GEODEZJA I KARTOGRAFIA
t. L, z.2, s.75-90 (2001)

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A model of deformation of a rock mass with respect to relations of a subsidence coefficient and a time co-ordinate and spatial co-ordinates

According to analyses of results of surveying measurements, description of undefined subsidence, performed with the use of S. Knothe's model is characterised by the sufficient coherence with results of measurements since the moment when the full subsiding trough becomes visible on the terrain surface. Lower coherence with results of measurements appears in the initial period after commencement of exploitation. In order to improve the quality of description within the initial phase of subsidence, a mathematical model has been developed, which is based on relations between the coefficient of the velocity of subsidence and the time co-ordinate, and from geometric co-ordinates. Obtained solutions have been verified basing on results of surveying measurements, with the use of a special computer software.

Lists of symbols

a	– the roof control factor
$\mathrm{tg}\,\beta$	– tangent of an angle of the range of basic influences
d	– exploitation periphery
$w(t, x)$ or w_t	– momentary subsidence
$w_k(t_k, x)$ or w_k	– the final subsidence
t_k	– the time interval, when momentary subsidence assumes values, which are practically equal to asymptotic values
x	– the spatial co-ordinate, $x \in R^2$
t	– the time variable
t_p	– the starting moment of the process
C_1	– the integration permanent
c	– the coefficient of subsidence velocity
$c_c = \int c(t)dt$	
Ω	– the area of deformation
$\partial\Omega$	– the edge of the area of deformation
$\varphi(t)$	– the function of the edge of the area of deformation
wrz	– subsidence conformed by measurements (symbols on drawings)
wrz stat	– asymptotic (final) subsidence confirmed by measurements (symbols on drawings)
wo	– calculated subsidence (symbols on drawings)

wo stat – calculated asymptotic subsidence (symbols on drawings)
P_i – a set of points

INTRODUCTION

The solution concerning calculation of momentary deformations of a rock mass, which is mostly applied in Poland, is the proposal [2], which is based on a differential equation, expressing the velocity of subsidence of a point located above an area of mining exploitation:

$$\frac{dw}{dt} = c \cdot (w_k(t) - w_t) \tag{1}$$

where:

$w_k(t)$ – the final (asymptotic) value of subsidence
w_t – a momentary value of subsidence
c – a factor of subsidence velocity (time) of a permanent value.

Obviously, due to the progress of exploitation, the asymptotic value of subsidence changes in time (a continuous model). This results in considerable complications of the solution of the equation (1). Therefore, for practical purposes, it is very convenient to assume that the sufficiently small background field has been assumed. Then, it may be also assumed that exploitation has been performed for the time approaching zero; this considerably simplifies the solution, since w_k = const. may be assumed. For such assumption, the solution of the equation (1) is obtained in the following form (the initial condition $w(0) = 0$ is met):

$$w_t = w_k \cdot (1 - e^{-ct}) \tag{2}$$

For the real mining exploitation the excavated field may be divided into elementary belts, for which momentary subsidence is calculated basing on the formula (2). Then, the total value of subsidence is calculated by adding subsidence calculated for particular elementary belts, selected for the given time horizon. It is the, so-called, discrete model, which is also presented in [2].

The equation (1) was the base for further investigations concerning mathematical description of the deformation process in time [1]. In order to make the quality of description close to results of surveying measurements, many scientists have proposed introduction of the function of time variable instead of the coefficient of subsidence velocity [3], [4], [9]. Another solution used for the similar purpose is the proposal specified in [5], which consists of distinguishing of four phases of the subsidence process, for which calculations are performed in a separate way. Own investigations [7], [8] point that the coefficient of subsidence velocity c is not characterised by constant values within the time of the deformation process. This is confirmed by the above works, as well as in the, so-called, adaptation model [6], which assumes the constancy of the parameter between measurement cycles and step changes of its value at the moment of measurements.

1. *Examples of analysis of results of measurements basing on a model which assumes the constancy of the c parametr*

In order to present the consistency of S. Knothe's model with results of measurements, examples of calculations of undetermined subsidence have been performed (the, so-called, re-forecast). As an example, analysis of results of surveying measurements performed within the area of the "Czeczott" Coal Mine has been performed. Exploitation was performed in a rock mass, which was not disarranged by mining activities, at the depth of 500 m, by means of the wall system with fall of roof at the height of 2.7 m. Measurements (technical levelling) were performed for earth benchmarks, stabilised at the depth of about 1.5 m; the average interval between benchmarks equalled to 30 m. Accuracy of measurements equalled to 1 cm per 1 km. The time interval between measurements was 14 days.

Basing on determined subsidence values of parameters of S.Knothe's theory were determined:

– the coefficient of roof control $a = 0.33$ (small width of an excavated field comparing to the radius of the range of influence),

– tangent of the angle of range of the basic influence $\mathrm{tg}\beta = 2.2$.

Then, the value of the subsidence velocity coefficient c was measured by means of the gradient-free Powell method basing on subsidence resulting from measurements for particular points located over cavings. Then calculations of values of subsidence for time horizons, which corresponded to successive measurements, for the value of the parameter $c = 6.57$ [1/year] specified basing on results of measurements, were performed. Figure 1 presents diagrams of subsidence confirmed by measurements and calculated in successive cycles. The symbol *wrz* marks subsidence confirmed by measurements, and *wo* – calculated subsidence. Numbers place after the symbols marks the number of the measuring cycle. Asymptotic subsidence is additionally marked by the "stat" symbol.

It should be stressed, that the exploitation borders d have not been considered in calculations. It was assumed that although the consistency of calculations with results of measurements is improved after introduction of this additional parameter, but it is not formally justified, particularly in the case of calculations of momentary subsidence. Figure 2 presents values of proportional errors of subsidence obtained in successive cycles.

Basing on analysis of calculations (Fig. 1 and 2) it may be noticed that for the first two measuring cycles the consistency between results of calculations with results of measurements is very low. Deeper consideration of this issue results in the conclusion that kinematics of the process should be different in the initial phase of the deformation process (the start-up period of the wall) and it should change in the period of causing the complete fall of roof. Considering this it may be concluded that, in order to increase the consistency between calculations and results of measurements, the value of the parameter c should vary following the progress of works, i.e. it should vary in time. This requires appropriate assumption in the initial equation (1).

2. *Assumption concerning the c variability in time*

Current works aiming at making the description of momentary values of post-exploitation subsidence of the rock mass closer to results of measurements, consisted mainly of introduction

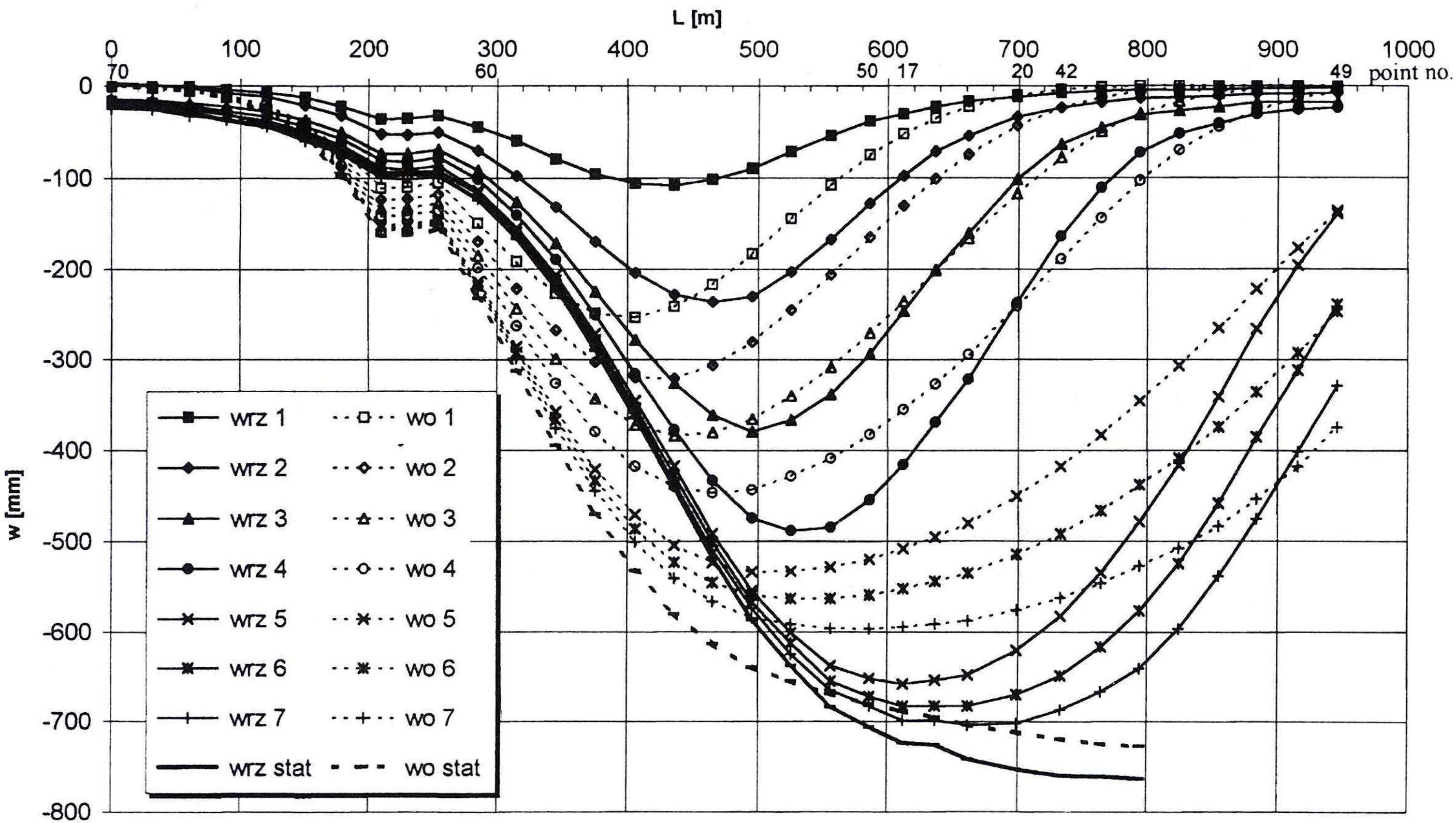
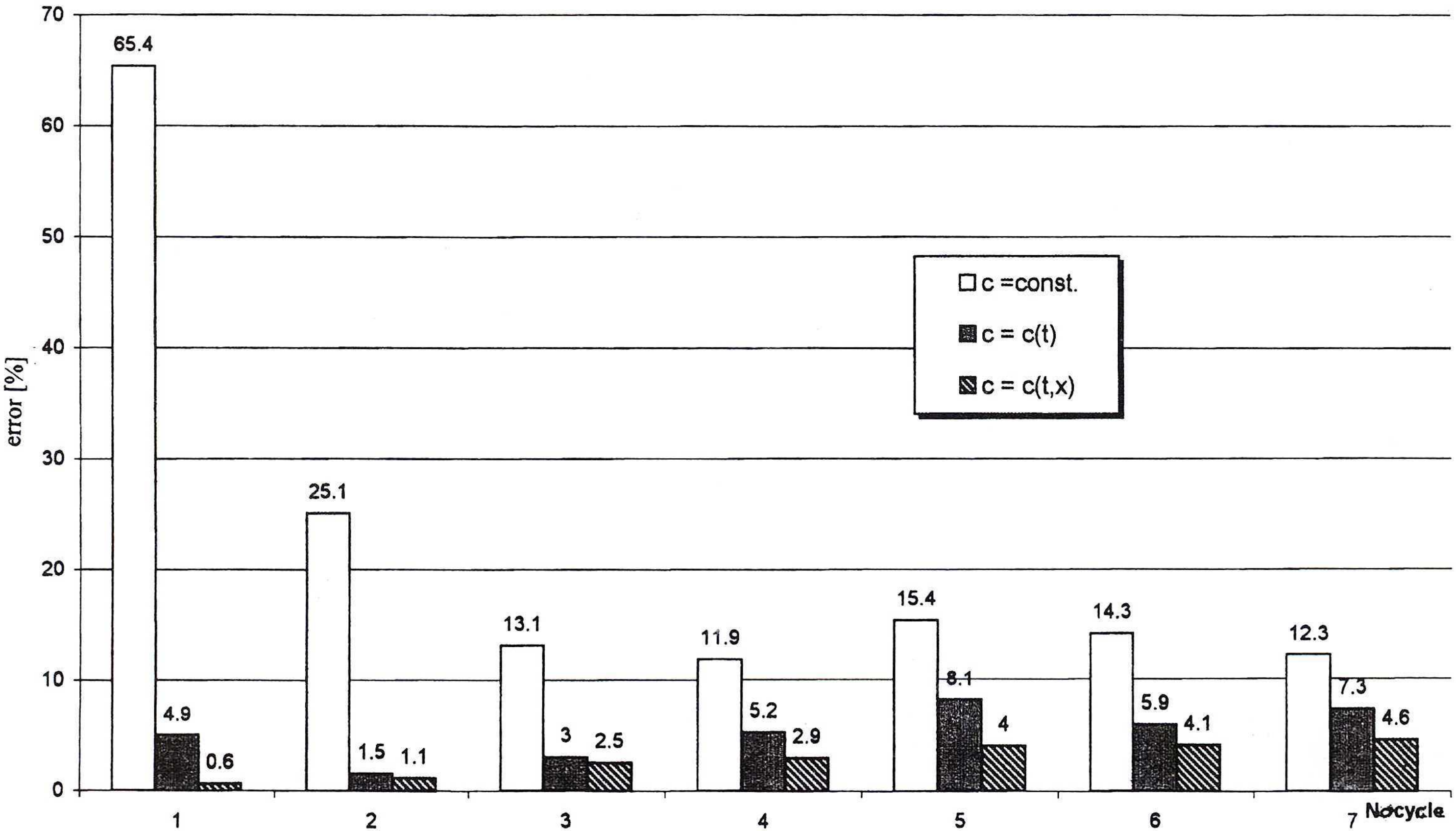


Fig. 1. Comparison of measured and calculated subsidence with the constant value of the time factor *c*



Fg. 2. Comparison of obtained values of proportional errors of subsidence

of some functions instead of the constant coefficient c. The form of such functions in the equation (1) or in its solution (2) was assumed in advance, with the constraints concerning the initial and the final conditions. The considerable progress in digital processing, which is taking place recently, was the reason of a slightly different approach to the problem – the variability of the parameter c in time may be assumed at the very beginning in the equation (1) and then identification of the parameter basing on results of measurements may be performed. At the final stage of investigations a function may be sought which would be the best approximation of the obtained distribution of the parameter, what will allow to apply the model in practice.

Modification of the equation (1), which is based on the assumption that $c = c(t)$ leads to the following relations for the discrete model ($w_k(t_k, x)$ = const.):

$$w(t,x) = w_k(t_k,x) + C_1 e^{-\int c(t)dt} \quad (3)$$

where:

$w(t,x)$ – momentary subsidence,
$w_k(t_k,x)$ – final subsidence,
t_k – time, for which momentary subsidence assumes values, which are practically equal to asymptotic values,
x – the spatial co-ordinate, $x \in R^2$,
C_1 – the constant of integration, which value was assumed as $C_1 = 1$ for simplification of further considerations.

3. *Verification of the proposed solution*

For the needs of verification of the obtained formula (3) the discrete algorithm of calculation of momentary subsidence has been used.

As a consequence of application in calculations a formula, which considers the variability of the coefficient c in time, was the achievement of an independent value of this parameter only for the first cycle of observations, performed for the measuring line. Values of the parameter c, determined basing on subsidence confirmed by measurements in successive cycles depended on values specified in all preceding cycles, according to the proposed solution. This required numerical calculation of the integral, which appears in the formula (3); therefore its values was calculated by means of Simpson formula. Processes of subsidence, confirmed by measurements and calculated, are presented in Fig.3 (symbols are the same as in Fig.1). Figure 4 presents values of the parameter $c = \int c(t)dt$ obtained as a result of optimisation and proportional errors of subsidence for the same results of optimisation and proportional errors of subsidence for the same results of measurement as in the previous case, presented in Fig.2.

Obtained values of the parameter c_c were approximated by the following function:

$$\int c dt = 6.35 \, \mathrm{tg}\, h(\frac{t}{77.837} - 1.18), [1/\text{year}] \quad (4)$$

where: t – time [days],
for the value of the coefficient of multiple correlation $R = 0.992$. Figure 4 also presents values of c_c calculated basing on the equation (4).

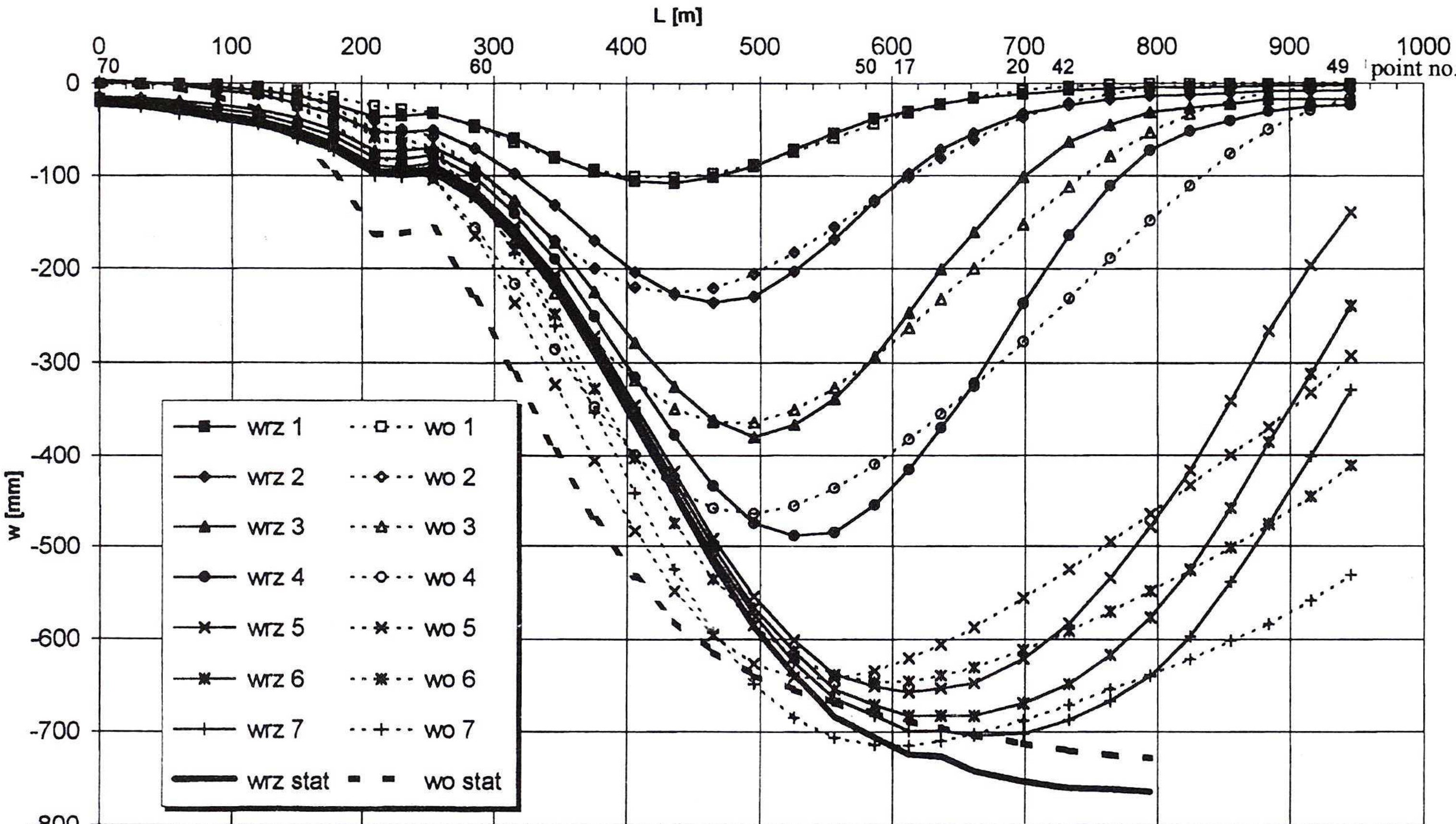


Fig. 3. Comparison of measured and calculated subsidence, with the assumption $c = c(t)$

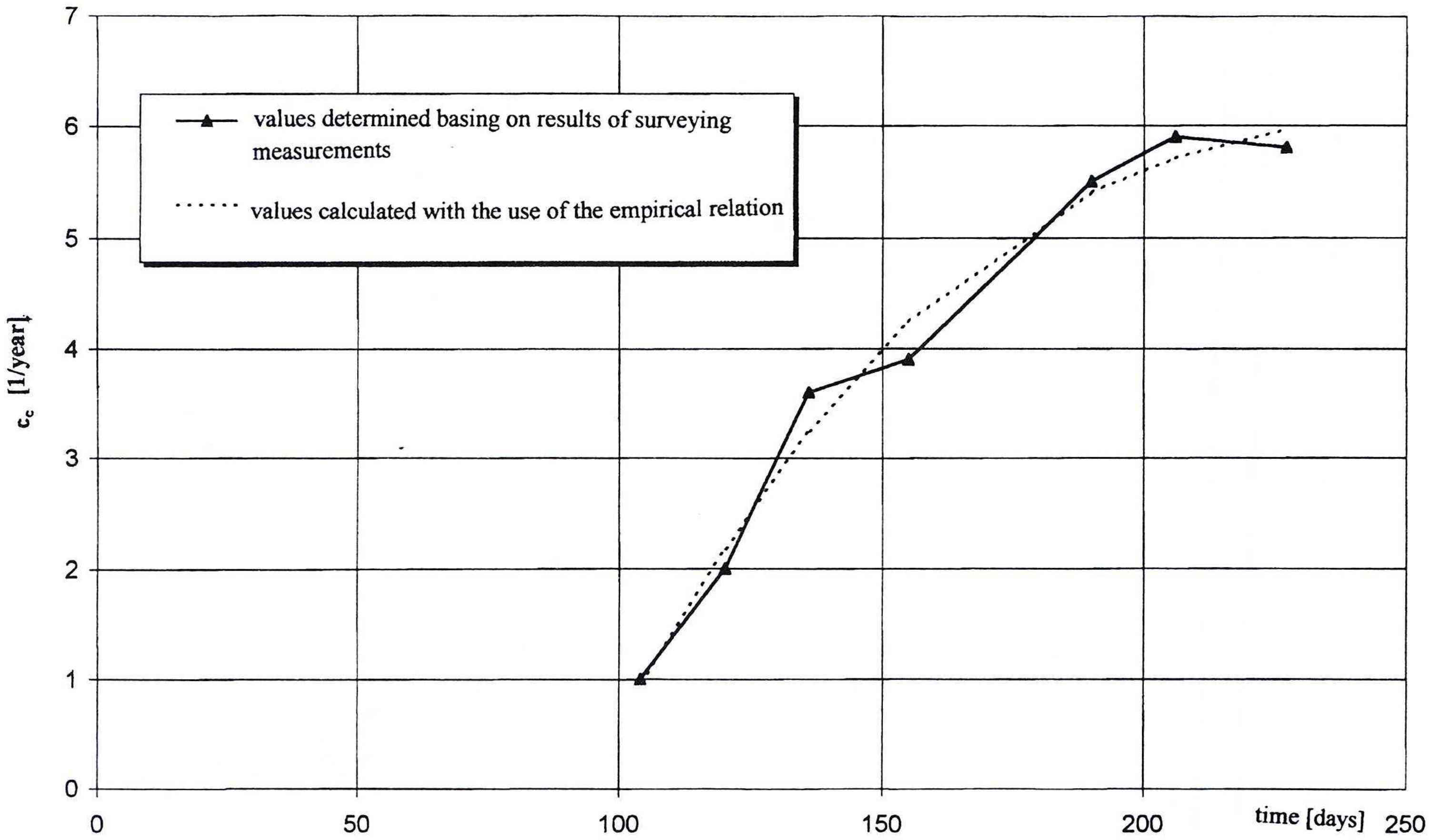


Fig. 4. Comparison of c_c values determined basing on measurements and calculated according to the proposed empirical formula (4)

4. *Relation between the coefficient of and spatial co-ordinates and time co-ordinates*

It results from performed analysis that the value of the coefficient c depends on the time co-ordinate as well as on geometric co-ordinates of the discussed point.

Considering this, the standard equation used for description of undetermined subsidence by S.Knothe [2] may be modified in the following way:

$$\frac{dw(t,x)}{dt} = c[w_k(t_k,x) - w(t,x)] \tag{5}$$

where:

c is a function $(t;x)$; $c = c(t;x)$,

t_k – time, for which momentary subsidence assumes values, which are practically equal to asymptotic values,

$w_k(t_k, x)$ – the final value of subsidence. Further consideration were performed for $w_k(t_k,x)$ = const. (the discrete model).

x – spatial co-ordinates, $x \in R^2$.

The equation (5) may be also written as the partial differential equation of the first order:

$$\frac{\partial w}{\partial t}(t,x) = c(t,x)[w_k(t_k,x) - w(t,x)] \tag{6}$$

The equation of characteristics has the form:

$$c(t,x)dt = \frac{dw}{w_k(t_k,x) - w} \tag{7}$$

By integration of the equation (7), the following formula is obtained:

$$\Gamma\lfloor \ln(w(t,x) - w_k(t_k,x)) + \int c(t,x)dt \rfloor = 0 \tag{8}$$

where: Γ – a function, which meets some initial conditions.

Introduction of the function Γ is connected with the necessity to specify the borders of the area of deformations; without such specification the mathematical model would *e* incomplete. This issue is reviewed in more details in further parts of the elaboration. After solution of the equation (8) the following formula is obtained:

$$w(t,x) = w_k(t_k,x) + C_1 e^{-\int c(t,x)dt} \tag{9}$$

for C_1 = const. >0.

As a result of analysis of the equation (9) the following comments may be noticed: We expect that for the time $t \rightarrow t_k$ the condition $w(t,x) \rightarrow w_k(t_k,x)$ will occur. Therefore the following relation must occur:

$$\int c(t,x)dt \Rightarrow \infty \tag{10}$$

Specification of the initial condition leads to the following relation, for $t = 0$:

$$w(0,x) = w_k(t_k, x) + C_1 e^{-\int c(t,x)dt/t=0} \tag{11}$$

Assuming the starting point of the process as the initial condition (i.e. lack of depression), i.e. $w(0,x) = 0$, the following relation is obtained:

$$w_k(t_k, x) = -C_1 e^{-\int c(t,x)dt/t=0} \tag{12}$$

Basing on the equation (12) $c(0,x)$ should be determined with the use of the relation:

$$\ln\left|w_k(t_k, x)\right| - \ln C_1 = -\int c(t,x)dt/_{t=0} \tag{13}$$

It should be explicitly stressed, that the relation (13) is not sufficiently physically justified, since the issue of the border of the area of deformation (the issue related to the border of the trough in the plane $x = \{x^1, x^2\}$), for which the following relations may be written with the use of Fig.5, has not been considered:

$$w(t,x) = 0 \text{ for } x = \Phi(t)$$
$$w(0,x) = 0 \text{ for } t = 0 \tag{14}$$

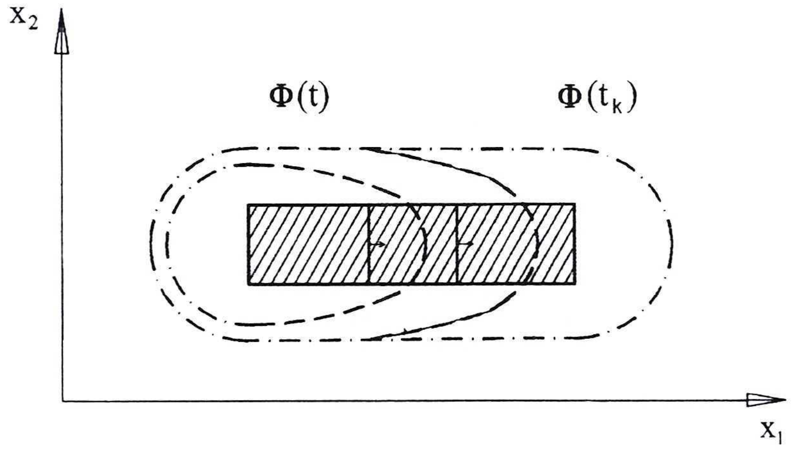


Fig. 5. Area of the deformation edge in the plane $x = \{x_1, x_2\}$ for the moments t and t_k

The function $\Phi(t)$ describes the motion of the border of the area of deformation – Fig.6. Therefore the relation (14) is useful for determination of the function $c(0,x)$, which would be carried by the point $x = 0$ at the moment $t = 0$. Since we cannot talk about a function in a point, so this leads to inconsistency with the formula (12). So the analysis of the initial issue would lead to a paradox.

In this context analysis of variability of the parameter $c=c(t,x)$ should lead to a conclusion that the starting moment of the process $t_p = t > 0$. For this assumption the carrier of the initial

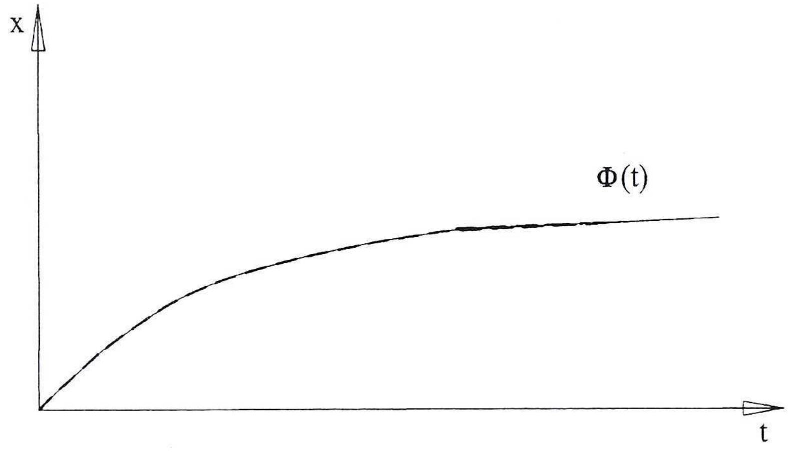


Fig. 6. Function of the edge of the deformation area

condition is not the point $x = 0$ but a certain spatial area $W\,x\,(t_p)$, where the initial condition is reasonable for $t = t_p$, so:

$$\ln\left|w_k(t_k,x)\right| - \ln C_1 = -\int c(t_p,x)dt \tag{15}$$

So, the solution (9) must meet the initial condition (15) within a certain area $\Omega \cap \{t_p\} \neq \emptyset$ and for the areas which is assigned in this way (for $t > t_p$) identification of $c(t,x)$ should be performed basing on results of surveying measurements. However, the problem arises which is related to consideration of the edge of the trough, which delimits the area where the equation (9) occurs. Let us consider the problem from the formal point of view, in order to obtain the description of the fact that some points are not involved in the process of subsidence of the medium, before it crosses the edge of the trough. Analysing the process of decrease of points of the medium for times $t > t_p$ we should consider that the area of the trough belongs to the certain set, which is delimited by the edge $\partial\Omega$. The edge may be determined by the function $\Phi(t)$. Therefore, for $t > t_p$, some points are included in the area Ω, and other points are not included within this area. So, in the vicinities of the edge $\partial\Omega$, we have the set of points $P_i \in \Omega \cap \{t\}$, the edge $P_i \notin \Omega \cap \{t_p\}$ (the logical product of the area Ω and the moment t_p). Some points P_i at the moment $t > t_p$ will be included in the set $\Omega \cap \{t\}$, for which $|w(t,x)| > 0$. So, at the moment t_p, $P_i \in \text{supp}\,(w \cap t_p\,\{t\})$. Therefore, determination of the starting parameter $c(t,x)$ for points P_i is not possible (besides, there is no need for such determination, since points are located outside the area of deformation). So (due to obvious reasons), the process of identification will be related to the limited area $\Omega = \text{supp}(w \cap \{t_i\})$ for successive t_i, for which successive points P_i will be included within the area of deformation, so they will become elements of the set P_i.

5. *Verification of the presented description basing on results of surveying measurements*

Verification of obtained solution has been performed for the same results of measurements, which were used previously. Values of the parameter c have been determined in accordance to the formula described by the equation (9), assuming $C_1 = 1$. In order to make the parameter c as a variable of geometric co-ordinates, the observation line has been divided into three sections for each cycle:

1) $c(x_1,t)$ when $x \notin P$,
2) $c(x_2,t)$ when x is located in the permanent edge of the mining field,
3) $c(x_3,t)$ when $\in P$,

where: P – the mining field.

Considering, that n = 7 measurement cycles were analysed, the matrix c_{ki} was obtained for a line, where: k= 1,2,3 and i=1,2, ... , 7.

For a measurement line, divided onto three sections, according to the above method, identification of the parameter value was performed basing on subsidence, stated for each measurement cycle, by means of the same computer software, which was previously used. Results of calculations are presented in Fig. 7, with the use of the same symbols, which are used in Fig. 1 and 3. Values of proportional errors of calculated subsidence are presented in Fig. 2.

CONCLUSIONS

This paper presents an original way of analysing the issue related to variability of the coefficient c, which occurs in the equation proposed by S. Knothe: first, as the variable of the time co-ordinate and then of geometric co-ordinates. Performed analyses enable to formulate the following conclusions:

1. In order to achieve the high consistency of calculations with results of measurements, the parameter, which describes kinematics of the process of deformation, should assume values changing in time. This is confirmed in works of other authors: [3], [4], [9]. Assumptions accepted in the presented work are also conformed by values of proportional errors of calculated subsidence, presented in Fig.2.

2. The above presented approach to the investigated issue, which consists of assumption of variability of $c = c(t)$ in the initial equation, has the general form. Due to assumption of a discrete model for the needs of description of the process of subsidence in time, the applied method of analysis, consisting of numerical calculation of an integral, which occurs in the equation (3), allows to obtain the most general form of variability of the parameter. The achieved variability may be the subject of furthèr statistical investigations. With the assumption of the discrete model, any form of relations may not be directly visible. One of possible functions, which approximate distribution of the parameter $\int c(t)dt$ is the function tg$h$ – formula (4).

3. Assumed division of a line onto sections, in the course of verification of a model with the parameter c ($c = c(t,x)$, as a function of time and geometric co-ordinates, resulted from conditions related to earlier investigations. It seems obvious, that arbitrary – even 100% - quality of projection may be obtained, with identification of the c value in every point. It is obvious, that assuming division of a line into three sections, a non-smooth curve of subsidence has been obtained. However, the aim of presented analyses was to prove the correctness of an idea concerning the dependence of the parameter from geometric co-ordinates and the time co-ordinate, what has been achieved.

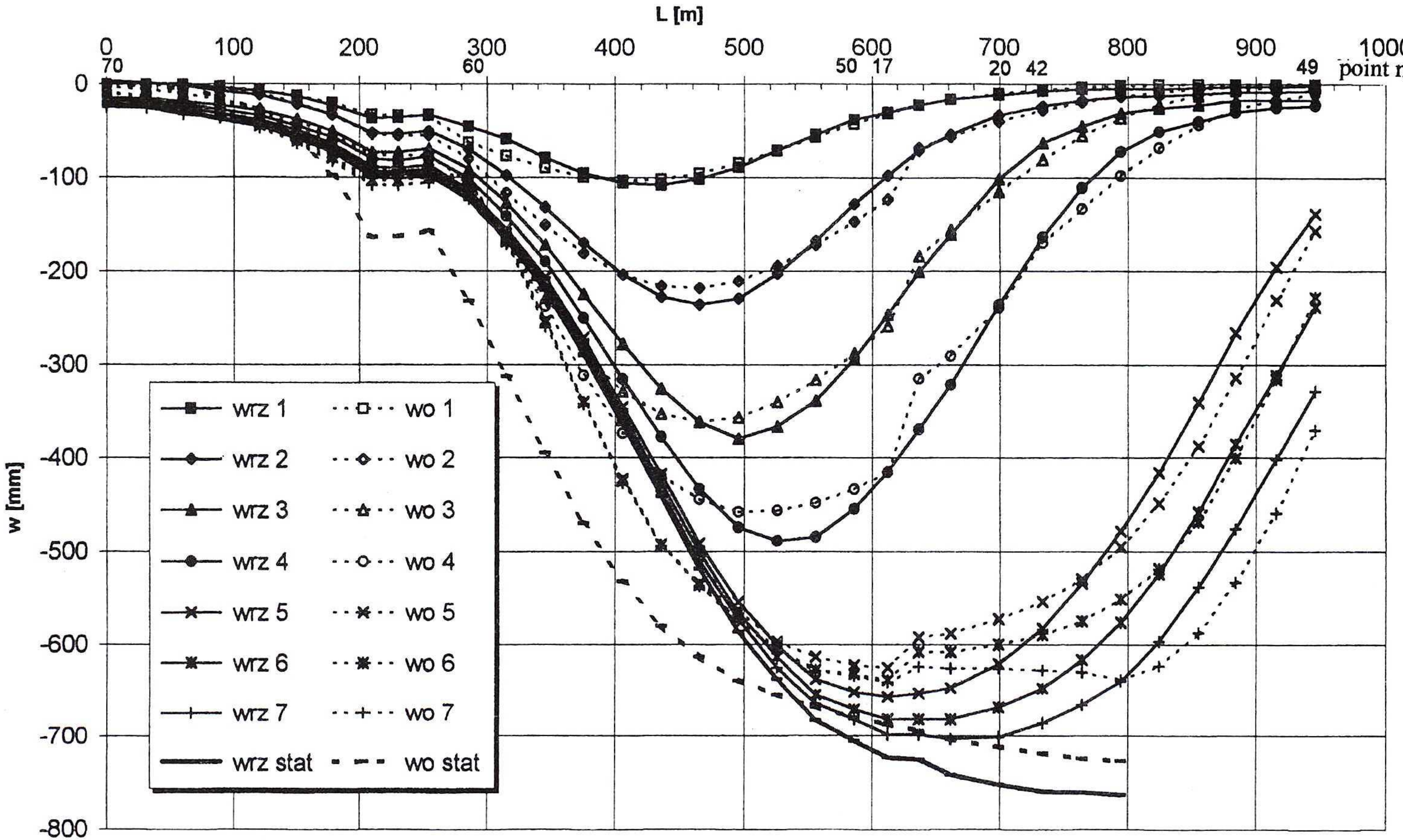


Fig. 7. Comparison of measured and calculated subsidence with the assumption c = $c(t, x)$

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Received May 15, 2001
Accepted August 2, 2001

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Model deformacji górotworu uwzględniający zależność współczynnika prędkości osiadania od współrzędnej czasowej i współrzędnych przestrzennych

S t r e s z c z e n i e

Analizy wyników pomiarów geodezyjnych wskazują, że opis osiadań nieustalonych przy zastosowaniu modelu S. Knothego charakteryzuje się dostateczną zgodnością z wynikami pomiarów od momentu ujawnienia się na powierzchni terenu pełnej niecki osiadania. Mniejsza zgodność z wynikami pomiarów występuje natomiast w początkowym okresie po rozpoczęciu eksploatacji. W celu zwiększenia jakości opisu w początkowej fazie procesu osiadania, opracowano model matematyczny, którego istotą jest uzmiennienie współczynnika prędkości osiadania od współrzędnej czasowej, a następnie również od współrzędnych geometrycznych. Otrzymane rozwiązanie poddano weryfikacji w oparciu o wyniki pomiarów geodezyjnych przy zastosowaniu specjalnie opracowanego oprogramowania komputerowego.

Пётр Стшалковски

Модель деформаций горных пород с учётом зависимости коэффициента скорости оседания от координаты времени и пространственных координат

Резюме

Анализ результатов геодезических измерений показывает, что описание оседаний, неопределённых с примиением модели С. Кноте, отличается удовлетворительным соответствием результатам измерений с момента обнаружения осадков на поверхности местности полной мульды. Более низкое совпедение с результатами измиерений происходит в первоначальном периоде, после начала эксплуатации. С целью повышения качества описания процесса оседания в начальной фазе, разработана математическая модель, суть которой это определение зависимости коэффициента скорости оседания от координаты времени, а затем от геометрических координат. Полученное решение проверено на основе результатов геодезических измерений с применением специально разработанной вычислительной программы.