10.24425/acs.2021.138689

Archives of Control Sciences
Volume 31(LXVII), 2021
No. 3*, pages 481–493*

The problem of control of rod heating process with nonseparated conditions at intermediate moments of time

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The problem of control of rod heating process by changing the temperature along the rod whose ends are thermally insulated is considered. It is assumed that, along with the classical boundary conditions, nonseparated multipoint intermediate conditions are also given. Using the method of separation of variables and methods of the theory of control of finite-dimensional systems with multipoint intermediate conditions, a constructive approach is proposed to build the sought function of temperature control action. A necessary and sufficient condition is obtained, which the function of the distribution of the rod temperature must satisfy, so that under any feasible initial, nonseparated intermediate, and final conditions, the problem is completely controllable. As an application of the proposed approach, control action with given nonseparated conditions on the values of the rod temperature distribution function at the two intermediate moments of time is constructed.

Key words: heating control, temperature, intermediate moments of time, nonseparated multipoint conditions, complete controllability

1. Introduction

When studying controlled thermal processes, mathematical problems of control of processes arise, which are described by differential equations with partial derivatives [1–6]. In many applications, the problems of control of the process of rod heating whose ends are thermally insulated occur. Issues of the development of regimes to control the heating of the rod are relevant. Theoretical investigations, as well as various statements of the problems of control and optimal control of processes described by parabolic equations, are given, in particular,

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Received 13.03.2021. Revised 01.07.2021.

in [1–17]. In the works [12–15] the problems of control and optimal control of heat (parabolic) equation with the help of distributed control are considered, in particular, allowing to take into account the constraints on the structure of the solution and control [12], the presence of integral constraint on the state [13], constraints on the control and states [14] and with point constraints of the control [15]. Control problems often arise, in which desired temperature state must be generated, satisfying multipoint intermediate conditions. A characteristic feature of multipoint boundary problems of control is the presence of nonseparated (non-local) multipoint intermediate conditions, along with the classical boundary (initial and final) conditions. Nonseparated multipoint boundary problems on the one hand arise as mathematical models of real processes, and on the other hand, for many processes, it is impossible to provide a correct setting of local boundary problems. In particular, the nonseparation of multipoint conditions may also be due to the impossibility in practice to instantly perform measurements of the measured parameters of the state of an object at its individual points. The problems of control of the process of heating the rod with given nonseparated multipoint conditions at intermediate moments of time have not been investigated yet.

The objective of this article is to develop a constructive approach to build the control function of the temperature state of the homogeneous rod to control the heating process with given initial, final conditions and nonseparated (nonlocal) values of the temperature of the rod points at intermediate moments of time. The article is related to the research in [18, 19].

2. Problem statement

Consider the controlled process of heating a homogenous rod with a length l, whose ends are fixed. Let the temperature distribution in the rod be described by the function $Z(x,t)$, $0 \leqslant x \leqslant l$, $0 < t < T$, which satisfies the equation

$$\frac{\partial Z}{\partial t} = a^2 \frac{\partial^2 Z}{\partial x^2} + u(x,t), \qquad 0 < x < l, \quad t > 0 \tag{1}$$

with initial condition

$$Z(x,0) = \varphi_0(x), \qquad 0 \leqslant x \leqslant l, \tag{2}$$

and homogenous boundary conditions

$$Z(0,t) = 0, \qquad Z(l,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{3}$$

In Eq. (1) $a^2 = \dfrac{k}{c\rho}$ – coefficient of thermal diffusivity of the rod material, ρ – material density, c – specific mass heat capacity, k – coefficient of thermal conductivity of the rod.

The function $u(x,t)$ is the control action. It is assumed that the control of the thermal process is performed as follows: allocate heat sources along the rod, for example, stretch a wire with a powerful current, then the temperature of the rod can be changed by changing over time the current strength. Mathematically, that is characterized by the fact that the control function $u(x,t)$ in Eq. (1) can be represented as the product of two functions, one of which determines the distribution of the heat source, and the other – the change in their power. Each of these functions can be considered as control.

Let at some intermediate moments of time $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$ on the values of the temperature function of the rod nonseparated (nonlocal) conditions are given in the form

$$\sum_{k=1}^{m} f_k Z(x,t_k) = \alpha(x), \tag{4}$$

where f_k – given values ($k = 1,\ldots,m$), $\alpha(x)$ – given function.

The condition (4) may also be due to the impossibility of instantaneous measurement of the temperature state in the rod at its individual points.

The problem of control of the heating process of a rod with given nonseparated (nonlocal) values of the temperature function at intermediate moments of time t_k ($k = 1,\ldots,m$) can be formulated as follows: among the possible feasible controls $u(x,t)$, $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$, it is required to find such a control that the corresponding solution $Z(x,t)$ to Eq. (1) with conditions (2) and (3) ensures the fulfillment of the nonseparated multipoint intermediate conditions (4) and satisfies the given final condition

$$Z(x,T) = \varphi_T(x) = \varphi_{m+1}(x), \qquad 0 \leqslant x \leqslant l. \tag{5}$$

If Ω denotes the set $\Omega = \{(x,t)\colon\ x \in [0,l],\ t \in [0,T]\}$, then the function $u(x,t) \in L_2(\Omega)$ is called a feasible control. By the solution of the stated problem, it is meant such a function of feasible control $u(x,t) \in L_2(\Omega)$ that the corresponding solution $Z(x,t) \in L_2(\Omega)$ to Eq. (1) with conditions (2) and (3) satisfies conditions (4) and (5).

It is assumed that the system (1) under the constraints (2)–(5) over the time interval [0, T] is completely controllable [5, 20]. This means that over the time interval [0, T] it is possible to choose control $u(x,t)$, under the influence of which the temperature function of the rod $Z(x,t)$ satisfies the equation (1) and the given conditions (2)–(5).

3. Solution of the problem

To construct solution of the stated problem, the solution of Eq. (1) with boundary conditions (3) is sought in the form

$$Z(x,t) = \sum_{n=1}^{\infty} Z_n(t) \sin \frac{\pi n}{l} x. \tag{6}$$

$u(x,t)$ and $\alpha(x)$ functions are represented in the form of Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{\pi n}{l} x, \qquad \alpha(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{\pi n}{l} x. \tag{7}$$

Substituting the expansions (6), (7) in the relations (1)–(5) and due to the orthogonality of the system of eigenfunctions, it follows that the Fourier coefficients $Z_n(t)$ satisfy a countable number of systems of ordinary differential equations

$$\dot{Z}_n(t) + \lambda_n^2 Z_n(t) = u_n(t), \qquad \lambda_n^2 = \left(\frac{a\pi n}{l}\right)^2 \qquad n = 1, 2, \ldots \tag{8}$$

and the following initial, nonseparated multipoint intermediate and final conditions:

$$Z_n(0) = \varphi_n^{(0)}, \tag{9}$$

$$\sum_{k=1}^{m} f_k Z_n(t_k) = \alpha_n, \tag{10}$$

$$Z_n(T) = \varphi_n^{(T)}, \tag{11}$$

where $\varphi_n^{(0)}$, $\varphi_n^{(T)}$ denote Fourier coefficients corresponding to the functions $\varphi_0(x)$, $\varphi_T(x)$.

The general solution of Eq. (8) with initial condition (9) has the form

$$Z_n(t) = \varphi_n^{(0)} e^{-\lambda_n t} + \int_0^t u_n(\tau) e^{-\lambda_n (t-\tau)} \, d\tau. \tag{12}$$

Now, taking into account intermediate nonseparated (10) and final (11) conditions, using the approaches given in [20, 21] from the equation (12), we find that the functions $u_n(\tau)$ for each n must satisfy the following system of equalities:

$$\begin{gathered}\int_0^T u_n(\tau)e^{-\lambda_n(T-\tau)}\mathrm{d}\tau = C_n(T),\\ \sum_{k=1}^m f_k\int_0^{t_k} u_n(\tau)e^{-\lambda_n(t_k-\tau)}\mathrm{d}\tau = C_n^{(m)}(t_1,\dots,t_m),\\ n = 1,\ 2,\dots,\end{gathered}\tag{13}$$

where

$$\begin{gathered}C_n(T) = \varphi_n^{(T)} - \varphi_n^{(0)}e^{-\lambda_n T},\\ C_n^{(m)}(T) = C_n^{(m)}(t_1,\dots,t_m) = \alpha_n - \sum_{k=1}^m f_k\varphi_n^{(0)}e^{-\lambda_n t_k}.\end{gathered}\tag{14}$$

Introducing the following functions

$$\begin{gathered}h_n(\tau) = e^{-\lambda_n(T-\tau)},\qquad 0\leqslant\tau\leqslant T,\\ h_n^{(m)}(\tau) = \sum_{k=1}^m f_k h_n^{(k)}(\tau),\\ h_n^{(k)}(\tau) = \begin{cases} e^{-\lambda_n(t_k-\tau)}, & 0\leqslant\tau\leqslant t_k,\\ 0, & t_k<\tau\leqslant t_{m+1}=T,\end{cases}\end{gathered}\tag{15}$$

the integral relations (13), using functions (15), will be written as follows

$$\begin{gathered}\int_0^T u_n(\tau)h_n(\tau)\mathrm{d}\tau = C_n(T),\\ \int_0^T u_n(\tau)h_n^{(m)}(\tau)\mathrm{d}\tau = C_n^{(m)}(T),\quad n=1,2,\dots\end{gathered}\tag{16}$$

Thus, the sought functions $u_n(\tau)$, $\tau\in[0,\ T]$ for each n must satisfy the integral relations (16).

Introducing the following notations

$$H_n(\tau) = \begin{pmatrix} h_n(\tau)\\ h_n^{(m)}(\tau)\end{pmatrix},\qquad \eta_n = \begin{pmatrix} C_n(T)\\ C_n^{(m)}(T)\end{pmatrix}\tag{17}$$

integral relations (16) are written as follows

$$\int_0^T H_n(t)u_n(t)\,\mathrm{d}t = \eta_n\,. \tag{18}$$

From the relation (18) (or (16)) it follows that for each n the process described by Eq. (8) with conditions (9)–(11) is completely controllable if and only if for any given vector η_n (17) the control $u_n(t)$, $t \in [0, T]$ can be found, satisfying condition (18) (or (16)).

Following [20, 21], for each $n = 1, 2, \dots$ the function $u_n(t)$, $t \in [0, T]$, satisfying the integral relation (18) can be represented as

$$u_n(t) = (H_n(t))'\, S_n^{-1}\eta_n + \nu_n(t). \tag{19}$$

where

$$S_n = \int_0^T H_n(t)\,(H_n(t))'\,\mathrm{d}t = \begin{pmatrix} s_{11}^{(n)} & s_{12}^{(n)} \\ s_{21}^{(n)} & s_{22}^{(n)} \end{pmatrix}, \tag{20}$$

and $\nu_n(t)$ – some function such that

$$\int_0^T H_n(t)\nu_n(t)\,\mathrm{d}t = 0. \tag{21}$$

Hereinafter, the superscript $'$ denotes the transpose operation.

The elements of the matrix S_n, according to (20) and the notations (15), (17), have the following forms

$$s_{11}^{(n)} = \int_0^T (h_n(\tau))^2\,\mathrm{d}\tau = \int_0^T e^{-2\lambda_n(T-\tau)}\,\mathrm{d}\tau = \frac{1}{2\lambda_n}\left(1 - e^{-2\lambda_n T}\right),$$

$$s_{12}^{(n)} = s_{21}^{(n)} = \int_0^T h_n(\tau)h_n^{(m)}(\tau)\,\mathrm{d}\tau = \sum_{k=1}^{m} f_k \frac{1}{2\lambda_n}\left(e^{-\lambda_n(T-t_k)} - e^{-\lambda_n(T+t_k)}\right), \tag{22}$$

$$s_{22}^{(n)} = \int_0^T \left(h_n^{(m)}(\tau)\right)^2\,\mathrm{d}\tau = \int_0^T \left(\sum_{k=1}^{m} f_k h_n^{(k)}(\tau)\right)^2\,\mathrm{d}\tau.$$

Note that according to the notation (15) we have

$$h_n^{(m)}(t) = \begin{cases} \sum_{k=1}^{m} f_k e^{-\lambda_n(t_k - t)}, & 0 \leqslant t \leqslant t_1, \\ \sum_{k=2}^{m} f_k e^{-\lambda_n(t_k - t)}, & t_1 < t \leqslant t_2, \\ \ldots, & \\ \sum_{k=m-1}^{m} f_k e^{-\lambda_n(t_k - t)}, & t_{m-2} < t \leqslant t_{m-1}, \\ f_m e^{-\lambda_n(t_m - t)}, & t_{m-1} < t \leqslant t_m, \\ 0, & t_m < t \leqslant t_{m+1} = T. \end{cases}$$

Therefore, given the notation (15), (17), the control action $u_n(t)$, $t \in [0, T]$, according to (19), is presented in the following form:

$$u_n(t) = \begin{cases} \left(e^{-\lambda_n(T-\tau)} \;\; \sum_{k=1}^{m} f_k e^{-\lambda_n(t_k - t)}\right) S_n^{-1}\eta_n + \nu_n(t), & 0 \leqslant t \leqslant t_1, \\ \left(e^{-\lambda_n(T-\tau)} \;\; \sum_{k=2}^{m} f_k e^{-\lambda_n(t_k - t)}\right) S_n^{-1}\eta_n + \nu_n(t), & t_1 < t \leqslant t_2, \\ \ldots & \\ \left(e^{-\lambda_n(T-\tau)} \;\; f_m e^{-\lambda_n(t_m - t)}\right) S_n^{-1}\eta_n + \nu_n(t), & t_{m-1} < t \leqslant t_m, \\ \left(e^{-\lambda_n(T-\tau)} \;\; 0\right) S_n^{-1}\eta_n + \nu_n(t), & t_m < t \leqslant t_{m+1} = T. \end{cases} \tag{23}$$

Thus, for the control $u(x, t)$, from the formula (7) we have

$$u(x,t) = \begin{cases} \sum_{n=1}^{\infty} \left[\left(e^{-\lambda_n(T-\tau)} \;\; \sum_{k=1}^{m} f_k e^{-\lambda_n(t_k - t)}\right) S_n^{-1}\eta_n + \nu_n(t)\right] \sin \frac{\pi n}{l} x, & 0 \leqslant t \leqslant t_1, \\ \sum_{n=1}^{\infty} \left[\left(e^{-\lambda_n(T-\tau)} \;\; \sum_{k=2}^{m} f_k e^{-\lambda_n(t_k - t)}\right) S_n^{-1}\eta_n + \nu_n(t)\right] \sin \frac{\pi n}{l} x, & t_1 \leqslant t \leqslant t_2, \\ \ldots & \\ \sum_{n=1}^{\infty} \left[\left(e^{-\lambda_n(T-\tau)} \;\; f_m e^{-\lambda_n(t_m - t)}\right) S_n^{-1}\eta_n + \nu_n(t)\right] \sin \frac{\pi n}{l} x, & t_{m-1} < t \leqslant t_m, \\ \sum_{n=1}^{\infty} \left[\left(e^{-\lambda_n(T-\tau)} \;\; 0\right) S_n^{-1}\eta_n + \nu_n(t)\right] \sin \frac{\pi n}{l} x, & t_m < t \leqslant t_{m+1} = T. \end{cases}$$

Substituting (23) into (12), $Z_n(t)$ on the time interval $t \in [0, T]$ is obtained, and from the formula (6) the temperature function $Z(x, t)$ can be found. Note that the functions $u(x,t)$ and $Z(x,t)$ are the solutions of the stated problem since the series constructed for them converges uniformly in Ω .

4. Example

Suppose that $m = 2$ (i.e. $0 < t_1 < t_2 < t_3 = T$), then for $\nu_n(t) = 0$ from (23) the following holds

$$u_n(t) = \begin{cases} \left(e^{-\lambda_n(T-\tau)} f_1 e^{-\lambda_n(t_1-t)}\right) S_n^{-1}\eta_n, & 0 \leqslant t \leqslant t_1, \\ \left(e^{-\lambda_n(T-\tau)} f_2 e^{-\lambda_n(t_2-t)}\right) S_n^{-1}\eta_n, & t_1 < t \leqslant t_2, \\ \left(e^{-\lambda_n(T-\tau)} 0\right) S_n^{-1}\eta_n, & t_2 < t \leqslant t_3 = T. \end{cases} \tag{24}$$

From the formula (22) we have

$$s_{11}^{(n)} = \frac{1 - e^{-2\lambda_n T}}{2\lambda_n},$$
$$s_{12}^{(n)} = s_{21}^{(n)} = \frac{1}{2\lambda_n}\left[f_1 e^{-\lambda_n(T+t_1)}\left(e^{2\lambda_n t_1} - 1\right) + f_2 e^{-\lambda_n(T+t_2)}\left(e^{2\lambda_n t_2} - 1\right)\right],$$
$$s_{22}^{(n)} = \frac{1}{2\lambda_n}\left[f_1^2\left(1 - e^{-2\lambda_n t_1}\right) + f_2^2\left(1 - e^{-2\lambda_n t_2}\right) + 2f_1 f_2 e^{-\lambda_n(t_1+t_2)}\left(e^{2\lambda_n t_1} - 1\right)\right].$$

Hence, we have that

$$\begin{aligned} \det S_n = -\frac{1}{4\lambda_n^2} e^{-2\lambda_n(T+t_1+t_2)} &\left[f_1^2\left(e^{2\lambda_n(T+t_2)} + e^{2\lambda_n(2t_1+t_2)}\right)\right. \\ &+ f_2^2\left(e^{2\lambda_n(T+t_1)} + e^{2\lambda_n(t_1+2t_2)}\right) \\ &+ 2f_1 f_2\left(e^{3\lambda_n(t_1+t_2)} + e^{\lambda_n(2T+t_1+t_2)} - e^{\lambda_n(2T+3t_1+t_2)} - e^{\lambda_n(t_1+3t_2)}\right) \\ &\left.-(f_1^2 + f_2^2)\left(e^{2\lambda_n(t_1+t_2)} + e^{2\lambda_n(T+t_1+t_2)}\right)\right]. \end{aligned}$$

Note that for $f_1 \neq 0$ and $f_2 \neq 0$ it follows that $\det S_n \neq 0$.

Now assuming that $t_1 = 1$, $t_2 = 2$, $T = 3$ and $f_1 = f_2 = 1$, we have

$$\det S_n = \frac{2e^{-3\lambda_n}\left(1 + 2\cosh\lambda_n\right)\sinh\lambda_n^2}{\lambda_n^2},$$

and from the above expressions, for the inverse matrix, the ollowing is obtained

$$S_n^{-1} = \begin{pmatrix} s_{11}^{-(n)} & s_{12}^{-(n)} \\ s_{21}^{-(n)} & s_{22}^{-(n)} \end{pmatrix},$$

where

$$s_{11}^{-(n)} = \frac{\lambda_n e^{\lambda_n} \left(2 + 3\cosh\lambda_n + \sinh\lambda_n\right)}{2\left(\sinh\lambda_n + \sinh\lambda_n\right)},$$
$$s_{12}^{-(n)} = s_{21}^{-(n)} = -\frac{\lambda_n}{2}\operatorname{csch}\lambda_n,$$
$$s_{22}^{-(n)} = \lambda_n \coth\lambda_n - \frac{\lambda_n}{2}\operatorname{csch}\lambda_n.$$

From (17), taking into account (14), the following is obtained

$$\eta_n = \begin{pmatrix} \eta_1^{(n)} \\ \eta_2^{(n)} \end{pmatrix} = \begin{pmatrix} \varphi_n^{(T)} - \varphi_n^{(0)} e^{-3\lambda_n} \\ \alpha_n - \varphi_n^{(0)}\left(e^{-\lambda_n} + e^{-2\lambda_n}\right) \end{pmatrix},$$

$$S_n^{-1}\eta_n = \begin{pmatrix} s_{11}^{-(n)}\eta_1^{(n)} + s_{12}^{-(n)}\eta_2^{(n)} \\ s_{12}^{-(n)}\eta_1^{(n)} + s_{22}^{-(n)}\eta_2^{(n)} \end{pmatrix}.$$

Substituting the value of the vector $S_n^{-1}\eta_n$ in (24), it is obtained that

$$u_n(t) = \begin{cases} e^{-\lambda_n(3-t)}\left(s_{11}^{-(n)}\eta_1^{(n)} + s_{12}^{-(n)}\eta_2^{(n)}\right) \\ \quad + \left(e^{-\lambda_n(1-t)} + e^{-\lambda_n(2-t)}\right)\left(s_{12}^{-(n)}\eta_1^{(n)} + s_{22}^{-(n)}\eta_2^{(n)}\right), & 0 \leqslant t \leqslant 1, \\ e^{-\lambda_n(3-t)}\left(s_{11}^{-(n)}\eta_1^{(n)} + s_{12}^{-(n)}\eta_2^{(n)}\right) \\ \quad + e^{-\lambda_n(2-t)}\left(s_{12}^{-(n)}\eta_1^{(n)} + s_{22}^{-(n)}\eta_2^{(n)}\right), & 1 < t \leqslant 2, \\ e^{-\lambda_n(3-t)}\left(s_{11}^{-(n)}\eta_1^{(n)} + s_{12}^{-(n)}\eta_2^{(n)}\right), & 2 < t \leqslant 3 \end{cases}$$

and from (7) explicit expressions for the control function $u(x,t)$ are obtained in the form
for $0 \leqslant t \leqslant 1$

$$u(x,t) = \sum_{n=1}^{\infty}\Big[e^{-\lambda_n(3-t)}\left(s_{11}^{-(n)}\eta_1^{(n)} + s_{12}^{-(n)}\eta_2^{(n)}\right) + \left(e^{-\lambda_n(1-t)} + e^{-\lambda_n(2-t)}\right)\left(s_{12}^{-(n)}\eta_1^{(n)} + s_{22}^{-(n)}\eta_2^{(n)}\right)\Big]\sin\frac{\pi n}{l}x;$$

for $1 < t \leqslant 2$

$$u(x,t) = \sum_{n=1}^{\infty}\Big[e^{-\lambda_n(3-t)}\left(s_{11}^{-(n)}\eta_1^{(n)} + s_{12}^{-(n)}\eta_2^{(n)}\right) + e^{-\lambda_n(2-t)}\left(s_{12}^{-(n)}\eta_1^{(n)} + s_{22}^{-(n)}\eta_2^{(n)}\right)\Big]\sin\frac{\pi n}{l}x;$$

for $2 < t \leqslant 3$

$$u(x,t) = \sum_{n=1}^{\infty} \left[e^{-\lambda_n(3-t)} \left(s_{11}^{-(n)} \eta_1^{(n)} + s_{12}^{-(n)} \eta_2^{(n)} \right) \right] \sin \frac{\pi n}{l} x.$$

Substituting the expressions of the function $u_n(t)$ in (12), explicit expressions for the function $Z_n(t)$ are obtained in the form
for $0 \leqslant t \leqslant 1$

$$Z_n(t) = \varphi_n^{(0)} e^{-\lambda_n t} + \frac{A_n}{2\lambda_n} e^{-\lambda_n(3+t)} \left(e^{6\lambda_n} - 1 \right) + \frac{B_n}{2\lambda_n} \left[e^{-\lambda_n(2+t)} \left(e^{4\lambda_n} - 1 \right) + e^{-\lambda_n(1+t)} \left(e^{2\lambda_n} - 1 \right) \right];$$

for $1 < t \leqslant 2$

$$Z_n(t) = \varphi_n^{(0)} e^{-\lambda_n t} + \frac{A_n}{2\lambda_n} e^{-\lambda_n(3+t)} \left(e^{6\lambda_n} - 1 \right) + \frac{B_n}{2\lambda_n} e^{-\lambda_n(2+t)} \left(e^{4\lambda_n} - 1 \right);$$

for $2 < t \leqslant 3$

$$Z_n(t) = \varphi_n^{(0)} e^{-\lambda_n t} + \frac{A_n}{2\lambda_n} e^{-\lambda_n(3+t)} \left(e^{6\lambda_n} - 1 \right),$$

where

$$A_n = s_{11}^{-(n)} \eta_1^{(n)} + s_{12}^{-(n)} \eta_2^{(n)}, \qquad B_n = s_{12}^{-(n)} \eta_1^{(n)} + s_{22}^{-(n)} \eta_2^{(n)}.$$

From (6), explicit expressions for the function of the rod temperature $Z(x,t)$ are obtained in the form
for $0 \leqslant t \leqslant 1$

$$Z(x,t) = \sum_{n=1}^{\infty} \left\{ \varphi_n^{(0)} e^{-\lambda_n t} + \frac{A_n}{2\lambda_n} e^{-\lambda_n(3+t)} \left(e^{6\lambda_n} - 1 \right) + \frac{B_n}{2\lambda_n} \left[e^{-\lambda_n(2+t)} \left(e^{4\lambda_n} - 1 \right) + e^{-\lambda_n(1+t)} \left(e^{2\lambda_n} - 1 \right) \right] \right\} \sin \frac{\pi n}{l} x;$$

for $1 < t \leqslant 2$

$$Z(x,t) = \sum_{n=1}^{\infty} \left[\varphi_n^{(0)} e^{-\lambda_n t} + \frac{A_n}{2\lambda_n} e^{-\lambda_n(3+t)} \left(e^{6\lambda_n} - 1 \right) + \frac{B_n}{2\lambda_n} e^{-\lambda_n(2+t)} \left(e^{4\lambda_n} - 1 \right) \right] \sin \frac{\pi n}{l} x;$$

for $2 < t \leqslant 3$

$$Z(x,t) = \sum_{n=1}^{\infty} \left[\varphi_n^{(0)} e^{-\lambda_n t} + \frac{A_n}{2\lambda_n} e^{-\lambda_n (3+t)} \left(e^{6\lambda_n} - 1 \right) \right] \sin \frac{\pi n}{l} x.$$

Thus, using the proposed approach, for $m = 2$ explicit expressions of the control function that solve the stated problem and an explicit expression of the corresponding function of the distribution of the rod temperature are obtained.

5. Conclusion

The problem of control of the rod heating process differs from the well-known statements of the problems in that along with the classical boundary conditions, nonseparated multipoint intermediate conditions are also given. The proposed constructive method can be used in the construction of control of temperature regime for other not one-dimensional heating processes. This determines the scientific novelty and practical significance of the obtained results.

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