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On elastic contact problems of micro-periodic slant layered composite pressed by a rigid punch with a parabolic or rectangular shape

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Abstract. The paper presents an analysis of the influence of the shape of the rigid body pressed into the micro-periodic composite half-space on the examples of two punch shapes – parabolic and rectangular. The presented material is a layered body that consists of infinitely many thin alternately arranged homogenous layers. Layers of the presented composite are oblique to the boundary surface. Two cases of punch tip shape are examined – parabolic and rectangular. The presented problem has been formulated within the framework of a homogenized model with microlocal parameters and solved using the elastic potentials method and averaged boundary condition. Fourier integral transform method has been used to obtain the solution and the inverse integrals have been calculated numerically. Solutions in terms of contact pressure and maximum pressure characteristics were shown in the form of graphs.

Key words: contact problem; micro-periodic composite; punch shape.

1. INTRODUCTION

The contact problems of elastic solids are widely studied due to their engineering applications, when considering the contact of friction pairs or other processes related to the transfer of load from one body to another, among others. The problems of the classical theory of elasticity are still intensively analyzed for both homogeneous [1–3] and heterogeneous media [4, 5] including materials with a structural gradation of properties [6,7], materials with coatings [8–11] or composites [12–14]. For these types of materials, solving problems and describing mechanical properties often comes down to the use of various averaging methods, e.g., homogenization methods. One of the examples of such methods is the homogenization method with microlocal parameters given by Woźniak [15] used, among others, for heterogeneous composite materials – see [13, 15]. One of the examples of these materials is micro-periodic composites with a laminated structure, which occur in nature in the form of layered sedimentary rocks. The process of machining layered rock is one engineering example showing the importance of the analysis of the presented problem. In these issues, an important role is played by the shape of the pressed body (punch) in contact with the considered material, because it significantly affects the formation of the contact zone, the value of contact pressure, and the stress distribution in a loaded body. For this reason, the presented work examines the two most common shapes of punches pressed into a composite half-space with layering inclined at any angle.

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2. PROBLEM FORMULATION

In this section, we will describe the presented contact problem and the approach to a mathematical description based on the use of two coordinate systems and the homogenization method. The presented problem is defined as pressing an infinitely long rigid punch into an elastic composite half-space without friction. Composite half-space consists of many thin alternately arranged homogenous layers as presented in Fig. 1. Layers of two kinds are stacked obliquely at an angle α to the boundary surface. Conditions of ideal contact between the layers were assumed. The plane contact problem has been formulated as pressing a rigid punch into the composite body for two examples of punch cross-section: the first case of a parabolic punch (Fig. 1a) and the second case of a rectangular punch (Fig. 1b).

For both cases (see Fig. 1) parameter a represents half-width of the contact zone. In the case of a punch with a parabolic tip, the width of the contact zone is unknown, which causes the mixed boundary problem to become non-linear and the width of

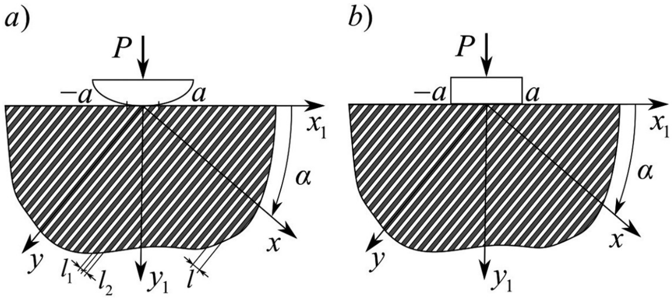

Fig. 1. Scheme of considered contact problems for: a) parabolic punch, b) rectangular punch

the contact zone a is calculated using the equilibrium condition, see for example [1, 16]

$$\int_{-a}^{a} p(x_1)\,\mathrm{d}x_1 = P, \tag{1}$$

where $p(x_1)$ is an averaged contact pressure.

In the presented layered composite, two kinds of isotropic layers have thicknesses defined as l_1 and l_2 (see Fig. 1), and the layer thickness ratio η is given by:

$$\eta = \frac{l_1}{l_1 + l_2} = \frac{l_1}{l}, \tag{2}$$

where l is the total thickness of a layer.

The presented problem has been solved using two coordinate systems: a (x, y, z) system associated with layering directions and the (x_1, y_1, z_1) system associated with the boundary surface. The system (x_1, y_1, z_1) is obtained by rotating the (x, y, z) system around the z-axis by the angle α, $\alpha \in \langle 0, 90^\circ \rangle$ as follows:

$$\begin{cases} x_1 = x\cos\alpha - y\sin\alpha, \\ y_1 = x\sin\alpha + y\cos\alpha, \\ z_1 = z. \end{cases} \tag{3}$$

The displacement vector in the (x, y, z) system is denoted as $\mathbf{u}(x, y) = [u, v, 0]$ and as $\mathbf{u}_1(x_1, y_1) = [u_1, v_1, 0]$ in (x_1, y_1, z_1) system. Similar nomenclature was used to designate the components of the stress tensor in the aforementioned systems – σ_{xx}, σ_{xy}, σ_{yy}, σ_{zz} and $\sigma_{x_1x_1}$, $\sigma_{x_1y_1}$, $\sigma_{y_1y_1}$, $\sigma_{z_1z_1}$, respectively. The presented problem was solved with the use of a homogenized model with microlocal parameters, wherein the components of a displacement vector are described as follows [17, 18]

$$\begin{aligned} u(x, y) &= U(x, y) + h(x)q_x(x, y), \\ v(x, y) &= V(x, y) + h(x)q_y(x, y), \end{aligned} \tag{4}$$

where $U = U(x, y)$, $V = V(x, y)$ are macro-displacement vector components in the x and y directions, respectively; $q_x = q_x(x, y)$, $q_y = q_y(x, y)$ are the unknown microlocal parameters and $h = h(x)$ is the shape function given as [17, 18]

$$h(x) = \begin{cases} x - 0.5l_1 & \text{for } 0 \le x < l_1, \\ -\eta x/(1-\eta) - 0.5l_1 + l_1(1-\eta) & \\ & \text{for } l_1 \le x \le l, \end{cases} \tag{5}$$
$$h(x + l) = h(x).$$

The shape function was given *a priori* so that the continuity conditions at the interfaces connecting individual layers were met.

Let λ_j and μ_j be Lame constants of the material of j-th layer. Using equations (4) in classical equations of the theory of elasticity and applying the homogenization procedures described in [17], we obtain:

$$\begin{aligned} &(\bar{\lambda} + \bar{\mu})\left(\frac{\partial^2 V}{\partial y \partial x} + \frac{\partial^2 U}{\partial x^2}\right) + \bar{\mu}\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial x^2}\right) \\ &\qquad + ([\lambda] + 2[\mu])\frac{\partial q_x}{\partial x} + [\mu]\frac{\partial q_y}{\partial y} = 0, \\ &(\bar{\lambda} + \bar{\mu})\left(\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 U}{\partial y \partial x}\right) + \bar{\mu}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \\ &\qquad + [\lambda]\frac{\partial q_x}{\partial y} + [\mu]\frac{\partial q_y}{\partial x} = 0, \end{aligned} \tag{6}$$

where microlocal parameters satisfy a system of equations:

$$\begin{aligned} \left(\hat{\lambda} + 2\hat{\mu}\right) q_x &= -[\lambda]\left(\frac{\partial V}{\partial y} + \frac{\partial U}{\partial x}\right) - 2[\mu]\frac{\partial U}{\partial x}, \\ \hat{\mu} q_y &= -[\mu]\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right), \end{aligned} \tag{7}$$

with the following notation:

$$\bar{f} \equiv \langle f \rangle, \quad [f] \equiv \langle f h^* \rangle, \quad \hat{f} \equiv \left\langle f\,(h^*)^2 \right\rangle, \tag{8}$$

where $\langle f \rangle$ stands for an average value defined as:

$$\langle f \rangle = \frac{1}{l}\int_0^l f(x)\,\mathrm{d}x \tag{9}$$

and $h^* = h^*(x)$ is a derivative of the shape function $h(x)$

$$h^*(x) = \frac{\mathrm{d}h}{\mathrm{d}x} = \begin{cases} 1 & \text{for } 0 \le x < l_1, \\ -\eta/(1-\eta) & \text{for } l_1 \le x \le l. \end{cases} \tag{10}$$

The homogenized model takes the following assumptions [17, 18]

$$\begin{aligned} &u \approx U, \quad \frac{\partial u}{\partial x} \approx \frac{\partial U}{\partial x} + h^* q_x, \quad \frac{\partial u}{\partial y} \approx \frac{\partial U}{\partial y}, \\ &v \approx V, \quad \frac{\partial v}{\partial x} \approx \frac{\partial V}{\partial x} + h^* q_y, \quad \frac{\partial v}{\partial y} \approx \frac{\partial V}{\partial y}. \end{aligned} \tag{11}$$

Stresses in individual layers are given by [17, 18]

$$\begin{aligned} \sigma_{xx}^{(j)} &= (\lambda_j + 2\mu_j)\left(\frac{\partial U}{\partial x} + h^* q_x + h\frac{\partial q_x}{\partial x}\right) \\ &\quad + \lambda_j\left(\frac{\partial V}{\partial y} + h\frac{\partial q_y}{\partial y}\right), \\ \sigma_{xy}^{(j)} &= \mu_j\left(\frac{\partial U}{\partial y} + h\frac{\partial q_x}{\partial y} + \frac{\partial V}{\partial x} + h^* q_y + h\frac{\partial q_y}{\partial x}\right) \\ &\quad + \lambda_j\left(\frac{\partial V}{\partial y} + h\frac{\partial q_y}{\partial y}\right), \\ \sigma_{yy}^{(j)} &= (\lambda_j + 2\mu_j)\left(\frac{\partial V}{\partial x} + h\frac{\partial q_y}{\partial y}\right) \\ &\quad + \lambda_j\left(\frac{\partial U}{\partial x} + h^* q_x + h\frac{\partial q_x}{\partial x}\right), \\ \sigma_{zz}^{(j)} &= \lambda_j\left(\frac{\partial V}{\partial y} + h\frac{\partial q_x}{\partial y} + \frac{\partial U}{\partial x} + h^* q_x + h\frac{\partial q_x}{\partial x}\right), \end{aligned} \tag{12}$$

where $j = 1,2$. For $j = 1$ stress component corresponds to the sublayer of the first kind and for $j = 2$ – the sublayer of the second kind of composite component.

Seeing that $|h(x)| < l$ for every x then for the small thicknesses l the terms containing $h(x)$ are small and will be neglected, It should be noted that the derivative $h^*(x)$ is not small and cannot be ignored. These conclusions and the use of equations (7) lead to the elimination of the microlocal parameters from equations (6) and (12). After eliminating the microlocal parameters (see [15]) in the plane strain state, the presented problem could be described by a system of equations as follows [17, 18]

$$\begin{cases} A_1\dfrac{\partial^2 U}{\partial x^2}+(B+C)\dfrac{\partial^2 V}{\partial x\partial y}+C\dfrac{\partial^2 U}{\partial y^2}=0, \\ A_2\dfrac{\partial^2 V}{\partial y^2}+(B+C)\dfrac{\partial^2 U}{\partial x\partial y}+C\dfrac{\partial^2 V}{\partial x^2}=0, \end{cases} \tag{13}$$

where A_1, A_2, B, C are the homogenized model parameters related to composite material properties [17, 18], see Appendix, equation (A.1).

Stress tensor in j-th component can be expressed as follows [17, 18]

$$\begin{aligned} &\sigma_{xx}^{(j)} = A_1\frac{\partial U}{\partial x}+B\frac{\partial V}{\partial y}, \quad \sigma_{yy}^{(j)} = D_j\frac{\partial U}{\partial x}+E_j\frac{\partial V}{\partial y}, \\ &\sigma_{xy}^{(j)} = C\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right), \quad \sigma_{zz}^{(j)} = \frac{\lambda_j}{\lambda_j+2\mu_j}\left(\sigma_{xx}^{(j)}+\sigma_{yy}^{(j)}\right), \end{aligned} \tag{14}$$

where D_j, E_j are the homogenized model parameters [17, 18]. The stress tensor components $\sigma_{xx}^{(j)}$ and $\sigma_{xy}^{(j)}$ in formula (14) are continuous and effective modulus A_1, B and C do not depend on j due to the continuity conditions at the interfaces connecting both composite components. This conclusion makes it possible to confirm with certainty that, within the homogenized model with microlocal parameters, the ideal conditions of continuity at the interfaces connecting the various components are satisfied.

For the plane problem of the theory of elasticity, the displacement condition is impossible to meet due to the infinity of the punch [19]. In this case, the condition for the derivative of the displacement has to be fulfilled. This is a standard approach in contact mechanics (see [19]).

For the presented contact problem, the following boundary conditions are defined:

$$\begin{aligned} &\frac{\partial V_1(x_1,0)}{\partial x_1} = -\frac{\partial g(x_1)}{\partial x_1}, && \text{for } |x_1| \le a, \\ &\sigma_{y_1y_1}^{(j)}(x_1,0) = 0, && \text{for } |x_1| > a, \\ &\sigma_{x_1y_1}^{(j)}(x_1,0) = 0, && \text{for } x_1 \in \mathbb{R}, \end{aligned} \tag{15}$$

where $g(x_1)$ represents the punch cross-section shape and V_1 represents a component of the macro-displacement vector in the direction of the y_1 axis.

Regularity conditions at infinity are as follows:

$$\sigma_{x_1x_1}^{(j)}, \ \sigma_{x_1y_1}^{(j)}, \ \sigma_{y_1y_1}^{(j)} \to 0 \quad \text{for} \quad x_1^2+y_1^2 \to \infty. \tag{16}$$

The function $g(x_1)$ represents a punch cross-section in the form:

$$\begin{aligned} &g(x_1) = D-\frac{x_1^2}{2R} && \text{(parabolic punch)}, \\ &g(x_1) = D && \text{(rectangular punch)}, \end{aligned} \tag{17}$$

where $R = \text{const}$ and D is the depth of punch penetration.

In the boundary conditions (15), the stress tensor components $\sigma_{y_1y_1}^{(j)}$, $j = 1,2$ are discontinuous on the interfaces between sublayers, which makes the solution problematic, hence we apply the averaged boundary condition in the form [13]:

$$\sigma_{y_1y_1}^{(*)} = \eta\sigma_{y_1y_1}^{(1)}+(1-\eta)\sigma_{y_1y_1}^{(2)}. \tag{18}$$

This condition was successfully used in similar problems [13]. Moreover, its accuracy was also verified in [12] and a conclusion was drawn that the averaged boundary condition can be used when the ratio of laminate thickness to the width of a contact zone is sufficiently small ($l/a < 0.05$) for a lamination perpendicular to the boundary. In the case of a layering parallel to the boundary, the solution using a homogenized model also gives good results (for $l/a < 0.2$), see [20]. In previous works [8, 12, 13], we can find conclusions on the applicability of the homogenized model to modeling micro-periodic layered composites. The works [12, 13] investigated examples of problems that can be solved using the homogenized model and an exact model, i.e., within the theory of elasticity. In these cases, an enormous amount of work is required to obtain a solution within the framework of the theory of elasticity. The use of direct numerical methods, like FEM, also entails significant computational complications, if we want to obtain a solution at the boundary. Therefore, the accuracy of the solution depends on the thickness of the layers and the authors refer to the solutions obtained for extreme cases, because the ratio l/a for the considered problem will depend on the angle α.

3. SOLUTION METHOD

In order to solve the presented problem, it is necessary to determine the solution of the system of equations (13) describing the macro-displacements, taking into account the boundary conditions (15). A solution has been obtained by the use of the elastic potentials method, where the potentials $\Psi_1 = \Psi_1(x_1,y_1)$, $\Psi_2 = \Psi_2(x_1,y_1)$ associated to the (x_1,y_1,z_1) system (3) are given as [12]:

$$U = \kappa_1\frac{\partial \Psi_1}{\partial x}+\kappa_2\frac{\partial \Psi_2}{\partial x}, \quad V = \frac{\partial \Psi_1}{\partial y}+\frac{\partial \Psi_2}{\partial y}, \tag{19}$$

where $\kappa_k = (A_2\gamma_k^2-C)/(B+C)$ and γ_k, $k = 1,2$ are the solutions to the following characteristic equation [13]:

$$A_2C\gamma_k^4+(B^2+2BC-A_1A_2)\gamma_k^2+A_1C = 0. \tag{20}$$

In this case, the following equations are obtained [13]:

$$\gamma_k^2 \frac{\partial^2 \Psi_k}{\partial x^2} + \frac{\partial^2 \Psi_k}{\partial y^2} = 0, \quad k = 1,2. \tag{21}$$

Equations (21) have real roots $\pm\gamma_1$, $\pm\gamma_2$ as follows [13]:

$$\gamma_{1,2} = \left(\frac{A_1 A_2 - 2BC - B^2 \mp \sqrt{\Delta}}{2A_2 C} \right)^{\frac{1}{2}}, \qquad \Delta = \left(B^2 + 2BC - A_1 A_2\right)^2 - 4A_1 A_2 C^2 > 0. \tag{22}$$

According to (14) and (19), we could determine the stress tensor components as functions of elastic potentials Ψ_k, $k = 1,2$ as [12]:

$$\begin{aligned}
\sigma_{xx}^{(j)} &= \sum_{k=1}^{2} \left(A_1 \kappa_k \frac{\partial^2 \Psi_k}{\partial x^2} + B \frac{\partial^2 \Psi_k}{\partial y^2} \right), \\
\sigma_{xy}^{(j)} &= C \sum_{k=1}^{2} (1+\kappa_k) \frac{\partial^2 \Psi_k}{\partial x \partial y}, \\
\sigma_{yy}^{(j)} &= \sum_{k=1}^{2} \left(D_j \kappa_k \frac{\partial^2 \Psi_k}{\partial x^2} + E_j \frac{\partial^2 \Psi_k}{\partial y^2} \right), \\
\sigma_{zz}^{(j)} &= \frac{\lambda_j}{2(\lambda_j + \mu_j)} \left(\sigma_{xx}^{(j)} + \sigma_{yy}^{(j)} \right), \quad j = 1,2.
\end{aligned} \tag{23}$$

Assuming the averaged contact pressure $p(\breve{x}_1)$ in the contact zone as:

$$p(x_1) = \eta p^{(1)}(x_1) + (1-\eta) p^{(2)}(x_1), \tag{24}$$

where $p^{(j)}(x_1)$, $j = 1,2$ is the contact pressure for j-th composite body component, we could move from the displacement problem to the boundary pressure load problem [21].

Fourier integral transform method was used to obtain the solution for (21), which is presented in detail in [22], where Fourier transform of the function $f(x,y)$ is denoted as:

$$\tilde{f}(s,y) = F\left[f(x,y); x \to s\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x,y) \exp(-ixs)\,\mathrm{d}x. \tag{25}$$

The solution which satisfies the condition (16) in the form of elastic potential can be found in [21] as:

$$\Psi_k(x_1,y_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-a_1^k |s| y_1 + is\left(x_1 - a_2^k y_1\right)\right) \cdot \left(A_k^{(1)p} + i\,\mathrm{sgn}(s) A_k^{(2)p}\right) \tilde{p}(s) \frac{1}{s^2}\,\mathrm{d}s, \quad k = 1,2. \tag{26}$$

where $A_k^{(1)p}, A_k^{(2)p}$ are the constants given in [22], see Appendix, equation (A.3).

To simplify notation, the solutions were considered in dimensionless coordinates related to half-width of the contact zone a in the form of $\breve{x}_1 = x_1/a$, $\breve{y}_1 = y_1/a$. Furthermore, pressures (and thus stress tensor components) were considered relative to the mean contact pressure $p_0 = P/2a$ as $\breve{p} = p/p_0$.

For the punch of any smooth shape, the general solution is given as a system of dual integral equations in the form of:

$$\begin{cases}
\sum_{k=1}^{2} \left(B_k^{(1)} \breve{p}(\breve{x}_1) + \frac{B_k^{(2)}}{\pi} \int_{-1}^{1} \frac{\breve{p}(t)\,\mathrm{d}t}{\breve{x}_1 - t} \right) = -\frac{\partial g(\breve{x}_1)}{\partial \breve{x}_1}, \quad \breve{x}_1 \in \langle -1, 1 \rangle, \\
\int_{-\infty}^{\infty} \breve{p}(\breve{x}_1) \exp(-i\breve{s}\breve{x}_1)\,\mathrm{d}\breve{x}_1 = 0, \quad \breve{x}_1 \notin \langle -1, 1 \rangle,
\end{cases} \tag{27}$$

where $B_k^{(1)}$, $B_k^{(2)}$, $k = 1,2$ are the parameters related to mechanical properties of the composite components and lamination angle α, see Appendix, Eq. (A.2).

System of equations (27) has a solution in the form [23]:

$$\begin{aligned}
\breve{p}(\breve{x}_1) &= \frac{4}{\pi} \left(1 - \breve{x}_1^2\right)^{0.5}, & \breve{x}_1 &\in \langle -1;\ 1 \rangle, \\
\breve{p}(\breve{x}_1) &= 0, & \breve{x}_1 &\notin \langle -1;\ 1 \rangle,
\end{aligned} \tag{28}$$

for a parabolic punch and:

$$\begin{aligned}
\breve{p}(\breve{x}_1) &= \frac{2}{\pi} \left(1 - \breve{x}_1^2\right)^{-0.5}, & \breve{x}_1 &\in \langle -1;\ 1 \rangle, \\
\breve{p}(\breve{x}_1) &= 0, & \breve{x}_1 &\notin \langle -1;\ 1 \rangle,
\end{aligned} \tag{29}$$

for a rectangular punch [24].

From (1) and (28), we obtain half-width of the contact zone as [25]:

$$a^2 = \frac{2}{\pi} \left(B_1^{(2)} + B_2^{(2)} \right) RP, \tag{30}$$

where $B_1^{(2)}$, $B_2^{(2)}$ are the constants given in Appendix in equation (A.2).

Having elastic potentials Ψ_k, $k = 1,2$, we could determine the Fourier transform of contact pressure $\tilde{p}(\breve{s})$ as [26]:

$$\tilde{p}(\breve{s}) = \sqrt{\frac{2}{\pi}} \frac{P \mathrm{J}_1(\breve{s})}{\breve{s}}, \tag{31}$$

for a parabolic punch and [26]:

$$\tilde{p}(\breve{s}) = \frac{P}{\sqrt{2\pi}} \mathrm{J}_0(\breve{s}), \tag{32}$$

for a rectangular punch, where $\mathrm{J}_0(\cdot)$, $\mathrm{J}_1(\cdot)$ are the Bessel functions of the first kind [26].

Having Fourier transform of the contact pressure, the stress tensor components can be easily calculated using equations (23) and (26). For the case of contact pressure $\breve{p}^{(j)}(\breve{x}_1)$ it could be calculated as:

$$\breve{p}^{(j)}(\breve{x}_1) = \lim_{\breve{y}_1 \to 0^+} \left(\breve{\sigma}_{\breve{y}_1 \breve{y}_1}^{(j)}(\breve{x}_1, \breve{y}_1) \right), \ j = 1,2, \tag{33}$$

where the stress tensor component $\breve{\sigma}^{(j)}_{\breve{y}_1\breve{y}_1} = \sigma^{(j)}_{\breve{y}_1\breve{y}_1}/p_0$ could be defined by:

$$\breve{\sigma}^{(j)}_{\breve{y}_1\breve{y}_1}(\breve{x}_1,\breve{y}_1) = \frac{4}{\pi}\sum_{k=1}^{2}\int_0^{\infty}\frac{\mathrm{J}_1(\breve{s})}{\breve{s}}\exp\left(-a_1^k\breve{s}\breve{y}_1\right)\cdot \cdot\left(P^{(1)k}_{\breve{y}_1\breve{y}_1 j}\cos\left(\left(\breve{x}_1 - a_2^k\breve{y}_1\right)\breve{s}\right) + - P^{(2)k}_{\breve{y}_1\breve{y}_1 j}\sin\left(\left(\breve{x}_1 - a_2^k\breve{y}_1\right)\breve{s}\right)\right)\mathrm{d}\breve{s}, \quad j = 1,2, \tag{34}$$

for a parabolic punch and:

$$\breve{\sigma}^{(j)}_{\breve{y}_1\breve{y}_1}(\breve{x}_1,\breve{y}_1) = \frac{2}{\pi}\sum_{k=1}^{2}\int_0^{\infty}\mathrm{J}_0(\breve{s})\exp\left(-a_1^k\breve{s}\breve{y}_1\right)\cdot \cdot\left(P^{(1)k}_{\breve{y}_1\breve{y}_1 j}\cos\left(\left(\breve{x}_1 - a_2^k\breve{y}_1\right)\breve{s}\right) + - P^{(2)k}_{\breve{y}_1\breve{y}_1 j}\sin\left(\left(\breve{x}_1 - a_2^k\breve{y}_1\right)\breve{s}\right)\right)\mathrm{d}\breve{s}, \quad j = 1,2, \tag{35}$$

for a rectangular punch, where $P^{(1)k}_{\breve{y}_1\breve{y}_1 j}$, $P^{(2)k}_{\breve{y}_1\breve{y}_1 j}$ are given in the Appendix, equation (A.9).

Thus, we have the contact pressure in the j-th composite component in the form of:

$$\breve{p}^{(j)}(\breve{x}_1) = \frac{4}{\pi}\sum_{k=1}^{2}\int_0^{\infty}\frac{\mathrm{J}_1(\breve{s})}{\breve{s}}\cdot \cdot\left(P^{(1)k}_{\breve{y}_1\breve{y}_1 j}\cos(\breve{x}_1\breve{s}) - P^{(2)k}_{\breve{y}_1\breve{y}_1 j}\sin(\breve{x}_1\breve{s})\right)\mathrm{d}\breve{s}, \quad j = 1,2, \tag{36}$$

for a parabolic punch and:

$$\breve{p}^{(j)}(\breve{x}_1) = \frac{2}{\pi}\sum_{k=1}^{2}\int_0^{\infty}\mathrm{J}_0(\breve{s})\cdot \cdot\left(P^{(1)k}_{\breve{y}_1\breve{y}_1 j}\cos(\breve{x}_1\breve{s}) - P^{(2)k}_{\breve{y}_1\breve{y}_1 j}\sin(\breve{x}_1\breve{s})\right)\mathrm{d}\breve{s}, \quad j = 1,2, \tag{37}$$

for a rectangular punch.

Integrals in (36) and (37) were calculated numerically, based on a proprietary numerical algorithm based on typical integration methods, i.e., using a Gaussian quadrature and Python SciPy package. The results of those calculations are presented below.

4. RESULTS

Results for each composite component were drawn in different colors – grey lines for the first material component ($j = 1$) and black for the second one ($j = 2$).

Figure 2 shows the distribution of dimensionless contact pressure $\breve{p}^{(j)}(\breve{x}_1) = p^{(j)}(\breve{x}_1)/p_0$, $j = 1,2$ for two different punches (rectangular punch on the left and parabolic punch on the right).

Figures show that both the shape of the punch and the layering angle have a significant impact on the pressure distribution and the maximum values obtained. For the stratification perpendicular to the edge, the graph corresponds to the results obtained by Perkowski *et al.* [13]. The solutions obtained in this paper are consistent with [13] and the presented results are a generalization for the layers arranged at any angle.

Figures 3a and 3b present the characteristics describing the pressure value in the center of the contact zone $\breve{p}^{(j)}(0,0) = p^{(j)}(0,0)/p_0$, $j = 1,2$ depending on the ratio of Young's modu-

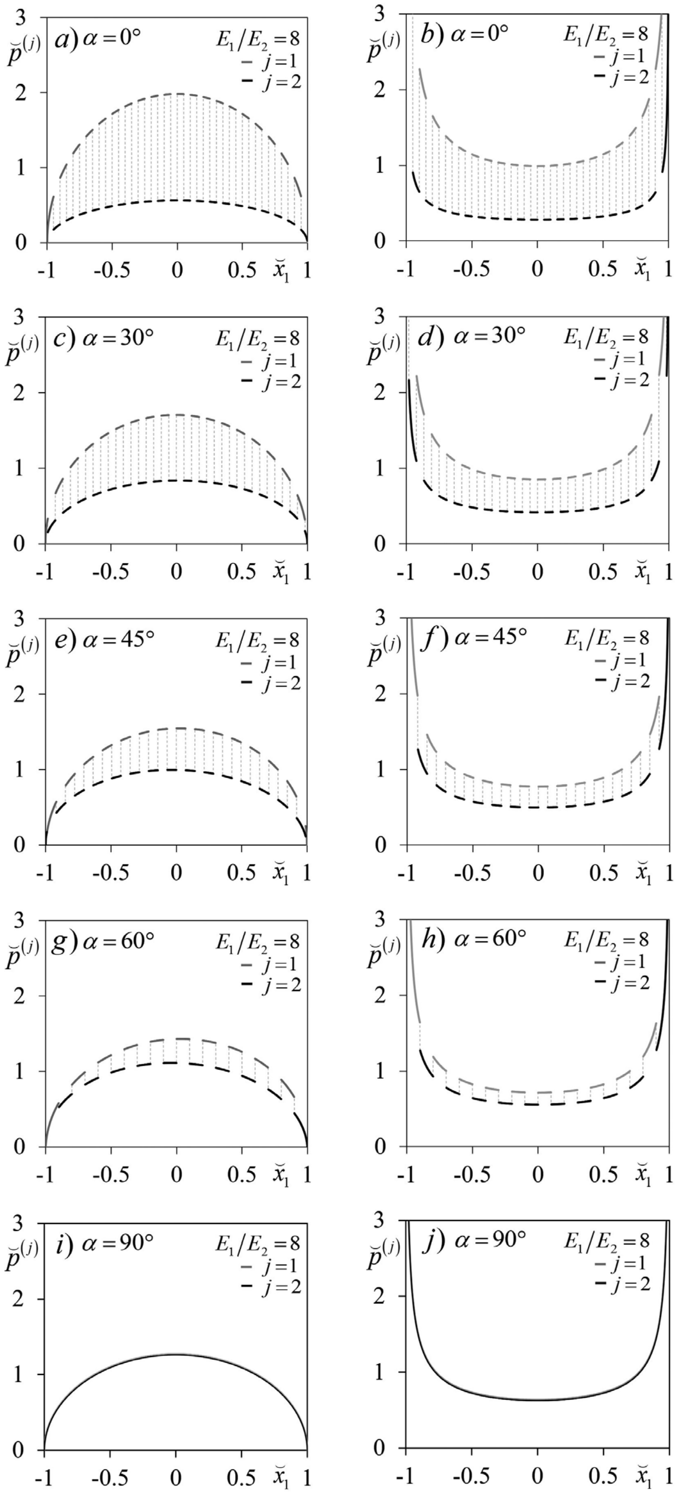

Fig. 2. Contact pressures $\breve{p}^{(j)}(\breve{x}_1) = p^{(j)}(\breve{x}_1)/p_0$, $j = 1,2$ for j-th component of the composite body at $E_1/E_2 = 8$, $\nu_1 = \nu_2 = 0.3$, $\eta = 0.5$ for different lamina angles and different punch shapes: a), c), e), g), i) – parabolic punch, b), d), f), h), j) – rectangular punch

lus of the components of the composite medium. They show that the greatest differences in pressure between the components of the composite occur at the perpendicular lamination to the edge.

From Fig. 3 it can also be seen that regardless of the angle, the values of this pressure in the center of the zone are twice as high for a parabolic punch than for a rectangular punch, which is consistent with the known solutions for the lamination perpendicular to the boundary, see [13]. Additionally, for the lamination parallel to the boundary, it can be seen that for both layers the results are in full compliance with the homogenous material, for which $p_{\max} = p(0,0) = 4p_0/\pi$ – parabolic punch and $p_{\min} = p(0,0) = 2p_0/\pi$ – rectangular punch.

Figures 4a and 4b present the contact pressure at the central point of the contact zone for different ratios of layer thickness for the composite with component Young's modulus ratio $E_1/E_2 = 8$. As can be seen, the values differ significantly with a changing angle α and the differences are the highest for lamination perpendicular to the boundary surface ($\alpha = 0^\circ$) and nonexistent for layering parallel to the surface. It could also be seen that the graphs converge to the specific values of contact pressure $(4p_0/\pi)$ and $(2p_0/\pi)$ for the values of layer thickness ratio $\eta \to 1$ and $\eta \to 0$ – for the first and second components, respectively.

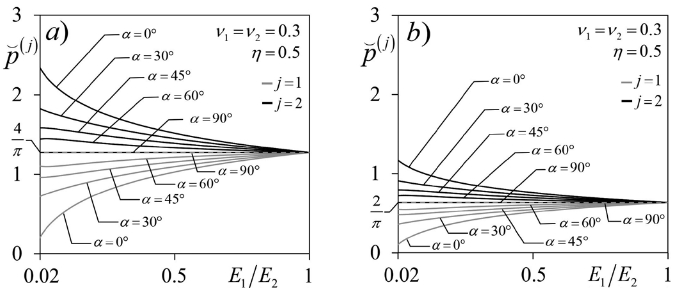

Fig. 3. Contact pressure $\breve{p}^{(j)}(0,0) = p^{(j)}(0,0)/p_0$, $j = 1,2$ for j-th component of the composite body at the central point of the contact zone ($x = 0$, $y = 0$) for different stiffness ratios E_1/E_2 and different lamina angles α for: a) parabolic punch, b) rectangular punch

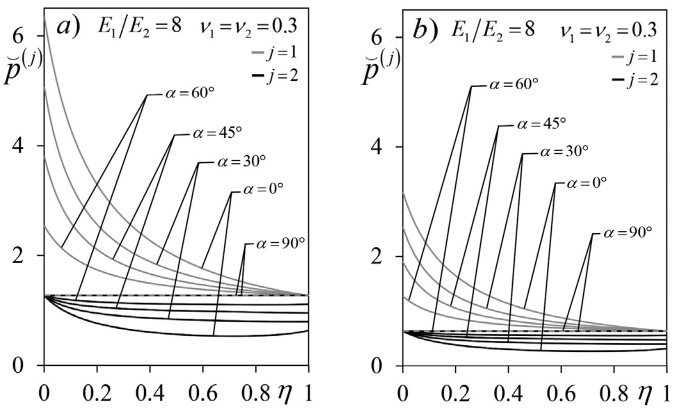

Fig. 4. Contact pressure $\breve{p}^{(j)}(0,0) = p^{(j)}(0,0)/p_0$, $j = 1,2$ for j-th component of the composite body at the central point of the contact zone ($x = 0$, $y = 0$) for different ratios of layer thickness η and different lamina angles α for the case of a) parabolic punch, b) rectangular punch

5. CONCLUSIONS

The presented work discusses the elastic contact problem of micro-periodic elastic half-space in a plane strain state pressed by a rigid punch. The homogenized model with microlocal parameters has been used to solve the considered problem. Two examples of punch cross-section have been considered: parabolic and rectangular. For both shapes of the punch cross-section, the contact pressure has been analyzed and the characteristics of its relation to the composite structure have been drawn.

From this analysis, the following conclusions can be drawn:

1. Results show that averaged contact pressure distribution for the contact problems for the composites with slant lamination is related to the shape of the punch in a similar fashion as it is for the homogenous bodies.
2. In the case of per-component contact pressures it could be seen that their maximum values differ the more the higher stiffness ratios E_1/E_2 are, so it is not justified to consider only the averaged values and an approach that allows us to determine the stresses in each layer separately should be used – as is possible with the homogenized model with microlocal parameters.
3. For each composite component, the contact pressures for both punch shapes differ in the same way – as they differ for the composite with lamination perpendicular to the boundary and the differences decrease with the transition to the composite with lamination parallel to the boundary surface.

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APPENDIX

Parameters used above in the text [21]:

$$
\begin{aligned}
A_1 &= \frac{(\lambda_1+2\mu_1)(\lambda_2+2\mu_2)}{(1-\eta)(\lambda_1+2\mu_1)+\eta(\lambda_2+2\mu_2)} > 0,\\
A_2 &= A_1 + \frac{4\eta(1-\eta)(\mu_1-\mu_2)(\lambda_1-\lambda_2+\mu_1-\mu_2)}{(1-\eta)(\lambda_1+2\mu_1)+\eta(\lambda_2+2\mu_2)} > 0,\\
B &= \frac{(1-\eta)\lambda_2(\lambda_1+2\mu_1)+\eta\lambda_1(\lambda_2+2\mu_2)}{(1-\eta)(\lambda_1+2\mu_1)+\eta(\lambda_2+2\mu_2)} > 0,\\
C &= \frac{\mu_1\mu_2}{(1-\eta)\mu_1+\eta\mu_2} > 0, \quad D_j = \frac{\lambda_j}{\lambda_j+2\mu_j}A_1 > 0,\\
E_j &= \frac{4\mu_j(\lambda_j+\mu_j)}{\lambda_j+2\mu_j} + \frac{\lambda_j}{\lambda_j+2\mu_j}B > 0, \quad j = 1,2.
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
B_k^{(1)} &= \left(A_k^{(1)}a_2^k + A_k^{(2)}a_1^k\right)\left(\kappa_k\sin^2\alpha+\cos^2\alpha\right) + \\
&\quad + A_k^{(1)}\sin\alpha\cos\alpha\,(1-\kappa_k),\\
B_k^{(2)} &= \left(A_k^{(2)}a_2^k - A_k^{(1)}a_1^k\right)\left(\kappa_k\sin^2\alpha+\cos^2\alpha\right) + \\
&\quad + A_k^{(2)}\sin\alpha\cos\alpha\,(1-\kappa_k), \quad k = 1,2.
\end{aligned}
\tag{A.2}
$$

$$A_k^{(1)p} = \frac{\Delta_1 \Delta_{11}^{(k)} + \Delta_2 \Delta_{12}^{(k)}}{\Delta_1^2 + \Delta_2^2}, \quad A_k^{(2)p} = \frac{\Delta_1 \Delta_{12}^{(k)} - \Delta_2 \Delta_{11}^{(k)}}{\Delta_1^2 + \Delta_2^2}, \quad k = 1,2. \tag{A.3}$$

$$\Delta_k = A_{11}^{(1)} A_{22}^{(k)} - A_{12}^{(1)} A_{21}^{(k)} + (-1)^{k+1} \left(A_{12}^{(2)} A_{21}^{(3-k)} - A_{11}^{(2)} A_{22}^{(3-k)} \right), \quad k = 1,2. \tag{A.4}$$

$$\begin{aligned}
\Delta_{1k}^{(1)} &= A_{12}^{(k)} \cos\alpha - A_{22}^{(k)} \sin\alpha, \\
\Delta_{1k}^{(2)} &= A_{21}^{(k)} \sin\alpha - A_{11}^{(k)} \cos\alpha, \\
\Delta_{2k}^{(1)} &= A_{22}^{(k)} \cos\alpha + A_{12}^{(k)} \sin\alpha, \\
\Delta_{2k}^{(2)} &= A_{11}^{(k)} \sin\alpha + A_{21}^{(k)} \cos\alpha, \quad k = 1,2.
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
A_{1k}^{(1)} &= \left(\left(a_1^k\right)^2 - \left(a_2^k\right)^2 \right) D_{11}^{(k)} + a_2^k D_{12}^{(k)} - D_{13}^{(k)}, \\
A_{1k}^{(2)} &= 2 a_1^k a_2^k D_{11}^{(k)} - a_1^k D_{12}^{(k)}, \\
A_{2k}^{(1)} &= \left(\left(a_1^k\right)^2 - \left(a_2^k\right)^2 \right) D_{21}^{(k)} + a_2^k D_{22}^{(k)} - D_{23}^{(k)}, \\
A_{2k}^{(2)} &= 2 a_1^k a_2^k D_{21}^{(k)} - a_1^k D_{22}^{(k)}, \quad k = 1,2.
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
D_{11}^{(k)} &= \big(A_1 \kappa_k \sin^2\alpha + B \cos^2\alpha + \\
&\quad + C(1+\kappa_k)\cos^2\alpha\big) \sin\alpha, \\
D_{12}^{(k)} &= \big(2(A_1\kappa_k - B)\sin^2\alpha + \\
&\quad + C(1+\kappa_k)\left(\cos^2\alpha - \sin^2\alpha\right)\big)\cos\alpha, \\
D_{13}^{(k)} &= \big(A_1 \kappa_k \cos^2\alpha + B\sin^2\alpha - \\
&\quad - C(1+\kappa_k)\cos^2\alpha\big)\sin\alpha, \\
D_{21}^{(k)} &= \big(B\kappa_k \sin^2\alpha + A_2\cos^2\alpha + \\
&\quad + C(1+\kappa_k)\sin^2\alpha\big)\cos\alpha, \\
D_{22}^{(k)} &= \big(2(B\kappa_k - A_2)\cos^2\alpha + \\
&\quad + C(1+\kappa_k)\left(\cos^2\alpha - \sin^2\alpha\right)\big)\sin\alpha, \\
D_{23}^{(k)} &= \big(B\kappa_k\cos^2\alpha + A_2\sin^2\alpha - \\
&\quad - C(1+\kappa_k)\sin^2\alpha\big)\cos\alpha, \quad k = 1,2.
\end{aligned} \tag{A.7}$$

$$a_1^k = \frac{\gamma_k}{\gamma_k^2 \sin^2\alpha + \cos^2\alpha}, \quad a_2^k = \frac{\left(\gamma_k^2 - 1\right)\sin\alpha\cos\alpha}{\gamma_k^2\sin^2\alpha + \cos^2\alpha}, \quad k = 1,2. \tag{A.8}$$

$$P_{\check{y}_1\check{y}_1 j}^{(l)k} = P_{xx\,j}^{(l))k}\sin^2\alpha + 2P_{xy\,j}^{(l))k}\sin\alpha\cos\alpha + P_{yy\,j}^{(l))k}\cos^2\alpha, \quad j,l,k = 1,2. \tag{A.9}$$

$$\begin{aligned}
P_{xx\,j}^{(l))k} &= A_1 \kappa_k C_{1k}^{(l))p} + B C_{3k}^{(l))p}, \\
P_{xy\,j}^{(l))k} &= C(1+\kappa_k) C_{2k}^{(l))p}, \\
P_{yy\,j}^{(l))k} &= D_j \kappa_k C_{1k}^{(l))p} + E_j C_{3k}^{(l))p}, \quad j,l,k = 1,2.
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
C_{1k}^{(1)p} &= A_k^{(1)} b_1^k - A_k^{(2)} b_2^k, \\
C_{1k}^{(2)p} &= A_k^{(2)} b_1^k + A_k^{(1)} b_2^k, \\
C_{2k}^{(1)p} &= A_k^{(1)} b_3^k - A_k^{(2)} b_4^k, \\
C_{2k}^{(2)p} &= A_k^{(2)} b_3^k + A_k^{(1)} b_4^k, \\
C_{3k}^{(1)p} &= A_k^{(1)} b_5^k - A_k^{(2)} b_6^k, \\
C_{3k}^{(2)p} &= A_k^{(2)} b_5^k + A_k^{(1)} b_6^k, \quad k = 1,2.
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
b_1^k &= \left(a_1^k\right)^2 \sin^2\alpha - \left(\cos\alpha - a_2^k \sin\alpha\right)^2, \\
b_2^k &= -2a_1^k \sin\alpha \left(\cos\alpha - a_2^k\sin\alpha\right), \\
b_3^k &= \left(a_1^k\right)^2 \sin\alpha\cos\alpha + \\
&\quad + \left(\cos\alpha - a_2^k\sin\alpha\right)\left(\sin\alpha + a_2^k\cos\alpha\right), \\
b_4^k &= a_1^k \sin\alpha\left(\sin\alpha + a_2^k\cos\alpha\right) + \\
&\quad - a_1^k\cos\alpha\left(\cos\alpha + a_2^k\sin\alpha\right), \\
b_5^k &= \left(a_1^k\right)^2\cos^2\alpha - \left(\sin\alpha + a_2^k\cos\alpha\right)^2, \\
b_6^k &= 2a_1^k\cos\alpha\left(\sin\alpha + a_2^k\cos\alpha\right), \quad k = 1,2.
\end{aligned} \tag{A.12}$$

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