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# Explicit modeling of multi-period setup times in proportional lot-sizing and scheduling problem with variable capacity

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Small bucket models with many short fictitious micro-periods ensure high-quality schedules in multi-level systems, i.e., with multiple stages or dependent demand. In such models, setup times longer than a single period are, however, more likely. This paper presents new mixed-integer programming models for the *proportional lot-sizing and scheduling problem* (PLSP) with setup operations overlapping multiple periods with variable capacity.

A new model is proposed that explicitly determines periods overlapped by each setup operation and the time spent on setup execution during each period. The model assumes that most periods have the same length; however, a few of them are shorter, and the time interval determined by two consecutive shorter periods is always longer than a single setup operation. The computational experiments show that the new model requires a significantly smaller computation effort than known models.

**Key words:** production, lot-sizing and scheduling, mixed-integer programming

## 1. Introduction

This paper addresses a class of *mixed-integer programming* (MIP) models for the lot-sizing and scheduling problems with discrete time scale. Multiple products have to be produced to satisfy deterministic, time-varying demand. There is one machine with limited capacity, i.e., during each period, the total workload assigned to the machine cannot exceed its length. When the machine is changed over from the production of one product to another, a setup operation must be executed. Each setup operation causes additional costs and takes some



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time that may be longer than a period, i.e., may overlap several periods. Neither the setup costs nor times are sequence-dependent. The objective is to minimize the sum of the setup and inventory holding costs.

Simultaneous lot-sizing and scheduling integrates two decision levels of the classic production planning system. First, capacitated lot-sizing determines sizes and completion dates of production orders (lots), i.e., assigns orders to periods (weeks, days, or shifts), to minimize the set-up and inventory (work-in-process) holding costs [7, 19]. Second, machine scheduling determines detailed production schedules for fixed lots during a single period, i.e., assignment to machines and sequences of orders, to ensure that all orders assigned to a period are completed during this period. Such integration is not possible when system requirements make scheduling hard to solve even as a stand-alone problem [20]. Then schedules are determined separately with set-up times aggregated with processing times [20] or modeled explicitly, to determine shorter product cycle times [4, 18], or because set-up times depend on the sequence of products [5].

Researchers divide lot-sizing and scheduling models into three classes, models with large or small time-buckets and hybrid models. Large bucket models allow many machine setups and lots within the same period. Small bucket models allow one machine setup and two lots per period at most. To ensure high-quality solutions of small bucket models, real periods (macro-periods) are usually split into several short fictitious micro-periods, e.g., [9, 16, 21], which makes setup times longer than a single period more likely. Hybrid models use both macro- and micro-periods. A more detailed discussion of large and small buckets pros and cons provide [7, 11, 14, 21].

This paper considers the *proportional lot-sizing and scheduling problem* (PLSP) [8, 10]. The PLSP is the most flexible small bucket model, as it allows for the processing of two products during a single period (one before and another after the setup operation).

The amount of research on small bucket models with setup times longer than period length is limited. There are a few papers on models with constant period capacity. Blocher et al. [3] and Cattrysse et al. [6] extended *the discrete lot-sizing and scheduling problem* [9], which assumes that processing times of lots and setups are integer multiples of a period. Kaczmarczyk [12] proposed three model formulations for the PLSP that explicitly determine the end of each setup operation for the known begin time. Kaczmarczyk [15] also proposed another PLSP model that explicitly determines the beginning of each setup operation for the known completion time. Haase [10] and Suerie [21] proposed models for the PLSP with variable capacity, which determine a setup schedule implicitly: an additional variable accumulates the time assigned to the setup operation during consecutive periods until it becomes equal to the setup time.

This paper proposes an adaptation of the explicit model for uniform periods [15] to the case with variable capacity of periods. It is based on the simple observation, most periods have the same capacity in the calendar of a short-term production schedule, and only a few are shorter. For example, there are three shifts on ordinary working days in certain companies (i.e., the working day is 24 hours long), but there are only one or two shifts during the day on Saturdays or days before certain holidays. Some other companies produce during two shifts per day but schedule some overtime hours at the end of certain days. If the shift is the basic period, then overtime constitutes an additional shorter period. Also, scheduled maintenance operations decrease the capacity of some distant periods.

Section 2 presents the model for uniform periods [15], including its parameters and variables. In this model, four different cases of schedules of a single setup operation are possible. Subsection 2.2 presents the approach of combining similar constraints into one more general one, enabling a more concise description of the model. In Section 3, the same approach is applied to formulate the new model with variable capacity and general constraints for twenty different cases of setup schedule. Subsection 3.5 presents the heuristics for the explicit models. Section 4 describes the computational experiments, data sets, tools, and results. Section 5 gives a summary. The appendix contains illustrative examples for all cases of the setup execution considered in the new model.

## 2. Models for uniform periods

The list below presents the notation: firstly, the basic parameters that define the PLSP; next, the derivative parameters necessary for the explicit description of the changeovers in the model; and finally, the continuous and binary variables.

*Parameters:*

$\mathcal{T} = \{1, \ldots, T_{\max}\}$ – set of periods;
$\mathcal{N} = \{1, \ldots, N_{\max}\}$ – set of products;
$d_{jt}$ – demand of product $j$ during period $t$;
$p_j$ – processing time of product $j$;
$I_{j0}$ – initial inventory of product $j$;
$h_{jt}$ – unit holding cost of product $j$ during period $t$;
$sc_j$ – setup cost of product $j$;
$st_j$ – setup time of product $j$;
$C_t$ – capacity (length) of single period $t$;

*Derivative parameters :*

$\overline{C}$ – ordinary period capacity; i.e., capacity upper limit ($C_t \leqslant \overline{C}$);
$Q_j = \lfloor st_j / \overline{C} \rfloor$ is (integer) *quotient of setup time* of product $j$ and capacity;

$R_j$ $= st_j - Q_j\overline{C}$ is the *remainder of setup time* of product $j$ divided by capacity; however, if setup time is integer multiply of capacity ($Q_j \geqslant 1$ and $st_j = Q_j\overline{C}$), then $Q_j = \lfloor st_j/\overline{C}\rfloor - 1$ and $R_j = \overline{C}$;

$\mathcal{P}_j$ $= \{-Q_j, \ldots, 0\} \cup \mathcal{T}$ – set of periods extended by past periods necessary to describe unfinished setup of product $j$ started in the past.

*Continuous variables:*

$x_{jt}$ – production volume of product $j$ during period $t$;
$I_{jt}$ – inventory of product $j$ at the end of period $t$;
$b_{jt}$ – time during period $t$ reserved for product $j$ *before* changeover;
$a_{jt}$ – time during period $t$ reserved for product $j$ *after* changeover;
$s_{jt}$ – time used to set up machine for product $j$ during period $t$ (but only when $t$ is last period overlapped by setup operation);
$s'_{jt}$ – time used to set up machine during any period $t$ overlapped by setup operation;

*Binary variables:*

$y_{jt}$ $= 1$ if machine is *set up* to process product $j$ at the end of period $t$ (i.e., machine is reserved [ready] for that product); 0 otherwise;
$z_{jt}$ $= 1$ if machine *starts up* to produce product $j$ during period $t$ (i.e., setup operation for this product is finished); 0 otherwise;
$w_{jt}$ $= 1$ if processing of product $j$ during period $t$ is *switched off*; 0 otherwise;
$v_{jt}$ $= 1$ if $z_{jt} = 1$ and time used to process setup operation of product $j$ during period $t$ is longer than remainder of setup time ($s_{jt} \geqslant R_j$); 0 otherwise;
$u_{jt}$ $= 1$ if setup operation of product $j$ during period $t$ is processed but not finished (i.e., it must be *continued* during the next period); 0 otherwise.

In the models described in this paper, all variables with period indices $t \leqslant 0$ or $t \geqslant T_{\max} + 1$ (i.e., from the past or future planning horizon) are set to be equal to zero. There are only two exceptions; firstly, inventory variables $I_{j0}$ may represent non-zero initial inventories; and secondly, variables with $t \leqslant 0$ describe the initial state of the machine, i.e., determine which product may be produced at the beginning of the first period or describe the state of a setup operation started i n the past. Variable $s'_{jt}$ is used only to describe schedules in the illustrative example and not in the considered models.

If the products are indivisible, demand, production, and inventory values should be integer. However, this article assumes that all these parameters and variables are continuous because dynamic lot-sizing models are used to plan the serial production of medium-volume products. The values of the period demand are then measured in hundreds or thousands of items. So, rounding up or down the production changes the daily or weekly workload by only a few minutes. If the demand per period is small, e.g., less than ten, the production volume must be integer, but the set of feasible solutions is also small and can be quickly examined by the branch and bound algorithm.

The basis for further considerations constitutes an explicit formulation of the PLSP model [17] (denoted here as PLSP/E), which uses variables $b_{jt}$ and $a_{jt}$ to describe the division of capacity among consecutive lots explicitly (i.e., the time before and after beginning each setup operation).

The example in Table 1 illustrates the basic idea of the model proposed by Kaczmarczyk [15]. The setup time for product $j$ is $st_j = 350$, and the capacity is $\overline{C} = 100$; i.e., the (integer) quotient of setup time and capacity is $Q_j = 3$, and the remainder of that division is $R_j = 50$.

Variable $z_{jt}$ points to the end of the setup and separates time intervals during which production is prevented and enabled. Variable $s_{jl}$ describes the capacity used to set up the machine during the last period $l$ overlapped by the setup operation. If $s_{jl}$ is longer or equal to the remainder of setup time $R_j$ (see Case a), then the setup operation overlaps the $Q + 1$ periods; otherwise, it overlaps the $Q + 2$ periods (see Case b). To complete the schedule of a setup operation, one must determine the first overlapped period $f$ and the capacity $a_{jf}$ used to process it during this period. During time interval $[f, l-1]$, production is prevented by binary variable $u_{jt}$, and during period $l$, the capacity available for production is decreased by $s_{jl}$.

Table 1: Scenarios of setup operation finished during period $l$ in the model with constant capacity

a) $s_{jl} \geqslant R_j$; i.e., $v_{jl} = 1$

| $t$ | $l-4$ | $l-3$ | $l-2$ | $l-1$ | $l$ | $l+1$ |
|---|---|---|---|---|---|---|
| $a_{jt}$ | | 70 | | | | |
| $s'_{jt}$ | | 70 | 100 | 100 | 80 | |
| $x_{jt}$ | | | | | 20 | 100 |
| $z_{jt}$ | | | | | 1 | |
| $v_{jt}$ | | | | | 1 | |
| $u_{jt}$ | | 1 | 1 | 1 | | |
| $y_{jt}$ | | | | | 1 | 1 |

b) for $s_{jl} < R_j$; i.e., $v_{jl} = 0$

| $t$ | $l-4$ | $l-3$ | $l-2$ | $l-1$ | $l$ | $l+1$ |
|---|---|---|---|---|---|---|
| $a_{jt}$ | 30 | | | | | |
| $s'_{jt}$ | 30 | 100 | 100 | 100 | 20 | |
| $x_{jt}$ | | | | | 80 | 100 |
| $z_{jt}$ | | | | | 1 | |
| $v_{jt}$ | | | | | 0 | |
| $u_{jt}$ | 1 | 1 | 1 | 1 | | |
| $y_{jt}$ | | | | | 1 | 1 |

The explicit formulation of the PLSP/E model with setup operations overlapping multiple periods proposed by Kaczmarczyk [15] (denoted as PLSP/E-MS/P) is presented below:

$$\min \quad \sum_{t \in \mathcal{T}} \sum_{j \in N} (sc_j z_{jt} + h_{jt} I_{jt}), \tag{1.1}$$

$$I_{j,t-1} + x_{jt} - d_{jt} = I_{jt}, \qquad j \in \mathcal{N},\ t \in \mathcal{T}, \tag{1.2}$$
$$C_t(y_{jt} - v_{jt}) + a_{jt} + b_{jt} \geqslant p_j x_{jt} + s_{jt}, \qquad j \in \mathcal{N}: Q_j = 0,\ t \in \mathcal{T}, \tag{1.3}$$
$$C_t y_{jt} + b_{jt} \geqslant p_j x_{jt} + s_{jt}, \qquad j \in \mathcal{N}: Q_j \geqslant 1,\ t \in \mathcal{T}, \tag{1.4}$$
$$C_t(1 - z_{j,t+1} + v_{j,t+1}) \geqslant p_j x_{jt} + s_{jt}, \qquad j \in \mathcal{N}: Q_j = 0,\ t \in \mathcal{T}, \tag{1.5}$$
$$\sum_{j \in \mathcal{N}} (b_{jt} + a_{jt}) = \sum_{j \in \mathcal{N}} C_t w_{jt}, \qquad t \in \mathcal{T}, \tag{1.6}$$
$$C_t w_{jt} \geqslant b_{jt}, \qquad j \in \mathcal{N},\ t \in \mathcal{T}, \tag{1.7}$$
$$C_t\left(v_{j,t+Q_j} + z_{j,t+Q_j+1} - v_{j,t+Q_j+1}\right) \geqslant a_{jt}, \qquad j \in \mathcal{N},\ t \in \mathcal{P}_j, \tag{1.8}$$
$$\sum_{j \in \mathcal{N}} (y_{jt} + u_{jt}) = 1, \qquad t \in \mathcal{T}, \tag{1.9}$$
$$z_{jt} - w_{jt} = y_{jt} - y_{j,t-1}, \qquad j \in \mathcal{N},\ t \in \mathcal{T}, \tag{1.10}$$
$$R_j v_{jt} \leqslant s_{jt}, \qquad j \in \mathcal{N},\ t \in \mathcal{T}, \tag{1.11}$$
$$s_{jt} \leqslant C_t z_{jt}, \qquad j \in \mathcal{N},\ t \in \mathcal{T}, \tag{1.12}$$
$$z_{j,t+1} - v_{j,t+1} \leqslant u_{jt}, \qquad j \in \mathcal{N}: Q_j = 0,\ t \in \mathcal{P}_j, \tag{1.13}$$
$$R_j(z_{jt} + v_{jt} - 1) \leqslant a_{jt}, \qquad j \in \mathcal{N}: Q_j = 0,\ t \in \mathcal{P}_j, \tag{1.14}$$
$$R_j(z_{j,t+1} - v_{j,t+1}) \leqslant a_{jt} + s_{j,t+1}, \qquad j \in \mathcal{N}: Q_j = 0,\ t \in \mathcal{P}_j, \tag{1.15}$$
$$\sum_{r=1}^{Q_j+1} z_{j,t+r} - v_{j,t+Q_j+1} \leqslant u_{jt}, \qquad j \in \mathcal{N}: Q_j \geqslant 1, t \in \mathcal{P}_j \tag{1.16}$$
$$(R_j + \overline{C})(z_{j,t+Q_j} + v_{j,t+Q_j} - 1) \leqslant a_{jt} + s_{j,t+Q_j}, \qquad j \in \mathcal{N}: Q_j \geqslant 1,\ t \in \mathcal{P}_j, \tag{1.17}$$
$$R_j(z_{j,t+Q_j+1} - v_{j,t+Q_j+1}) \leqslant a_{jt} + s_{j,t+Q_j+1}, \qquad j \in \mathcal{N}: Q_j \geqslant 1,\ t \in \mathcal{P}_j, \tag{1.18}$$
$$I_{jt} \geqslant 0, \qquad j \in \mathcal{N},\ t \in \mathcal{T}, \tag{1.19}$$
$$x_{jt}, a_{jt}, b_{jt}, s_{jt} \in [0, \overline{C}], \qquad j \in \mathcal{N},\ t \in \mathcal{T}, \tag{1.20}$$
$$w_{jt} \in [0, 1], \qquad j \in \mathcal{N},\ t \in \mathcal{T}, \tag{1.21}$$
$$z_{jt}, v_{jt}, y_{jt}, u_{jt} \in \{0, 1\}, \qquad j \in \mathcal{N},\ t \in \mathcal{T}. \tag{1.22}$$

The above model is correct only for a constant period capacity over the planning horizon ($C_t = \overline{C}$ for all $t$). Nevertheless, the capacity is often denoted as $C_t$ and not $\overline{C}$ for compatibility with the considerations in the following sections.

The model objective is to minimize the total setup and inventory holding costs (1.1). Equalities (1.2) ensure the balance of production, inventory, and demand.

Constraints (1.3)–(1.5) limit the production volume depending on the machine state. Constraints (1.6) ensure that the whole period capacity is distributed among the products. Constraints (1.7) and (1.8) allow non-zero values of $b_{jt}$ and $a_{jt}$ only if the machine switches off ($w_{jt} = 1$) or starts up ($v_{jt} = 1$) the processing of product $j$, respectively.

Constraint (1.9) ensures the unique state of the machine at the end of each period. Constraint (1.10) forces start-up variables $z_{jt}$ and switch-off variables $w_{jt}$ to take a value of 1 after each change of the machine state $y_{jt}$.

Constraints (1.11)–(1.18) model the execution of the setup operations. During the last period overlapped by setup operation ($z_{jt} = 1$), constraint (1.11) ensures that the finish variable $v_{jt}$ takes a value of 1 only if $s_{jt} \geqslant R_j$. In such a case, the setup operation overlaps only $Q_j + 1$ periods (Example a) in Table 1). Constraint (1.12) ensures that there is no setup time when $z_{jt} = 0$. Otherwise, it could be "profitable" to set $v_{jt} = 1$, as it would ensure some (infeasible) capacity for production according to (1.8).

Constraints (1.13) and (1.16) force the variables $u_{jt}$ to take value 1 during the periods in which the setup operation is continued. Constraints (1.14) and (1.17) reserve the capacity $a_{jt}$ for the setup in the first overlapped period when the setup operation overlaps $Q_j + 1$ periods, and constraints (1.15) and (1.18) when it overlaps $Q_j + 2$ periods.

### 2.1. Valid inequalities

The valid inequalities presented below (denoted as ILB) determine the minimal inventory necessary at the end of period $t-1$ if, during some time interval $[t, t+\Delta]$, the machine is never set up to produce product $j$, e.g., [1, 2, 19, pp. 217-220]. They are added a priori to the model here.

$$I_{j,t-1} \geqslant \sum_{\tau=t}^{t+\Delta} d_{j\tau} \left[ 1 - y_{j,t-1} - \sum_{k=t}^{\tau} z_{jk} \right], \qquad j \in \mathcal{N}, \;\; t \in \mathcal{T}, \;\; \Delta \in [0, T_{\max} - t]. \tag{2}$$

In models with fictitious *micro-periods*, demand is usually non-zero only at the end of the last micro-period of each macro-period [11]. For such a demand pattern, inventory lower bound (2) should be replaced by constraints (3) (denoted here as ILBW), which are defined only for macro-periods [14].

*Additional parameters in cases with macro- and micro-periods:*

$\mathcal{W}$ $= (1, \ldots, W_{\max})$ – set of macro-periods; where $W_{\max}$ is the number of macro-periods,

$\mathcal{T}_w$ $= F(w), \ldots, L(w)$ – set of periods (micro-periods) in macro-period $w$; where $F(w)$ and $L(w)$ are the first and last period in macro-period $w$,

$Y_{jw}$ $= 1$ if machine is setup to process product $j$ during macro-period $w$; 0 otherwise.

Macro-period based inventory lower bound:

$$Y_{jw} \in \{0,1\}, \qquad j \in \mathcal{N},\ w \in \mathcal{W}, \tag{3.1}$$

$$Y_{jw} = y_{j,L(w-1)} + \sum_{t \in \mathcal{T}_w} z_{jt}, \qquad j \in \mathcal{N},\ w \in \mathcal{W}, \tag{3.2}$$

$$I_{j,L(w-1)} \geqslant \sum_{\tau=w}^{w+\Delta} d_{j\tau} \left[1 - \sum_{k=w}^{\tau} Y_{jk}\right], \quad j \in \mathcal{N},\ w \in \mathcal{W}, \Delta \in [0, W_{\max} - w]. \tag{3.3}$$

### 2.2. Generalization of constraints

In the model for uniform periods (1), several constraints are either for products with setup times shorter or longer than a single period length, i.e., for $Q_j = 0$ or $Q_j \geqslant 1$. Below, they are generalized into constraints suitable for all products to make the model more concise.

The generalization of constraints that determine the values of continuation variables $u_{jt}$ is simple. For $Q_j = 0$, constraint (1.16) reduces to (1.13); i.e., we may replace them by the following generalized constraint:

$$\sum_{k=1}^{Q_j+1} z_{j,t+k} - v_{j,t-(Q_j+1)} \leqslant u_{jt}, \qquad j \in \mathcal{N},\ t \in \mathcal{T}. \tag{4.1}$$

Capacity constraints (1.3) and (1.4) differ only in variables $v_{jt}$ and $a_{jt}$, so it is easy to replace them by the following generalized constraint using additional binary (logical) parameter $q_j = \operatorname{sgn}(Q_j)$:

$$C_t \left(y_{jt} - (1 - q_j) v_{jt}\right) + (1 - q_j) a_{jt} + b_{jt} \geqslant p_j x_{jt} + s_{jt}, \quad j \in \mathcal{N}\ t \in \mathcal{T}, \tag{4.2}$$

Constraints (1.14)–(1.15) and (1.17)–(1.18) ensure accurate values of variables $a_{jf}$ for a given $s_{jl}$. Their generalization is not difficult; however, a general approach introduced below is used later to formulate the model with variable periods, which is much more complex and requires some systematic procedure.

In the problem with uniform periods, there are four different cases of setup execution: for $Q_j = 0$ or $Q_j \geqslant 1$, and $s_{jl} \geqslant R_j$ or $s_{jl} < R_j$; denoted a, b, A, and B. The description of these cases is summarized in Table 2, where $f$ is the f irst and $l$ is the last period overlapped by the setup operation. For known period $l$ and setup execution time $s_{jl}$, distance $l - f$ points to the period during which the setup must start, and $a_{jf}$ gives the necessary time during this period $f$ to complete the setup operation.

First, all four of the cases given in Table 2 can be generalized with the help of parameter $q_j$ to the two general cases, denoted aA and bB, presented in Table 3.

Second, one has to apply the standard description of big-M (indicator) constraints, e.g., [22, p. 158]. Condition $\delta = 0 \rightarrow \sum_j a_j x_j \leqslant b$ may be represented

Table 2: Summary of the model with uniform periods

| Case | $q_j$ | Condition: | $v_{jl}$ | $l-f$ | $a_{jf}$ |
|---|---|---|---|---|---|
| a | 0 | $s_{jl} \geqslant R_j$ | 1 | 0 | $R_j$ |
| b | 0 | $s_{jl} < R_j$ | 0 | 1 | $R_j - s_{jl}$ |
| A | 1 | $s_{jl} \geqslant R_j$ | 1 | $Q_j + 0$ | $R_j + \overline{C} - s_{jl}$ |
| B | 1 | $s_{jl} < R_j$ | 0 | $Q_j + 1$ | $R_j - s_{jl}$ |

Table 3: Summary of the generalized model with uniform periods

| Case | Condition: | $v_{jl}$ | $l-f$ | $a_{jf}$ |
|---|---|---|---|---|
| aA | $s_{jl} \geqslant R_j$ | 1 | $Q_j + 0$ | $R_j + q_j(\overline{C} - s_{jl})$ |
| bB | $s_{jl} < R_j$ | 0 | $Q_j + 1$ | $R_j - s_{jl}$ |

in a MIP model by inequality $\sum_j a_j x_j \leqslant b + M\delta$, where $\delta \in \{0,1\}$ and $M$ is the upper bound of $\sum_j a_j x_j - b$. To improve readability of the following sections, all expressions replacing $M$ are written in square brackets $[M]$ and $\delta$ in pointy brackets $\langle\delta\rangle$. If $\delta$ is an integer that takes values greater than 1, the big-M constraint remains valid but is less tight.

According to Table 3, if in period $l$ ends setup operation for $j$ ($z_{jl} = 1$), then the following constraints ensure the setup time during the first overlapped period:

$$\text{Case aA: } R_j + q_j(\overline{C} - s_{jl}) \leqslant a_{jf} + \left[R_j + q_j\overline{C}\right]\left\langle(1 - z_{jl}) + (1 - v_{jl})\right\rangle, \quad \text{where } f = l - Q_j,$$

$$\text{Case bB: } R_j - s_{jl} \leqslant a_{jf} + \left[R_j\right]\left\langle(1 - z_{jl}) + v_{jl}\right\rangle, \quad \text{where } f = l - Q_j + 1.$$

Substituting $f$ by $t$, and $l$ by $t + Q_j$ or $t + Q_j + 1$, after reduction, one gets the constraints in their final forms:

$$(R_j + q_j\overline{C})(z_{j,t+Q_j} + v_{j,t+Q_j} - 1) \leqslant a_{jt} + q_j s_{j,t+Q_j}, \qquad j \in \mathcal{N},\ t \in \mathcal{P}_j, \tag{4.3}$$

$$R_j\left(z_{j,t+Q_j+1} - v_{j,t+Q_j+1}\right) \leqslant a_{jt} + s_{j,t+Q_j+1}, \qquad j \in \mathcal{N},\ t \in \mathcal{P}_j, \tag{4.4}$$

To summarize, four constraints (4) may replace eight constraints in model (1): (1.3)–(1.4), (1.13)–(1.15), and (1.16)–(1.18). For example, for $q_j = 1$, (4.3) reduces to (1.17) and (4.4) to (1.18).

## 3. Model with variable period capacity

In this section presented is a new model for problems with periods $t \in \mathcal{T}$ of different length $C_t$. Model for uniform periods (1) is for such problems incorrect, which may be explained with the help of the example presented in Table 4. Here, the ordinary period capacity is equal to $\overline{C} = 100$, and the processing time of the setup operation is equal to $st_j = 350$ (i.e., the quotient is $Q_j = 3$, and the remainder is $R_j = 50$). The processing of the setup operation ends during period $l = 7$, the most recent shorter period $r(l)$ before $l$ is 3, and its length is equal to $C_3 = 30$.

Table 4: Examples of processing long setup operation ($st_j = 350$)

| | | t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $v_{j7}$ | $v''_{j7}$ | $C_t$ | 100 | 100 | 30 | 100 | 100 | 100 | 100 | 100 |
| 1 | - | $s'_{jt}$ | – | – | - | 60 | 100 | 100 | 90 | – |
| 0 | 0 | $s'_{jt}$ | – | – | 20 | 100 | 100 | 100 | 30 | – |
| 0 | 1 | $s'_{jt}$ | – | 10 | 30 | 100 | 100 | 100 | 10 | – |

"–" is unimportant or replaces zero

Depending on the time reserved for the setup operation during the last period (i.e., $s_{j8} = 90$, 30, or 10), there are three different cases; i.e., the setup operation may be started during periods $l - Q_j = 4$, $l - (Q_j + 1) = 3$ or $l - (Q_j + 2) = 2$. To distinguish between them, one binary variable $v_{jt}$ from model (1) is not enough; another one is necessary (denoted as $v''_{jt}$).

The proposed model extends the model for uniform periods (1). The concept of this new model is based on a simple observation. In the calendar of short-term production planning, most time buckets (periods) are of the same length – only a few are shorter, and none of them is longer than an ordinary period.

### 3.1. Basic assumption

It is assumed that shorter periods occur so rarely that the schedule of each setup operation may be determined by considering at most one shorter period. More precisely, if $\alpha$ and $\omega$ are two consecutive shorter periods, then it is assumed that time interval $[\alpha, \omega]$ is longer or equal to the longest setup time; i.e.,

$$\sum_{t=\alpha}^{\omega} C_t \geqslant \max_j st_j \,. \tag{5}$$

This assumption may be justified with the help of the examples presented in Table 5. There are four schedules of setup operations on a calendar with two shorter periods: $\alpha = 4$ and $\omega = 7$. The total capacity of time interval $[\alpha, \omega]$ is equal to $\sum_{t=\alpha}^{\omega} C_t = 300$; it is smaller than the setup time of product $i$ (with $st_i = 350$), greater than the setup time of product $j$ (with $st_j = 250$), and equal to the setup time of product $k$ (with $st_k = 300$).

Table 5: Setup operations overlapping two consecutive shorter periods

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_t$ | 100 | 100 | 100 | 60 | 100 | 100 | 40 | 100 | 100 | 100 |
| $s'_{it}$ | – | – | 49 | 60 | 100 | 100 | 40 | 1 | – | – |
| $s'_{jt}$ | – | – | – | 9 | 100 | 100 | 40 | 1 | – | – |
| $s'_{kt}$ | – | – | – | 59 | 100 | 100 | 40 | 1 | – | – |
| $s'_{kt}$ | – | – | 1 | 60 | 100 | 100 | 39 | – | – | – |

"–" replaces zero.

If the setup of product $i$ ends during Period 8, then it may start during Period 3; the time $a_{i3}$ used to process it during this period depends on the capacity of both shorter periods ($C_4$ and $C_7$); i.e., $a_{i3} = st_i - \sum_{t=4}^{7} C_t - s_{i8}$. In the same situation, the setup for product $j$ must end during Period 4, and $a_{j4}$ depends only on $C_7$; i.e, $a_{j4} = st_j - \sum_{t=5}^{7} C_t - s_{j8}$.

For product $k$, the situation is similar as for $j$; however, when $s_{k8}$ tends towards zero, $a_{k4}$ goes to $C_4$, and any increment of $st_k$ could extend the setup for Period 3. If the setup for $k$ starts during Period 7, it must end during Period 3. To calculate $a_{k3}$, one must consider only $C_4$ while $C_7$ is unimportant, as $a_{k3} = st_k - \sum_{t=4}^{6} C_t - s_{k7}$ and (4.2) ensure that $s_{k7} \leqslant C_7$.

Therefore, $st_k = \sum_{t=\alpha}^{\omega} C_t$ is the longest setup operation whose schedule (value of variable $a_{kf}$) depends on the capacity of one shorter period at most.

In the introduction, two examples justifying such an assumption have already been presented. Moreover, when it is not satisfied in certain processes, it may be possible to modify the planning calendar to fulfill it. Let us see an example. In some companies, the ordinary period capacity is equal to three shifts (24 hours); however, the company works two shifts on weekends (16 hours). This means that there are two shorter periods (Saturday and Sunday), one after another. However, for the needs of the optimization model, one can create two fictitious periods on weekends: the first being three shifts long and the second only one shift long. The optimized schedule may be easily recalculated to match the real calendar.

### 3.2. Notation

The list below presents all of the additional parameters and variables necessary in the new model.

*Indices:*

$f$ – *first* period overlapped by setup operation;
$l$ – *last* period overlapped by setup operation;

*Distance to recent shorter period:*

$r(t)$ – most *recent* shorter period before current period $t$; i.e., $t - r(t) \geqslant 1$, $C_{r(t)} < \overline{C}$, and $C_{r(t)+1} = \ldots = C_{t-1} = \overline{C}$; if $C_1 = \ldots = C_{t-1} = \overline{C}$ then $r(t) = 0$;

$\pi^L_{jt} = 1$ if $Q_j \geqslant 1$ and $1 \leqslant t - r(t) < Q_j$; 0 otherwise;

$\pi^0_{jt} = 1$ if $Q_j \geqslant 1$ and $1 \leqslant t - r(t) \leqslant Q_j$; 0 otherwise;

$\pi^1_{jt} = 1$ if $t - r(t) = Q_j + 1$; 0 otherwise;

$\pi^2_{jt} = 1$ if $t - r(t) \geqslant Q_j + 2$; 0 otherwise;

*Capacity of the recent shorter period:*

$\eta_{jt} = 1$ if $C_{r(t)} \geqslant R_j$; 0 otherwise;

$v''_{jt} = 1$ if $z_{jt} = 1$ and $s_{jt} \geqslant R_j + \eta_{jt}\overline{C} - C_{r(t)}$; 0 otherwise.

Binary parameters $\pi^L_{jt}$, $\pi^0_{jt}$, $\pi^1_{jt}$, and $\pi^2_{jt}$ describe the distance between the last period $l$ overlapped by a setup operation and the recent shorter period $r(l)$. $v''_{jt}$ is the only additional variable.

### 3.3. Description of cases

In the assumed capacity pattern, one can distinguish 20 different cases of setup schedules. In this section, only four cases are discussed to explain the approach used to derive the constraints of the new model. Table 7 presents a concise but complete description of all cases, making it easy to recognize similarities and differences between them. The appendix contains illustrative examples for all cases.

In the considered cases, the setup time is longer than the ordinary period length ($Q_j \geqslant 1$), and distance $l - r(l)$ between last period $l$ overlapped by the setup operation and recent shorter period $r(l)$ is equal to $Q_j + 1$. This discussion illustrate examples with $l = 9$, capacity $\overline{C} = 100$, and setup time $st = 350$ ($Q_j = 3$, $R_j = 50$). The recent shorter period is $r(l) = 5$, and its capacity $C_{r(l)}$ in Case D is 70, while it is 30 in Cases E and F. Scenarios with extreme values for each case are presented in Table 6.

Case C: In the example, for $100 \geqslant s_{j9} \geqslant 50$, the setup must start during Period 6 with $50 \leqslant a_{j6} \leqslant 100$ because variable $s_{j9}$ cannot take values greater than 100, and for values smaller than 50, the capacity of Period 6 is not enough to complete the setup operation; i.e., it must overlap earlier periods. Generalizing, if $l - r(l) = Q_j + 1$ and $s_{jl} \geqslant R_j$ (or $v_{jf} = 1$), then the setup operation starts during period $f = l - Q_j$ and the capacity needed to complete the setup during this period is equal to $a_{jf} \geqslant R_j + \overline{C} - s_{jl}$.

Table 6: Examples for all scenarios of setup operation that ends in $Q_j + 1$ periods from a recent shorter period

| Case | $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | $s'_{jt}$ | – | – | – | – | – | 50 | 100 | 100 | 100 | – |
| | $s'_{jt}$ | – | – | – | – | – | 100 | 100 | 100 | 50 | – |
| D | $s'_{jt}$ | – | – | – | – | 1 | 100 | 100 | 100 | 49 | – |
| | $s'_{jt}$ | – | – | – | – | 49 | 100 | 100 | 100 | 1 | – |
| E | $s'_{jt}$ | – | – | – | – | 1 | 100 | 100 | 100 | 49 | – |
| | $s_{jt}$ | – | – | – | – | 30 | 100 | 100 | 100 | 20 | – |
| F | $s'_{jt}$ | – | – | – | 1 | 30 | 100 | 100 | 100 | 19 | – |
| | $s'_{jt}$ | – | – | – | 19 | 30 | 100 | 100 | 100 | 1 | – |

Case D: If the remainder of setup operation $R_j$ was not completed during Period 9 (i.e., $s_{j9} < 50$) but it fits in the recent shorter period (e.g., $C_5 = 70$), then the setup starts during Period 5, and requires capacity $0 < a_{j5} < 50$. Generalizing, if $l - r(l) = Q_j + 1$, $s_{jl} < R_j$ (or $v_{jl} = 0$), and $C_{r(l)} \geqslant R_j$ (or $\eta_{jl} = 1$), then $f = l - (Q_j + 1)$ and $a_{jf} \geqslant R_j - s_{jl}$.

Case E: If the setup time during the last period is smaller than setup remainder $R_j$ (i.e., $s_{j9} < 50$) but greater than the setup remainder less the capacity of the recent shorter period (e.g., $C_5 = 30$ and $s_{j9} \geqslant 50-30$), then the rest of the setup fits in the recent shorter Period 5 and requires $0 < a_{j5} \leqslant 30$. Generalizing, if $l-r(l) = Q_j+1$, $C_{r(l)} < R_j$ (or $\eta_{jl} = 0$), and $R_j > s_{jl} \geqslant R_j - C_{r(l)}$ (or $v_{jl} = 0$ and $v''_{jl} = 1$), then $f = l - (Q_j + 1)$ and $a_{jf} \geqslant R_j - s_{jl}$.

Case F: If $C_5 = 30$ and $50 - 30 > s_{j9} > 0$, then the setup starts during Period 4 and requires $0 < a_{j4} < 20$. Generalizing, if $l - r(l) = Q_j + 1$, $C_{r(l)} < R_j$ (or $\eta_{jl} = 0$), and $s_{jl} < R_j - C_{r(l)}$ (or $v''_{jl} = 0$), then $f = l - (Q_j + 2)$ and $a_{jf} \geqslant R_j - C_{r(l)} - s_{jl}$.

All cases are summarized in Table 7. There are 6 cases for $Q_j = 0$ (or $q_j = 0$) (denoted as a to f) and 14 cases for $Q_j \geqslant 1$ (or $q_j = 1$) (denoted as A to P). These are categorized into four groups according to the distance between the last period overlapped by the setup operation and recent shorter period: $l - r(l)$ equal or longer than $Q_j + 2$, equal to $Q_j + 1$ or $Q_j$, or shorter than $Q_j$ but longer than or equal to 1. Due to the definition of $r(t)$, the difference $t - r(t)$ cannot be zero.

### 3.4. New model

In Table 8, all cases of the setup processing are generalized with the help of binary parameters. One should note that setup finish variables $v_{jt}$ and $v''_{jt}$ are

Table 7: Detailed summary of cases for the model with variable capacity

| Case | $q_j$ | $l-r(l)$ | $\pi^L_{jl}$ | $\pi^0_{jl}$ | $\pi^1_{jl}$ | $\pi^2_{jl}$ | $v_{jl}$ | $\eta_{jl}$ | $v''_{jl}$ | $l-f$ | $a_{jf}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a | 0 | $\geqslant 0+2$ | 0 | 0 | 0 | 1 | 1 | – | – | 0 | $R_j$ |
| b | 0 | | 0 | 0 | 0 | 1 | 0 | – | – | 1 | $R_j - s_{jl}$ |
| c | 0 | $=0+1$ | 0 | 0 | 1 | 0 | 1 | – | – | 0 | $R_j$ |
| d | 0 | | 0 | 0 | 1 | 0 | 0 | 1 | – | 1 | $R_j - s_{jl}$ |
| e | 0 | | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | $R_j - s_{jl}$ |
| f | 0 | | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | $R_j - C_{r(l)} - s_{jl}$ |
| A | 1 | $\geqslant Q_j+2$ | 0 | 0 | 0 | 1 | 1 | – | – | $Q_j+0$ | $R_j + \overline{C} - s_{jl}$ |
| B | 1 | | 0 | 0 | 0 | 1 | 0 | – | – | $Q_j+1$ | $R_j - s_{jl}$ |
| C | 1 | $=Q_j+1$ | 0 | 0 | 1 | 0 | 1 | – | – | $Q_j+0$ | $R_j + \overline{C} - s_{jl}$ |
| D | 1 | | 0 | 0 | 1 | 0 | 0 | 1 | – | $Q_j+1$ | $R_j - s_{jl}$ |
| E | 1 | | 0 | 0 | 1 | 0 | 0 | 0 | 1 | $Q_j+1$ | $R_j - s_{jl}$ |
| F | 1 | | 0 | 0 | 1 | 0 | 0 | 0 | 0 | $Q_j+2$ | $R_j - C_{r(l)} - s_{jl}$ |
| G | 1 | $=Q_j$ | 0 | 1 | 0 | 0 | – | 1 | 1 | $Q_j+0$ | $R_j + \overline{C} - s_{jl}$ |
| H | 1 | | 0 | 1 | 0 | 0 | – | 1 | 0 | $Q_j+1$ | $R_j + \overline{C} - C_{r(l)} - s_{jl}$ |
| K | 1 | | 0 | 1 | 0 | 0 | – | 0 | 1 | $Q_j+1$ | $R_j + \overline{C} - C_{r(l)} - s_{jl}$ |
| L | 1 | | 0 | 1 | 0 | 0 | – | 0 | 0 | $Q_j+2$ | $R_j - C_{r(l)} - s_{jl}$ |
| M | 1 | $<Q_j$ | 1 | 1 | 0 | 0 | – | 1 | 1 | $Q_j+0$ | $R_j + 2\overline{C} - C_{r(l)} - s_{jl}$ |
| N | 1 | | 1 | 1 | 0 | 0 | – | 1 | 0 | $Q_j+1$ | $R_j + \overline{C} - C_{r(l)} - s_{jl}$ |
| O | 1 | | 1 | 1 | 0 | 0 | – | 0 | 1 | $Q_j+1$ | $R_j + \overline{C} - C_{r(l)} - s_{jl}$ |
| P | 1 | | 1 | 1 | 0 | 0 | – | 0 | 0 | $Q_j+2$ | $R_j - C_{r(l)} - s_{jl}$ |

"–" denotes unimportant value

Table 8: Generalized summary of cases for the model with variable capacity

| Cases | $l-r(l)$ | $\pi^0_{jl}$ | $\pi^1_{jl}$ | $\pi^2_{jl}$ | $v_{jl}$ | $\eta_{jl}$ | $v''_{jl}$ | $l-f$ | $a_{jf}$ | Group |
|---|---|---|---|---|---|---|---|---|---|---|
| aA | $\geqslant Q_j+2$ | 0 | 0 | 1 | 1 | – | – | $Q_j+0$ | $R_j + q_j\overline{C} + \pi^L_{jl}(\overline{C} - C_{r(l)}) - q_j s_{jl}$ | 1) |
| bB | | 0 | 0 | 1 | 0 | – | – | $Q_j+1$ | $R_j + \pi^0_{jl}(\overline{C} - C_{r(l)}) - s_{jl}$ | 2) |
| cC | $=Q_j+1$ | 0 | 1 | 0 | 1 | – | – | $Q_j+0$ | $R_j + q_j\overline{C} + \pi^L_{jl}(\overline{C} - C_{r(l)}) - q_j s_{jl}$ | 1) |
| dD | | 0 | 1 | 0 | 0 | 1 | – | $Q_j+1$ | $R_j + \pi^0_{jl}(\overline{C} - C_{r(l)}) - s_{jl}$ | 2) |
| eE | | 0 | 1 | 0 | 0 | 0 | 1 | $Q_j+1$ | $R_j + \pi^0_{jl}(\overline{C} - C_{r(l)}) - s_{jl}$ | 2) |
| fF | | 0 | 1 | 0 | 0 | 0 | 0 | $Q_j+2$ | $R_j - C_{r(l)} - s_{jl}$ | 3) |
| GM | $\leqslant Q_j+0$ | 1 | 0 | 0 | – | 1 | 1 | $Q_j+0$ | $R_j + q_j\overline{C} + \pi^L_{jl}(\overline{C} - C_{r(l)}) - q_j s_{jl}$ | 1) |
| HN | | 1 | 0 | 0 | – | 1 | 0 | $Q_j+1$ | $R_j + \pi^0_{jl}(\overline{C} - C_{r(l)}) - s_{jl}$ | 2) |
| KO | | 1 | 0 | 0 | – | 0 | 1 | $Q_j+1$ | $R_j + \pi^0_{jl}(\overline{C} - C_{r(l)}) - s_{jl}$ | 2) |
| LP | | 1 | 0 | 0 | – | 0 | 0 | $Q_j+2$ | $R_j - C_{r(l)} - s_{jl}$ | 3) |

"–" denotes unimportant value

unnecessary for all products during all periods. Variable $v_{jt}$ is needless for each product $j$ during each period $t$, which is fewer than $Q_j + 1$ periods from recent shorter period $r(t)$; i.e., for $t - r(t) \leqslant Q_j$ (or $\pi^0_{jt} = 1$), or for $\pi^0_{jt}$ in short.

Variable $v''_{jt}$ is needless for $\pi^2_{jt}$ (or $t - r(t) \geqslant Q_j + 2$) and for products with $\eta_{jt} = 1$ (or $C_{r(t)} \geqslant R_j$) for $\pi^1_{jt}$ (or $t - r(t) = Q_j + 1$). Only for $\eta_{jt} = 0$ and $\pi^1_{jt}$ are both variables necessary to identify the periods overlapped by the setup operation. Therefore, these may be determined with the help of the following constraints (replacing (1.11)):

$$s_{jt} \geqslant R_j v_{jt}, \qquad j \in \mathcal{N}\colon Q_j = 0,\ t \in \mathcal{T}\colon\ \pi^0_{jt} = 0, \tag{6.1}$$

$$s_{jt} \geqslant (R_j + \eta_{jt}\overline{C} - C_{r(t)})\, v''_{jt}, \quad j \in \mathcal{N},\ t \in \mathcal{T}\colon\ \pi^0_{jt} = 1 \vee \pi^1_{jt} = 1 \wedge \eta_{jt} = 0. \tag{6.2}$$

In all other cases, $v_{jt}$ and $v''_{jt}$ are useless; i.e., they may be set to zero or simply ignored. Summarizing, in the model with variable capacity, there is one additional binary variable necessary for each shorter period and each product with $\eta_{jt} = 0$.

Independent from the distance to the recent shorter period, one can distinguish three groups of cases that require the same general constraints: 1) aAcCGM, 2) abBdDeEHNKO, and 3) fFLP. The general constraints for all three groups are determined below.

First, we need constraints that set the continuation variables $u_{jt}$ equal to 1 in time interval $[f, l - 1]$ to replace (4.1). There are three groups of cases:

1) in cases aAcCGM: $\quad z_{jl} \leqslant u_{j,l-k}, \quad k = 1, \ldots, Q_j,$
2) in cases bBdDeEHNKO: $\quad z_{jl} \leqslant u_{j,l-k}, \quad k = 1, \ldots, Q_j + 1,$
3) in cases fFLP: $\quad z_{jl} \leqslant u_{j,l-k}, \quad k = 1, \ldots, Q_j + 2,$

where $1, \ldots, 0$ denotes an empty set of indices $k$ for Group 1. One can reformulate the description of these inequalities:

I. in all cases: $\quad z_{jl} \leqslant u_{j,l-k}, \quad k = 1, \ldots, Q_j,$
II. not in cases aAcCGM: $\quad z_{jl} \leqslant u_{j,l-k}, \quad k = Q_j + 1,$
III. only in cases fFLP: $\quad z_{jl} \leqslant u_{j,l-k}, \quad k = Q_j + 2.$

Bearing in mind that all variables are fixed and equal to zero for a period smaller than or equal to zero, one may determine the following constraint (replacing (1.13) and (1.16), or (4.1) ):

$$\begin{aligned} u_{jt} \geqslant q_j \sum_{k=1}^{Q_j} z_{j,t-k} &+ \left(z_{jl_1} - (\pi^2_{jl_1} + \pi^1_{jl_1}) v_{jl_1} - \eta_{jl_1}\pi^0_{jl_1} v''_{jl_1}\right) \\ &+ (1 - \eta_{jl_2})(\pi^1_{jl_2} + \pi^0_{jl_2})(z_{jl_2} - v''_{jl_2}), \\ &j \in \mathcal{N},\ t \in \mathcal{T} \cup \mathcal{P}_j,\ l_i = t + Q_j + i, \end{aligned} \tag{6.3}$$

On the right side, there are three elements modeling conditions I–III. If $z_{jl} = 1$, then in $Q_j$ previous periods $u_{jt} = 1$ (Condition I). Moreover, the setup may be processed during period $t$ if it finishes during period $l_1 = t + (Q_j + 1)$. This will be so for $z_{jl_1} = 1$ in all cases except for aAcCGM (Condition II.); i.e., for $\pi^2_{jl_1}$ or $\pi^1_{jl_1}$ when $z_{jl_1} = 0$ (neither aA nor cC);and also for $\eta_{jl_1} = 1$ and $\pi^0_{jl_1}$ when $z_{jl_1} = 0$ (not GM). The setups finished in $l_2 = t + (Q_j + 2)$ may start in $t$ (Condition III.) for cases fF and GM; i.e., for $\eta_{jl_2} = 0$, $\pi^1_{jl_2}$ or $\pi^0_{jl_2}$ when $z_{jl_2} = 1$ but $v''_{jl_2} = 0$.

Constraint (1.8) allows $a_{jt} > 0$ only when the setup operation for product $j$ starts during period $t$. This first setup period must be identified with the help of variables pointing to the end of the setup. The constraint below replaces (1.8) and takes all cases from Table 8 into account:

$$
\begin{aligned}
a_{jt} \leqslant C_t \Big\langle & (\pi^2_{jl_0} + \pi^1_{jl_0}) v_{jl_0} + \eta_{jl_0} \pi^0_{jl_0} v''_{jl_0} \\
& + (\pi^2_{jl_1} + \pi^1_{jl_1})(z_{jl_1} - v_{jl_1}) + \eta_{jl_1} \pi^0_{jl_1} (z_{jl_1} - v''_{jl_1}) \\
& + (1 - \eta_{jl_1})(\pi^1_{jl_1} + \pi^0_{jl_1}) v''_{jl_1} + (1 - \eta_{jl_2})(\pi^1_{jl_2} + \pi^0_{jl_2})(z_{jl_2} - v''_{jl_2}) \Big\rangle, \\
& \qquad j \in \mathcal{N}, \ t \in \mathcal{T}, \quad l_i = t + Q_j + i,
\end{aligned}
\tag{6.4}
$$

The three rows on the right side take a value of 1 if the setup starts during period $t$, and time interval $l - f$ is equal to $Q_j + 0$, $Q_j + 1$, and $Q_j + 2$; i.e., if the setup finishes during period $l_0 = t + Q_j$, $l_1 = t + Q_j + 1$ or $l_2 = t + Q_j + 2$.

According to Table 8, the setup operation started during period $t$ may end in $l_0 = t + Q_j$: for the distance from $l_0$ to its recent shorted period $r(l_0)$ denoted by $\pi^2_{jl_0}$ and $\pi^1_{jl_0}$ only when $v_{jl_0} = 1$ (cases aAcC); and for $\eta_{jl_0} = 1$ and $\pi^0_{jl_0}$ when $v''_{jl_0} = 1$ (cases GM).

The setup started in $t$ may finish in $l_1 = t + (Q_j + 1)$: for $\pi^2_{jl_1}$ and $\pi^1_{jl_1}$ when $v_{jl_1} = 0$ (cases bBdDeE); for $\eta_{jl_1} = 1$ and $\pi^0_{jl_1}$ when $v''_{jl_1} = 0$ (cases HN); and for $\eta_{jl_1} = 0$ and $\pi^1_{jl_1}$ or $\pi^0_{jl_1}$ when $v''_{jl_1} = 1$ (cases eEKO).

The setup started in $t$ may end in $l_2 = t + (Q_j + 2)$: for $\eta_{jl_2} = 0$ and $\pi^1_{jl_2}$ or $\pi^0_{jl_2}$ when $v''_{jl_2} = 0$ (cases fFLP). In cases fF, condition $v_{jl_2} = 0$ is omitted because, if $v''_{jl_2} = 0$ (i.e., $s_{jl_2} < R_j - C_{r(l_2)}$ ), then also $v_{jl_2} = 0$ (i.e., $s_{jl_2} < R_j$).

Next, we need constraints that replace constraints (4.3)–(4.4). They determine the time $a_{jf}$ required to start the setup operation during the first overlapped period $f$. According to Table 8, there are three groups of cases:

1) in cases aAcCGM: $l - f = Q_j + 0$ and
$$a_{jf} \geqslant \left[R_j + q_j \overline{C} + \pi^L_{jf} (\overline{C} - C_{r(f)})\right] - q_j s_{jl};$$

2) in cases bBdDeEHNKO: $l - f = Q_j + 1$ and $a_{jf} \geqslant \left[R_j + \pi^0_{jf} (\overline{C} - C_{r(f)})\right] - s_{jl}$;

3) in cases fFLP: $l - f = Q_j + 2$ and $a_{jf} \geqslant \left[R_j - C_{r(f)}\right] - s_{jl}$.

In the above inequalities, $s_{jl}$ and $a_{jf}$ are the only variables, and their minimum value is zero; therefore, the expressions in square brackets represent the big-M number of the indicator constraints: $\sum_j a_j x_j - b \leqslant M\,\delta$; i.e., the upper bound of its left side. The inequalities below are slightly reduced big-M constraints that implement the conditions of all groups:

$$\begin{aligned} a_{jt} + q_j s_{jl} \geqslant \left[R_j + q_j\overline{C} + \pi^L_{jl}(\overline{C} - C_{r(l)})\right]\times \\ \times\left\langle 1 - (\pi^2_{jl} + \pi^1_{jl})(1 - v_{jl}) - \pi^0_{jl}(1 - v''_{jl})\right\rangle \\ j \in \mathcal{N}\colon Q_j,\ t \in \mathcal{T},\ l = t + Q_j\colon \\ l \leqslant T_{\max},\ \pi^0_{jl} = 0 \vee \eta_{jl} = 1; \end{aligned} \tag{6.5}$$

$$\begin{aligned} a_{jt} + s_{jl} \geqslant \left[R_j + \pi^0_{jl}(\overline{C} - C_{r(f)})\right]\left\langle z_{jl} - (\pi^2_{jl} + \pi^1_{jl})v_{jl} - \right. \\ \left. - \eta_{jl}\pi^0_{jl}v''_{jl} - (1 - \eta_{jl})(\pi^1_{jl} + \pi^0_{jl})(1 - v''_{jl})\right\rangle \\ j \in \mathcal{N},\ t \in \mathcal{T},\ \ l = t + (Q_j + 1)\colon\ l \leqslant T_{\max}; \end{aligned} \tag{6.6}$$

$$\begin{aligned} a_{jt} + s_{jl} \geqslant \left[R_j - C_{r(l)}\right]\left\langle z_{jl} - \pi^1_{jl}(v_{jl} + v''_{jl}) - \pi^0_{jl}v''_{jl})\right\rangle \\ j \in \mathcal{N},\ t \in \mathcal{T},\ l = t + (Q_j + 2)\colon \\ l \leqslant T_{\max},\ \pi^2_{jl} = \eta_{jl} = 0. \end{aligned} \tag{6.7}$$

For all cases, condition $z_{jl} = 1$ is necessary (i.e., 1-$z_{jl} = 0$). However, in constraint (6.5) for group 1), it is omitted because, when $v_{jl} = 1$ or $v''_{jl} = 1$, then $z_{jl} = 1$ holds as well (see (6.1) or (6.2) and (1.12)).

Constraint (6.5) describes the condition for Group 1) when the setup overlaps exactly $Q_j$ periods; i.e., $l - f = Q_j$. According to Table 8, for $l - r(l) \leqslant Q_j$ (or for $\pi^0_{jl}$ in short), this is possible only for $\eta_{jl} = 1$; hence, condition $\pi^0_{jl} = 0$ or $\eta_{jl} = 1$. Moreover, for $\pi^2_{jl}$ and $\pi^1_{jl}$, this occurs only when $v_{jl} = 1$ (cases aAcC) and for $\pi^0_{jl}$ when $v''_{jl} = 1$ (cases GM).

Constraint (6.6) describes the condition for Group 2) when the setup overlaps $Q_j + 1$ periods. For $\pi^2_{jl}$ and $\pi^1_{jl}$, this occurs only when $v_{jl} = 0$ (cases bBdDeE); for $\eta_{jl} = 1$ and $\pi^0_{jl}$ when $v''_{jl} = 0$ (cases HN); and for $\eta_{jl} = 0$ and $\pi^1_{jl}$ or $\pi^0_{jl}$ when $v''_{jl} = 1$ (cases eEKO).

Constraint (6.7) describes the condition for Group 3) when the setup overlaps $Q_j + 2$ periods. This is possible only for $\eta_{jl} = 0$ and $\pi^1_{jl}$ or $\pi^0_{jl}$; i.e., $\pi^2_{jl} = 0$. For distance $\pi^1_{jl}$, this occurs when $v_{jl} = v''_{jl} = 0$ (cases fF) and for $\pi^0_{jl}$ when $v''_{jl} = 0$ (cases LP).

To summarize, to reformulate the model with uniform periods (1) into the model with variable periods, constraints (6.1)–(6.2) must replace (1.11); (6.3) must replace (1.13) and (1.16) (or their generalized equivalent (4.1)); (6.4) must replace (1.8); (6.5)- (6.7) must replace (1.14)–(1.15) and (1.17)–(1.18) (or their generalization (4.3)–(4.4)).

### 3.5. MIP heuristic

Start-up variables $z_{jt}$ determine the crucial decisions of when to switch the machine from one product to another. Variables $v_{jt}$ make only some fine-tuning. Therefore Kaczmarczyk [15] proposed a MIP heuristic (denoted as HR) that uses heuristic cuts (7) to set the values of $v_{jt}$. The following derived parameters simplify the description of the HR heuristic:

$$\begin{aligned}
\widetilde{\mathcal{T}}_j &= \{Q_j + 2, \dots, T_{\max}\}, && j \in \mathcal{N}; \\
\mu_{jt} &= R_j / C_t, && j \in \mathcal{N},\ t \in \widetilde{\mathcal{T}}_j; \\
\mu''_{jt} &= \left(R_j + \eta_{jt}\overline{C} - C_{r(t)}\right)/C_t, && j \in \mathcal{N},\ t \in \widetilde{\mathcal{T}}_j.
\end{aligned}$$

Heuristics HR consist of two steps:

Step 1. Optimize the explicit model (1) extended by constraints (7):

$$v_{jt} \geqslant z_{jt}, \quad j \in \mathcal{N}\colon \mu_{jt} \leqslant 0.5 \quad t \in \widetilde{\mathcal{T}}_j; \tag{7.1}$$

$$v_{jt} = 0, \quad j \in \mathcal{N}\colon \mu_{jt} > 0.5 \quad t \in \widetilde{\mathcal{T}}_j. \tag{7.2}$$

Step 2. Optimize the explicit model (1) again, now without constraints (7), but with the start-up variables $z_{jt}$ fixed according to the solution obtained in Step 1.

Cuts (7) is an attempt to assign more likely values to variables $v_{jt}$. If $R_j$ is smaller than $0.5\,C$, then it is more likely that the setup operation will overlap only $Q_j + 1$ periods rather than $Q_j + 2$ periods. In the model with variable capacity, heuristic cuts of HR may be defined as follows:

$$v_{jt} \geqslant z_{jt}, \ \ j \in \mathcal{N},\ t \in \widetilde{\mathcal{T}}_j\colon \ (\pi^2_{jt} \vee \pi^1_{jt}) \wedge (\mu_{jt} \leqslant 0.5), \tag{8.1}$$

$$v_{jt} = 0, \quad j \in \mathcal{N},\ t \in \widetilde{\mathcal{T}}_j\colon \ (\pi^2_{jt} \vee \pi^1_{jt}) \wedge (\mu_{jt} > 0.5), \tag{8.2}$$

$$v''_{jt} \geqslant z_{jt}, \ \ j \in \mathcal{N},\ t \in \widetilde{\mathcal{T}}_j\colon \ \left(\pi^1_{jt} \wedge (\eta_{jt} = 0) \wedge (\mu_{jt} > 0.5) \vee \pi^0_{jt}\right) \wedge (\mu''_{jt} \leqslant 0.5), \tag{8.3}$$

$$v''_{jt} = 0, \quad j \in \mathcal{N},\ t \in \widetilde{\mathcal{T}}_j\colon \ \left(\pi^1_{jt} \wedge (\eta_{jt} = 0) \wedge (\mu_{jt} > 0.5) \vee \pi^0_{jt}\right) \wedge (\mu''_{jt} > 0.5). \tag{8.4}$$

## 4. Results of experiments

To verify the impact of the new explicit model formulation the following experiments were executed. The data sets used in the experiments imitate the division of real periods (macro-periods; e.g., weeks) into many short fictitious micro-periods (e.g., days, shifts, or fractions of shifts), with non-zero demand only in the last micro-period of each macro-period. In such a case, it is more likely that a single setup operation will overlap multiple periods.

All instances were solved with the help of the CPLEX 12.8.0 solver on an Intel Core i9-7900X processor with 3.3 GHz clock speed, 16 GB RAM, and 10 cores. All computations were performed with a ten-minute computation time limit with the standard-setting of the solver.

In the data set, there are eight macro-periods divided into 5, 10, 15, or 20 micro-periods. Macro-period capacity is always 1200. In the basic calendar with five micro-periods, the regular capacity $\overline{C}$ is 240. There are five products with uniform processing times $p_j$ and unit holding costs $h_j$ equal to 1. Holding costs are accounted only at the end of each macro-period.

There are three patterns of setup times: *mix*, *short*, and *long*. In the first one, there is a *mix* of different setup times equal to 48, 96, 144, 192, and 240. For $\overline{C} = 240$, these setup times constitute 0.2, 0.4, 0.6, 0.8, and 1.0 of $\overline{C}$, respectively. The ratios of the setup times and period capacity $st/C_t$ for a greater number of micro-periods are presented in Table 9. The setup-time quotients $Q_j$ are within a range of $[0, 3]$; in each instance, the setup remainders $R_j$ are within a range of $[0.2, 1.0]$. Two other setup time patterns assume uniform setup times for all products equal to 96 or 192 (i.e., 0.4 or 0.8 of $\overline{C} = 240$).

Table 9: Ratios of setup times to capacity

| | | Micro-periods | | | |
|---|---|---|---|---|---|
| $j$ | $st_j$ | 5 | 10 | 15 | 20 |
| 1 | 48 | 0.2 | 0.4 | 0.6 | 0.8 |
| 2 | 96 | 0.4 | 0.8 | 1.2 | 1.6 |
| 3 | 144 | 0.6 | 1.2 | 1.8 | 2.4 |
| 4 | 192 | 0.8 | 1.6 | 2.4 | 3.2 |
| 5 | 240 | 1.0 | 2.0 | 3.0 | 4.0 |
| | $\overline{C}$ | 240 | 120 | 80 | 60 |

A different set of five demand scenarios was generated with the following procedure for each setup time pattern. At first, average demand values $\overline{d}_j$ for all products $j$ were randomly drawn from a uniform distribution over an interval of $[10, 90]$ to create high- and low-demand products. Next, all macro-period demands $d_{jw}$ were randomly generated with the help of the gamma distribution, with a shape parameter equal to $\overline{d}_j$ and a scale parameter equal to 1. Initial inventory values $I_{j0}$ were generated with the same distribution as the demand. Finally, the demand and initial inventory have been rescaled proportionally for all products in such a way as to keep the machine's workload equal to 60%.

The setup time has been set so that the expected time devoted to the setup operations is about 20% of the total capacity. For the pattern with setup times $st_j = 96$, average cycle time $T_{\text{EOQ}}$ (i.e., the time interval between lots of the same product) was assumed to be equal to two macro-periods. If each of the five products has a setup operation every two macro-periods, then the average setup time per macro-period is $5 \times 0.5 \times 96 = 240$, equal to $0.2 \times 1200$. For $st_j = 192$, the cycle was assumed to be $T_{\text{EOQ}} = 4$, which gives a total setup time of $5 \times 0.25 \times 193 = 240$. For the mixed pattern, this was $T_{\text{EOQ}} = 3$, and the total setup time was $(48 + 96 + 144 + 192 + 240) \times 1/3 = 240$.

The setup cost was set to $sc_j = T^2_{\text{EOQ}}(\sum_{jt} d_{jt} - \sum_j I_{j0})/W_{\max}/2$, according to the EOQ formula. This way, the total expected machine utilization is 80% for each instance. In optimal solutions, the utilization was within a range of 74.2-75.8% The design of the data set is based on experience with lot-sizing and scheduling in the electronics industry [13].

The construction of the micro-period calendar within a single macro-period is presented in Table 10. For regular capacity $\overline{C} = 240$, 120, 80, or 60, there are 7, 11, 17, or 21 micro-periods, respectively.

Table 10: Capacity with shorter micro-periods

| 7 | | | 11 | | | 17 | | | 21 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | $C_t$ | $\frac{st_j}{\overline{C}}$ | $t$ | $C_t$ | $\frac{st_j}{\overline{C}}$ | $t$ | $C_t$ | $\frac{st_j}{\overline{C}}$ | $t$ | $C_t$ | $\frac{st_j}{\overline{C}}$ |
| $*$ | 240 | 1.0 | $*$ | 120 | 1.0 | $*$ | 80 | 1.0 | $*$ | 60 | 1.0 |
| 2 | 96 | 0.4 | 3 | 96 | 0.8 | 5 | 16 | 0.2 | 6 | 36 | 0.6 |
| 4 | 96 | 0.4 | 6 | 96 | 0.8 | 10 | 16 | 0.2 | 12 | 36 | 0.6 |
| 6 | 48 | 0.2 | 9 | 48 | 0.4 | 14 | 48 | 0.6 | 17 | 48 | 0.8 |

$t$ – period, "$*$" regular periods

For regular capacity $\overline{C} = 240$, there are four regular micro-periods (1, 3, 5, and 7) and three shorter (2, 4, and 6) with a capacity equal to 0.4, 0.4, and 0.2 of the regular capacity, respectively.

Calendars for smaller regular capacity are constructed by a division of these basic micro-periods. For example, three periods arise for $\overline{C} = 80$ from each micro-period with a capacity of 240, and two periods arise from a micro-period with $C_t = 98$, one with $C_t = 80$, and another with $C_t = 16$.

Data instances with the same demand and setup scenario differ only by a micro-period calendar, and therefore their solutions may be directly compared.

Experiments were executed on two formulations of the PLSP/E model:

POST1 – the easier to solve implicit model proposed by Suerie [21],

E-MS/P – the new explicit model.

The explicit model was also solved with the help of heuristics HR with cuts (8).

Two characteristics are used to assess the quality of the solutions. The *objective gap [%]* is the average relative difference between objective value $f$ for best solution obtained with considered model and the best objective value $f^*$ for best solution obtained any model (in most cases, some of them were solved to optimality), i.e., *the objective gap [%]* $= (f - f^*)/f^* \times 100\%$. The *MIP gap [%]* is the average relative difference between objective value $f$ for the best solution to the best lower bound LB obtained with the given solver and model formulation; i.e., *MIP gap [%]* $= (f - \text{LB})/f \times 100\%$. The number of instances for which the solver found a feasible solution and completed the search (MIP gap was zero) is given in the *Sol.* and *Opt.* columns, respectively.

The computational effort may be evaluated with help of the following characteristics. *Time* [s] is the computation time. *Iter.* is the mean number of iterations, and *It. time* is the average time per single iteration.

Table 11 presents the average results. First, note that models with macro-period-based ILBW inventory lower bounds are easier to solve than with classic ILB micro-period-based bounds. The time-saving for POST1 is equal to 37%, and for E-MS/P even 53%. For ILB, the MIP gap is higher, and there are some unsolved instances; which suggests that the difference would be greater for a higher time limit. Therefore, only models with ILBW bounds are considered further.

Table 11: Average results

| Cuts | Model | Gap [%] obj. | Gap [%] MIP | Sol. | Opt. | Time [s] | Nodes [$10^3$] | Iter. [$10^6$] | It.time [µs] |
|---|---|---|---|---|---|---|---|---|---|
| ILB | POST1 | 2.49 | 3.78 | 58 | 24 | 411 | 25.4 | 6.7 | 74 |
| | E-MS/P | 0.87 | 1.30 | 59 | 41 | 262 | 8.0 | 2.3 | 111 |
| ILBW | POST1 | 1.03 | 1.93 | 58 | 41 | 258 | 23.7 | 9.3 | 28 |
| | E-MS/P | 0.21 | 0.17 | 60 | 57 | 124 | 11 | 3.7 | 32 |
| ILBW, HR | E-MS/P | 0.62–0.58† | 0.00 | 60 | 60 | 48 | 9 | 2.4 | 24 |

† results after the first and second step of the heuristic

Next, the E-MS/P explicit model was solved 52% faster than the POST1 model on average, and the heuristic HR even 81% faster. The numbers of nodes were 53% and 63% smaller, respectively. Moreover, for POST1, the solver failed to find a feasible solution with the time limit for 2 instances and failed to find optimal solutions in another 17 cases. For E-MS/P, the solver always found a feasible solution, and only in 3 cases failed to complete the search, and therefore provided the best average quality of solutions.

Finally, the average objective value obtained using heuristics was worse then for E-MS/P with ILBW only by 0.37% for two reasons. First, with HR, the solver always completed the search within the time limit, so the MIP gap was 0%. Second, the limitations of the decision space imposed by the heuristics do not seem to have a destructive impact on solution quality.

Figure 1 shows a rapid increase in computation time as the number of microperiods increases. The heuristic needs ten times more time for 21 than for 7 microperiods. Still, the heuristic is only 6 seconds slower for 21 periods than the POST1 for 7 microperiods.

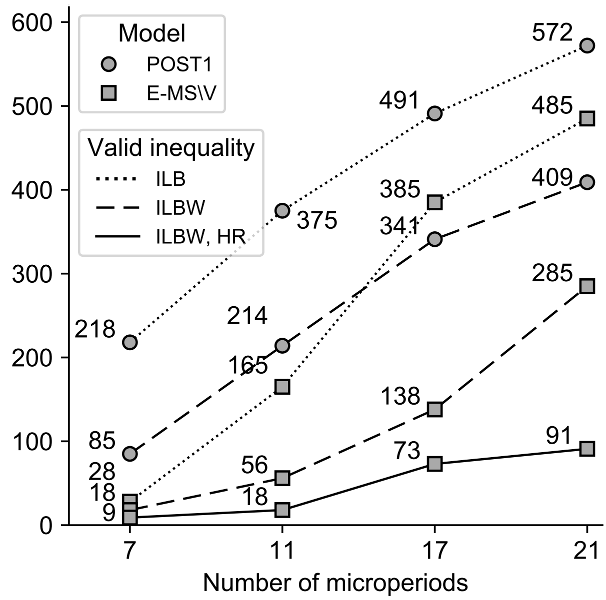


Figure 1: Computation time [s]

The objective gap is not zero only if the MIP gap is also not zero. In other words, all solutions with an optimality guarantee (and some others as well) have exactly the same objective value, for both POST1 and E-MS/P models, independent of the number of micro-periods. This observation is not surprising, as both the POST1 and E-MS/P models allow any schedule of setup and production operations; i.e., they do not impose any constraints on their beginnings or endings.

Table 12 presents the average objective gap [%] of the HR heuristic for various values of setup times and micro-period numbers. The average gap is 0.58%, and the maximal is 2.54%. It is two times higher for the long setups than for the mixed ones. For the short setups, it is almost zero. The calendar of micro-periods does not seem to have any impact on the solution quality.

Table 12: Average objective gap [%] obtained with HR heuristic

| setup | Micro-periods | | | | Mean |
|---|---|---|---|---|---|
| | 7 | 11 | 17 | 21 | |
| Short | 0.06 | – | – | 0.16 | 0.05 |
| Long | 1.31 | 1.32 | 0.97 | 1.13 | 1.18 |
| Mix | 0.31 | 0.70 | 0.29 | 0.75 | 0.58 |
| Mean | 0.56 | 0.67 | 0.42 | 0.68 | |

## 5. Summary

This paper presents new mixed-integer programming (MIP) model of the proportional lot-sizing problem (PLSP) with setup operations overlapping multiple periods and variable period length (capacity). The new model explicitly determines a schedule of each setup operation, i.e., for the known ending of the setup operation, the constraints explicitly point to its beginning. It is based on two assumptions: first – most periods have the same length, and only a few of them are shorter; second – the time interval determined by two consecutive shorter periods is always equal to or longer than a single setup operation.

Computational experiments confirmed that the explicit MIP model requires 52% less time than the implicit formulation proposed by Suerie [21] at least. Besides, the proposed heuristic requires 81% less time than the implicit model, while the average and maximal increments of the objective value were 0.58% and 2.54%, respectively.

The new explicit model for setup operations overlapping multiple periods enables practical applications of models with many short micro-periods in multi-level systems, which enables reduction of production cycles and work-in-process. Future research should evaluate this opportunity.

## Appendix A: Examples for all cases

The approach used to derive all cases is explained in Section 3.3. All cases are completely described in Table 7. Here they are illustrated by examples with ordinary capacity $\overline{C} = 100$, the remainder after division of set-up time by ordinary capacity $R_j = 50$, and the recent shorter period $r(l) = 5$. The capacity of recent shorter period $C_{r(l)}$ in examples with $C_{r(l)} < R_j$ is equal to 30; in examples with $C_{r(l)} \geqslant R_j$, the capacity is equal to 70.

### 1. Short set-up time ($Q_j = 0$ or $q_j = 0$)

In this section, all possible cases of the set-up process are presented for set-up times shorter than period length ($Q_j = 0$).

**For distance** $l - r(l) \geqslant 2$:

Case a: If $s_{jl} \geqslant R_j$ (or $v_{jl} = 1$) then $f = l - 0$ and $a_{jf} \geqslant R_j$

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | – | – | 50 | – | – | – |

Case b: If $s_{jl} < R_j$ (or $v_{jl} = 0$) then $f = l - 1$ and $a_{jf} \geqslant R_j - s_{jl}$

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | – | 1 | 49 | – | – | – |
| $s'_{jt}$ | – | – | – | – | – | 49 | 1 | – | – | – |

**For distance** $l - r(l) = 1$:

Case c: If $s_{jl} \geqslant R_j$ (or $v_{jl} = 1$) then $f = l - 0$ and $a_{jf} \geqslant R_j$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | – | 50 | – | – | – | – |

Case d: For $C_{r(l)} \geqslant R_j$ (or $\eta_{jl} = 1$), e.g., $C_{r(l)} = 70$:
if $s_{jl} < R_j$ (or $v_{jl} = 0$) then $f = l - 1$ and $a_{jf} \geqslant R_j - s_{jl}$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | 1 | 49 | – | – | – | – |
| $s'_{jt}$ | – | – | – | – | 49 | 1 | – | – | – | – |

Case e: For $C_{r(l)} < R_j$ (or $\eta_{jl} = 0$), e.g., $C_{r(l)} = 30$:
if $R_j > s_{jl} \geqslant R_j - C_{r(l)}$ (or $v_{jl} = 0$ and $v''_{jl} = 1$),
then $f = l - 1$ and $a_{jf} \geqslant R_j - s_{jl}$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | 1 | 49 | – | – | – | |
| $s'_{jt}$ | – | – | – | – | 30 | 20 | – | – | – | |

Case f: For $C_{r(l)} < R_j$ (or $\eta_{jl} = 0$), e.g., $C_{r(l)} = 30$:
if $s_{jl} < R_j - C_{r(l)}$ (or $v''_{jl} = 0$),
then $f = l - 2$ and $a_{jf} \geqslant R_j - C_{r(l)} - s_{jl}$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | 1 | 30 | 19 | – | – | – | |
| $s'_{jt}$ | – | – | – | 19 | 30 | 1 | – | – | – | |

If the setup operation ends in a shorter period, then constraint (4.2) ensures that: $s_{jl} \leqslant C_l$; the recent shorter period is not the fifth period but an earlier one; because of assumption (5) holds $l - r(l) \geqslant 2$, i.e., the beginning of the set-up is determined as in the a or b cases.

## 2. Long set-up time ($Q_j \geqslant 1$ or $q_j = 1$)

In this section, all possible cases of the set-up process are presented for set-up times longer or equal to period length ($Q_j \geqslant 1$). All cases are illustrated by examples with $st_j = 350$; i.e., $Q_j = 3$, $R_j = 50$.

**For distance** $l - r(l) \geqslant Q_j + 2$:

Case A: If $s_{jl} \geqslant R_j$ (or $v_{jl} = 1$),
then $f = l - Q_j$ and $a_{jf} \geqslant R_j + \overline{C} - s_{jl}$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | – | – | 50 | 100 | 100 | 100 |
| $s'_{jt}$ | – | – | – | – | – | – | 100 | 100 | 100 | 50 |

Case B: If $s_{jl} < R_j$ (or $v_{jl} = 0$),
then $f = l - (Q_j + 1)$ and $a_{jf} \geqslant R_j - s_{jl}$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | – | 1 | 100 | 100 | 100 | 49 |
| $s'_{jt}$ | – | – | – | – | – | 49 | 100 | 100 | 100 | 1 |

**For distance** $r(l) - l = Q_j + 1$:

Case C: If $s_{jl} \geqslant R_j$ (or $v_{jl} = 1$),
then $f = l - Q_j$ and $a_{jf} \geqslant R_j + \overline{C} - s_{jl}$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | – | 50 | 100 | 100 | 100 | – |
| $s'_{jt}$ | – | – | – | – | – | 100 | 100 | 100 | 50 | – |

Case D: For $C_{r(l)} \geqslant R_j$ (or $\eta_{jl} = 1$),
if $a_{jf} < R_j$ (or $v_{jl} = 0$),
then $f = l - (Q_j + 1)$ and $a_{jf} \geqslant R_j - s_{jl}$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | 1 | 100 | 100 | 100 | 49 | – |
| $s'_{jt}$ | – | – | – | – | 49 | 100 | 100 | 100 | 1 | – |

Case E: For $C_{r(l)} < R_j$ (or $\eta_{jl} = 0$),
if $R_j > s_{jl} \geqslant R_j - C_{r(l)}$ (or $v_{jl} = 0$ and $v''_{jl} = 1$ ),
then $f = l - (Q_j + 1)$ and $a_{jf} \geqslant R_j - s_{jl}$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | 1 | 100 | 100 | 100 | 49 | – |
| $s'_{jt}$ | – | – | – | – | 30 | 100 | 100 | 100 | 20 | – |

Case F: For $C_{r(l)} < R_j$ (or $\eta_{jl} = 0$),
if $s_{jl} < R_j - C_{r(l)}$ (or $v''_{jl} = 0$),
then $f = l - (Q_j + 2)$ and $a_{jf} \geqslant R_j - C_{r(l)} - s_{jl}$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | 1 | 30 | 100 | 100 | 100 | 19 | – |
| $s'_{jt}$ | – | – | – | 19 | 30 | 100 | 100 | 100 | 1 | – |

**For distance** $l - r(l) = Q_j$:

Case G: For $C_{r(l)} \geqslant R_j$ (or $\eta_{jl} = 1$),
if $s_{jl} \geqslant R_j + \overline{C} - C_{r(l)}$ (or $v''_{jl} = 1$)
then $f = l - Q_j$ and $a_{jf} \geqslant R_j + \overline{C} - s_{jl}$

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | – | 50 | 100 | 100 | 100 | – | – |
| $s'_{jt}$ | – | – | – | – | 70 | 100 | 100 | 80 | – | – |

Case H: For $C_{r(l)} \geqslant R_j$ (or $\eta_{jl} = 1$),
if $s_{jl} < R_j + \overline{C} - C_{r(l)}$ (or $v''_{jl} = 0$)
then $f = l - (Q_j + 1)$ and $a_{jf} \geqslant R_j + \overline{C} - C_{r(l)} - s_{jl}$

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | 1 | 70 | 100 | 100 | 79 | – | – |
| $s'_{jt}$ | – | – | – | 79 | 70 | 100 | 100 | 1 | – | – |

Case K: For $C_{r(l)} < R_j$ (or $\eta_{jl} = 0$),
if $s_{jl} \geqslant R_j - C_{r(l)}$ (or $v''_{jl} = 1$)
then $f = l - (Q_j + 1)$ and $a_{jf} \geqslant R_j + \overline{C} - C_{r(l)} - s_{jl}$.

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | 20 | 30 | 100 | 100 | 100 | – | – |
| $s'_{jt}$ | – | – | – | 100 | 30 | 100 | 100 | 20 | – | – |

Case L: For $C_{r(l)} < R_j$ (or $\eta_{jl} = 0$),
if $s_{jl} < R_j - C_{r(l)}$ (or $v''_{jl} = 0$)
then $f = l - (Q_j + 2)$ and $a_{jf} \geqslant R_j - C_{r(l)} - s_{jl}$

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | 1 | 100 | 30 | 100 | 100 | 19 | – | – |
| $s'_{jt}$ | – | – | 19 | 100 | 30 | 100 | 100 | 1 | – | – |

**For distance** $1 \leqslant l - r(l) < Q_j$:

Case M: For $C_{r(l)} \geqslant$ (or $\eta_{jl} = 1$),
if $s_{jl} \geqslant R_j + \overline{C} - C_{r(l)}$ (or $v''_{jl} = 1$)
then $f = l - Q_j$ and $a_{jf} \geqslant R_j + 2\overline{C} - C_{r(l)} - s_{jl}$

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | – | 80 | 70 | 100 | 100 | – | – | – |
| $s'_{jt}$ | – | – | – | 100 | 70 | 100 | 80 | – | – | – |

Case N: For $C_{r(l)} \geqslant R_j$ (or $\eta_{jl} = 1$)
if $s_{jl} < R_j + \overline{C} - C_{r(l)}$ (or $v''_{jl} = 0$)
then $f = l - (Q_j + 1)$ and $a_{jf} \geqslant R_j + \overline{C} - C_{r(l)} - s_{jl}$

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | 0 | 100 | 70 | 100 | 80 | – | – | – |
| $s'_{jt}$ | – | – | 79 | 100 | 70 | 100 | 1 | – | – | – |

Case O: For $C_{r(l)} < R_j$ (or $\eta_{jl} = 0$)
if $s_{jl} \geqslant R_j - C_{r(l)}$ (or $v''_{jl} = 1$)
then $f = l - (Q_j + 1)$ and $a_{jf} \geqslant R_j + \overline{C} - C_{r(l)} - s_{jl}$

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | – | 20 | 100 | 30 | 100 | 100 | – | – | – |
| $s'_{jt}$ | – | – | 100 | 100 | 30 | 100 | 20 | – | – | – |

Case P: For $C_{r(l)} < R_j$ (or $\eta_{jl} = 0$)
if $s_{jl} < R_j - C_{r(l)}$ (or $v''_{jl} = 0$)
then is $f = l - (Q_j + 2)$ and $a_{jf} \geqslant R_j - C_{r(l)} - s_{jl}$

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s'_{jt}$ | – | 1 | 100 | 100 | 30 | 100 | 19 | – | – | – |
| $s'_{jt}$ | – | 19 | 100 | 100 | 30 | 100 | 1 | – | – | – |

## References

[1] I. Barany, T.J. van Roy and L.A. Wolsey: Uncapacitated lot-sizing: The convex hull of solutions. *Mathematical Programming Studies*, **22** (1984), 32–43, DOI: 10.1007/BFb0121006.

[2] G. Belvaux and L.A. Wolsey: Modelling practical lot-sizing problems as mixed-integer programs. *Management Science*, **47**(7), (2001), 993–1007, DOI: 10.1287/mnsc.47.7.993.9800.

[3] J.D. Blocher, S. Chand and K. Sengupta: The changeover scheduling problem with time and cost considerations: Analytical results and a forward algorithm. *Operations Research*, **47**(7), (1999), 559-569, DOI: 10.1287/opre.47.4.559.

[4] W. Bożejko, M. Uchroński and M. Wodecki: Multi-machine scheduling problem with setup times. *Archives of Control Sciences*, **22**(4), (2012), 441–449, DOI: 10.2478/v10170-011-0034-y.

[5] W. Bożejko, A. Gnatowski, R. Idzikowski and M. Wodecki: Cyclic flow shop scheduling problem with two-machine cells. *Archives of Control Sciences*, **27**(2), (2017), 151–167, DOI: 10.1515/acsc-2017-0009.

[6] D. Cattrysse, M. Salomon, R. Kuik and L. van Wassenhove: A dual ascent and column generation heuristic for the discrete lotsizing and schedul-

ing problem with setup times. *Management Science*, **39**(4), (1993), 477–486, DOI: 10.1287/mnsc.39.4.477.

[7] K. Copil, M. Wörbelauer, H. Meyr and H. Tempelmeier: Simultaneous lotsizing and scheduling problems: a classification and review of models. *OR Spectrum*, **39**(1), (2017), 1–64, DOI: 10.1007/s00291-015-0429-4.

[8] A. Drexl and K. Haase: Proportional lotsizing and scheduling. *International Journal of Production Economics*, **40**(1), (1995), 73–87, DOI: 10.1016/0925-5273(95)00040-U.

[9] B. Fleischmann: The discrete lot-sizing and scheduling problem. *European Journal of Operational Research*, **44**(3), (1990), 337-348, DOI: 10.1016/0377-2217(90)90245-7.

[10] K. Haase: *Lotsizing and scheduling for production planning*. Number 408 in Lecture Notes in Economics and Mathematical Systems. Springer-Verlag, Berlin, 1994.

[11] W. Kaczmarczyk: Inventory cost settings in small bucket lot-sizing and scheduling models. In *Total Logistic Management Conference*, Zakopane, Poland, November 25-28 2009.

[12] W. Kaczmarczyk: Modelling multi-period set-up times in the proportional lot-sizing problem. *Decision Making in Manufacturing and Services*, **3**(1-2), (2009), 15–35, DOI: 10.7494/dmms.2009.3.2.15.

[13] W. Kaczmarczyk: Proportional lot-sizing and scheduling problem with identical parallel machines. *International Journal of Production Research*, **49**(9), (2011), 2605–2623, DOI: 10.1080/00207543.2010.532929.

[14] W. Kaczmarczyk: Valid inequalities for proportional lot-sizing and scheduling problem with fictitious microperiods. *International Journal of Production Economics*, **219**(1), (2020), 236–247, DOI: 10.1016/j.ijpe.2019.06.005.

[15] W. Kaczmarczyk: Explicit modelling of multi-period setup times in proportional lot-sizing problem with constant capacity. (2021), Preprint available at Research Square, DOI: 10.21203/rs.3.rs-1086310/v1.

[16] U.S. Karmarkar and L. Schrage: The deterministic dynamic product cycling problem. *Operations Research*, **33**(2), (1985), 326–345, DOI: 10.1287/opre.33.2.326.

[17] A. Kimms and A. Drexl: Proportional lot sizing and scheduling: Some extensions. *Networks*, **32**(2), (1998), 85–101, DOI: https://doi.org/10.1002/(SICI)1097-0037(199809)32:2<85::AID-NET2>3.0.CO;2-E.

[18] J. Krystek and M. Kozik: Analysis of the job shop system with transport and setup times in deadlock-free operating conditions. *Archives of Control Sciences*, **22**(4), (2012), 417–425, DOI: 10.2478/v10170-011-0032-0.

[19] Y. Pochet and L.A. Wolsey: *Production planning by mixed integer programming*. Series in Operations Research and Financial Engineering. Springer, New York, 2006.

[20] J. Rudy, J. Pempera and Cz. Smutnicki: Improving the TSAB algorithm through parallel computing. *Archives of Control Sciences*, **30**(2), (2020), 411–435, DOI: 10.24425/acs.2020.134672.

[21] C. Suerie: Modeling of period overlapping setup times. *European Journal of Operational Research*, **174**(2), (2006), 874–886, DOI: 10.1016/j.ejor.2005.03.033.

[22] H. P. Williams. *Model building in Mathematical Programming*. Wiley, 2006.