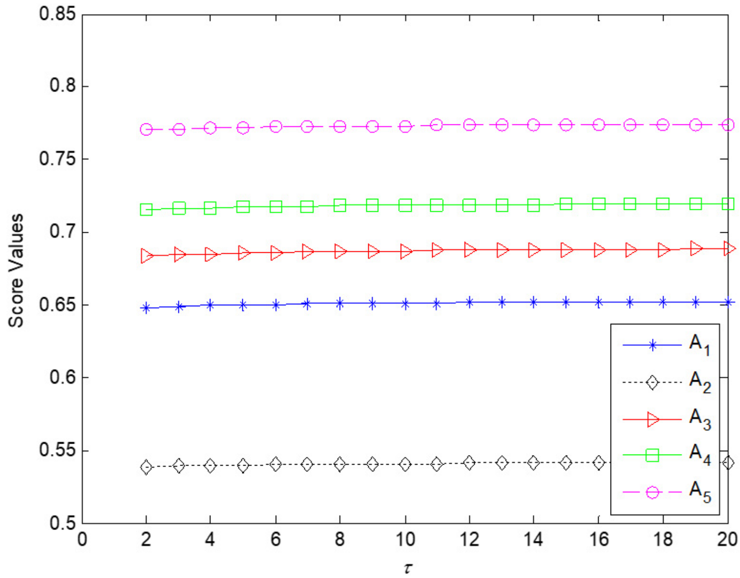

Figure 4: Changes of the score values applying IVDHFPFWG

7. Conclusion

The main contributions of this article is to define a score function of IVDHFE and to propose two prioritized aggregation operators AIVDHFPWA and AIVDHFPWG based on At-CN&t-Ns under the IVDHF context. Most of the prioritized-based aggregation operators can be constructed from AIVDHFPWA and AIVDHFPWG operators. Some desirable properties, such as idempotency, monotonicity, and boundedness of the proposed operators, are investigated. An approach for solving MCDM problem is presented in which the criteria are in different preference level. Through the illustrative example, it has been established the fact that the proposed method not only captures the existing Einstein operation based aggregation operators for IVHFEs [30] but also extends the scope of using aggregation operators in IVDHF environment. In future, the proposed operators may be extended to other domains, viz., q-rung orthopair fuzzy [34], Neutrosophic set [35,36], cubic bipolar fuzzy [37] and Pythagorean fuzzy [38–42] environments. Several types of AOs based on Schweizer-Sklar [43], Yager [44] and many other classes of t-CN&t-Ns can also be developed in IVDHF contexts.

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