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Research paper

An efficient iteration procedure for form finding of slack cables under concentrated forces

Igor Orynyak [1], Federico Guarracino[2], Mariano Modano[3], Roman Mazuryk[4]

Abstract: The goal of paper is the development and demonstration of efficiency of algorithm for form finding of a slack cable notwithstanding of the initial position chosen. This algorithm is based on product of two sets of coefficients, which restrict the rate of looking for cable geometry changes at each iteration. The first set restricts the maximum allowable change of absolute values of positions, angles and axial forces. The second set takes into account whether the process is the converging one (the signs of maximal change of parameters remain the same), so that it increases the allowable changes; or it is a diverging one, so that these changes are discarded. The proposed procedure is applied to two different methods of simple slack cable calculation under a number of concentrated forces. The first one is a typical finite element method, with the cable considered as consisting of number of straight elements, with unknown positions of their ends, and it is essentially an absolute coordinate method. The second method is a typical Irvine's like analytical solution, which presents only two unknowns at the initial point of the cable; due to the peculiarity of implementation it is named here a shooting method. Convergence process is investigated for both solutions for arbitrary chosen, even very illogical initial positions for the ACM, and for angle and force at the left end for SM as well. Even if both methods provide the same correct convergent results, it is found that the ACM requires a much lower number of iterations.

Keywords: concentrated force, cable, iteration procedure, shooting method, absolute coordinate method, accuracy

[1] Prof., Department of Applied Mathematics at National Technical University Kiev Polytechnic Institute, Peremohy str, 37, Kyiv 03056, Ukraine, e-mail: igor_orinyak@yahoo.com, ORCID: 0000-0003-4529-0235
[2] Prof., Department of Structural Engineering at University of Naples "Federico II", via Claudio, 21-80125 Napoli, Italy, e-mail: fguarrac@unina.it, ORCID: 0000-0001-7017-5170
[3] Associate Prof., Department of Structural Engineering at University of Naples "Federico II", via Claudio, 21-80125 Napoli, Italy, e-mail: modano@unina.it, ORCID: 0000-0003-0619-3360
[4] PhD. student, Department of Applied Mathematics at National Technical University Kiev Polytechnic Institute, Peremohy str, 37, Kyiv 03056, Ukraine, e-mail: r.mazuryk.ua@gmail.com, ORCID: 0000-0003-4309-824X

1. Introduction

Analysis of cables is a strongly nonlinear problem from a geometrical standpoint and requires a careful formulation of the element, as well as a robust and efficient algorithm for the solution of assemblages of cables. In most case there are no clear borders between these problems, because the type of element might demand a certain organization of the numerical scheme. So, it is hardly to judge which formulation can be more efficient and contribute mostly to the overall efficiency of analysis.

When the cable weight can be neglected, the behavior of a system of cables reduces to a set of straight lines which are connected under some angle at the point of force application. The governing equations for the isolated cable under concentrated loads are very simple. The first one is equilibrium equation:

$$\vec{T}_{e,i} = \vec{T}_{b,i+1} + \vec{F}^{i,i+1} \tag{1.1}$$

where $\vec{T}_{e,i}$ and $\vec{T}_{b,i+1}$ are the tangent forces at the end (index e) of the i-th section, and at the beginning (index b) of the (i+1)-th section, and $\vec{F}^{i,i+1}$ is the outer force applied between them. The second equation is relationship between the elongation and tangent force applied to each element:

$$\mathrm{d}s_i = \left(1 + \left|\vec{T}_i\right| / k_i\right) \mathrm{d}l_i \tag{1.2}$$

$\mathrm{d}s_i$, $\mathrm{d}l_i$ are the final and the initial lengths of the i-th element and k_i is its stiffness. These two equations represent, in fact, the basis for the theory of cables under concentrated forces. They are supplemented by boundary conditions and possible restrictions.

There are very few works which are devoted to the investigation of the actions of concentrated forces. Most of them take origin from the work of Irvine and Sinclair [1, 2]. They have shown that a cable element can be described by two independent unknowns at its origin (beginning), e.g., the horizontal and vertical components of the axial force or equivalently, by the value of axial force and its angle of inclination. In this sense the approach to concentrated forces is identical to the scheme of solution for the catenary equation (uniformly distributed loading) initially proposed in [3, 4]. The generalization for cable in space and for uniformly distributed load generically oriented in space was given in [5, 6], which requires to choose three (instead of two) trial unknowns at the beginning.

The simplicity of governing equations and their identity with those of truss element pose no problem with mathematical formulation of element and, in fact, guarantee the overall accuracy of solution, provided it converges and boundary conditions are fulfilled. So, the main question is whether or not the solution converges. Two choices, which are concerned with the algorithm, are “responsible” for the convergence. First is the choice of initial (trial) geometry (two initial parameters at the beginning for 2D and three- for 3D problem); second is the form finding procedure.

Starting from the seminal work of Irvine and Sinclair [1] the modified Newton or Newton Raphson [7] methods are used. But it is known [5] that Newton’s method will converge only if the starting values are sufficiently close to the actual solution of the problem. So, the trial initial geometry becomes the central problem in any numerical implementation. For one cable systems the trial solution can usually be obtained graphically [8]. The problem becomes more complex for a slack cable, whose initial length is much larger than the distance between the fixed end points [9, 10]. Therefore, the common approach is to start from a tensed geometry and iteratively adjust the position of either of supports to attain the correct position. This is generally done in combination with the Newton Raphson method [10]. Also, it is possible to artificially reduce the initial length of cable and successively increase its length iteratively.

All these methods are subjective and depend on choice of the initial configuration. Of course, for one cable systems the problem is not complicated because a guess can always be made about the initial position. The problem arises for a net of cables, where it is hardly to envisage the plausible initial geometry, including prestress. So, the cable net geometry determination can be a complex procedure, which requires first a form finding for separate elements, provided that their end positions are known and the internal forces are in equilibrium with the applied load and, successively, an adjustment of the position of mutual points of the system and a form finding for separate cables, iteratively [11].

The goal of the present study is the development of a robust form finding algorithm regardless the initial position of the single slack cable. The algorithm proposed in [12] and elaborated in [13, 14] for geometrically nonlinear problems is used. It is based on the notion of basic geometry and correction geometry, where the correction is used to refine the basic geometry. The coefficient of corrections depends on whether the solution converge or diverge.

Two analytical approaches are developed. The first approach considers as unknowns the position of each point of force application. This method resembles the absolute coordinate method [15, 16], and can be effective for the solution of systems with many cables, some points of which are connected.

The second method is derived from the classical Irvine's solution, with the difference that the initial values are updated on the basis of analytically (not numerically) determined derivatives for the positions of end points. Just because the implementation of this method resembles the general idea of the universal Shooting Method, such a name will be retained here [17].

The case study is that of a single slack cable under the action of several vertical forces. This geometry allows us to investigate the effectiveness of the different approaches with respect to the refinement of the geometry after each iteration.

2. Absolute coordinate method

2.1. Designations and basic equation

The equations are written in the form which will be used in iteration process. For convenience make the following identification. Consider the cable of initial length L_0, and account for n points of force applications on the cable. So, at whole, including the end boundaries, it is $n+2$ points, which are numerated as $i = 0, 1, 2, \ldots, n{+}1$, where points and $n+1$ are the boundary ones. The positions of these points are designated through the progressive lengths as $l_1, l_2, \ldots, l_i, \ldots, l_{n+1}$, where, of course, $l_{n+1} = L_0$. So, the i section (element) lies between points $i-1$ and i. We use the following designation for initial length at each section $s_{0,1} = l_1 - 0$, $s_{0,2} = l_2 - l_1$, $s_{0,n+1} = l_{n+1} - l_n$.

Formulate the boundary conditions. With no restrictions on the generality assume that at left boundary the cable is attached to point with initial position $B(0,0)$, and at right boundary – to the point $E\ (X_L, Y_L)$, Fig. 1. For convenience connect points B and E by a straight line and determine the geometrical length of vector $\overrightarrow{EB} = \vec{E} - \vec{B}$. Let assume that distance $\left|\overrightarrow{EB}\right| = L$.

$$\overrightarrow{EB} = \left(\vec{i}\sin\beta_0 + \vec{j}\cos\beta_0\right)L = \vec{v}L = \left(\vec{i}X_L + \vec{j}Y_L\right) \tag{2.1}$$

where β_0 is angle of inclination between the boundary points.

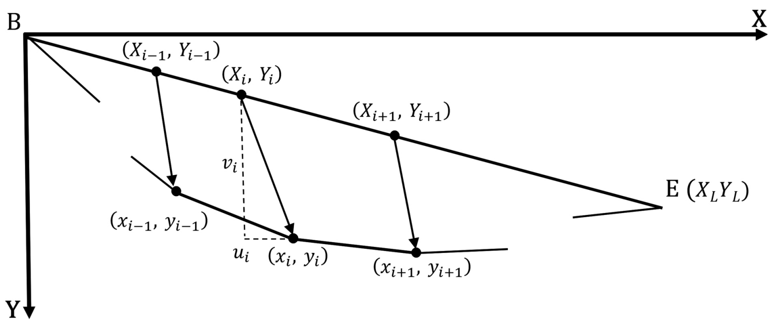


Fig. 1. Initial and deformed positions of force application points

Introduce four governing parameters for each point of the cable. They are: (scalar) tension force T, angle of inclination to the horizontal line α, horizontal position x and vertical position y. It is more convenient to relate the last two values to some initial reference position, connected to the vector $\overrightarrow{EB}$. So, define the reference position as:

$$\vec{g}_i = \vec{v}g_i = \left(\vec{i}X_i + \vec{j}Y_i\right) \tag{2.2}$$

Here we introduced some reference position of force application points, $\vec{i}X_i + \vec{j}Y_i$, with respect to the boundary points. Write down two equations of equilibrium (vertical and horizontal directions) at any points of force application:

$$T_i\sin\alpha_i - T_{i+1}\sin\alpha_{i+1} = F_i \qquad T_i\cos\alpha_i - T_{i+1}\cos\alpha_{i+1} = Q_i \tag{2.3}$$

Assume that we have some "guess in" preliminary geometry after j iteration. It is characterized by values of additional displacements u^j and v^j, where upper index means the iteration number. So, the positions of points on horizontal axis, x_i^j and vertical, y_i^j, axes, are:

$$x_i^j = X_i + u_i^j \qquad y_i^j = Y_i + v_i^j \tag{2.4}$$

Designate the length of each section (element) as s_i^j:

$$s_i^j = \sqrt{\left(x_i^j - x_{i-1}^j\right)^2 + \left(y_i^j - y_{i-1}^j\right)^2} \tag{2.5}$$

Assume that at next iteration step these values need to be refined. The incremental values are:

$$v^{j+1} = v^j + \varepsilon^{j+1} \qquad u^{j+1} = u^j + \gamma^{j+1} \tag{2.6}$$

where ε^{j+1} and γ^{j+1} are the correction displacements, the determination of which is the goal of each iteration. There is no need to keep the upper indexes for each iteration. Starting from here present the search for ultimate solution at the next iteration $\vec{U}^{j+1}$ as the sum of basic solution at the previous iteration $\vec{B}^j$ and the correction one $\vec{C}^{j+1}$, i.e.:

$$\vec{U}^{j+1} = \vec{B}^j + \vec{C}^{j+1} \tag{2.7}$$

Designations s_i, x_i, y_i, u_i, v_i pertain to basic solutions. Designations ε_i and γ_i pertain to correction solution. It is implicitly assumed that their values are much smaller than x_i and y_i. Present the expressions for the Ultimate lengths at the next iteration through their projections:

$$s_i^U = \sqrt{(x_i - x_{i-1} + \gamma_i - \gamma_{i-1})^2 + (y_i - y_{i-1} + \varepsilon_i - \varepsilon_{i-1})^2} \tag{2.8}$$

Evidently expressions for sinus and cosines functions from the angles of inclination are given by:

$$\sin\alpha_i^{j+1} = \frac{y_i + \varepsilon_i - \varepsilon_{i-1}}{s_i^U} \qquad \cos\alpha_i^{j+1} = \frac{x_i - x_{i-1} + \gamma_i - \gamma_{i-1}}{s_i^U} \tag{2.9}$$

Then according to physical relation between the tangent force and increment of lengths of the element, for the simplest case of linearly elastic material according to (1.2) it can be written:

$$T_i = k_i \frac{(s_i^U - l_i)}{l_i} \tag{2.10}$$

By inserting (2.8)–(2.10) into (2.3), we get two governing equations:

$$k_i \frac{s_i^U - l_i}{l_i} \frac{y_i - y_{i-1} + \varepsilon_i - \varepsilon_{i-1}}{s_i^U} - k_{i+1} \frac{s_{i+1}^U - l_{i+1}}{l_{i+1}} \frac{y_{i+1} - y_i + \varepsilon_{i+1} - \varepsilon_i}{s_{i+1}^U} = F_i \tag{2.11}$$

$$k_i \frac{s_i^U - l_i}{l_i} \frac{x_i - x_{i-1} + \gamma_i - \gamma_{i-1}}{s_i^U} - k_{i+1} \frac{s_{i+1}^U - l_{i+1}}{l_{i+1}} \frac{x_{i+1} - x_i + \gamma_{i+1} - \gamma_i}{s_{i+1}^U} = Q_i \tag{2.12}$$

Once all preliminary positions are set, the solution of the problem can be pursued. For simplicity assume that $k_i = k_{i+1} = k$, which allows us to simplify equilibrium expressions.

2.2. Linearization

There are n points with two unknown parameters (horizontal and vertical positions), so at each iteration we have $2n$ unknowns, represented by two sets of ε and γ. There are also n equations (2.11) and n equations (2.12). Thus, the number of unknowns and equations are the same.

The proposed strategy for solution consists in linearizing expressions (2.8), (2.9) and (2.10) with respect to ε and γ. Then these linearized expressions are substituted to (2.11) and (2.12) with subsequent linearization (i.e., by neglecting the products of ε and γ).

Retaining in expression for s_i^U (2.8) only the values of first order of smallness, it can be rewritten as:

$$s_i^U = s_i + (\gamma_i - \gamma_{i-1}) \frac{(x_i - x_{i-1})}{s_i} + (\varepsilon_i - \varepsilon_{i-1}) \frac{(y_i - y_{i-1})}{s_i} \tag{2.13}$$

From (2.13) follows:

$$\frac{1}{s_i^U} = \frac{(s_i)^2 - (\gamma_i - \gamma_{i-1})(x_i - x_{i-1}) - (\varepsilon_i - \varepsilon_{i-1})(y_i - y_{i-1})}{(s_i)^3} \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.11), neglecting the products of small unknown values, and grouping the knowns and unknown variables, we get the simplified equation:

$$\varepsilon_{i-1}\left(-Z_{2,i}^j\right) + \varepsilon_i\left(Z_{2,i}^j - Z_{3,i}^j\right) + \varepsilon_{i+1}\left(Z_{3,i}^j\right) + \gamma_{i-1}\left(-Z_{4,i}^j\right) + \gamma_i\left(Z_{4,i}^j - Z_{5,i}^j\right) + \gamma_{i+1}\left(Z_{5,i}^j\right) = Z_{1,i}^j \tag{2.15}$$

where

$$Z_{1,i}^j = \left(\frac{1}{l_i} - \frac{1}{s_i}\right)(y_i - y_{i-1}) - \left(\frac{1}{l_{i+1}} - \frac{1}{s_{i+1}}\right)(y_{i+1} - y_i) - F_i/k \tag{2.16}$$

$$Z_{2,i}^j = -\left(\frac{1}{l_i} - \frac{1}{s_i} + \frac{(y_i - y_{i-1})^2}{(s_i)^3}\right) \qquad Z_{3,i}^j = \left(\frac{1}{l_{i+1}} - \frac{1}{s_{i+1}} + \frac{(y_{i+1} - y_i)^2}{(s_{i+1})^3}\right) = -Z_{2,i+1}^j \tag{2.17}$$

$$Z_{4,i}^j = \frac{-(y_i - y_{i-1})(x_i - x_{i-1})}{(s_i)^3} \qquad Z_{5,i}^j = \frac{(y_{i+1} - y_i)(x_{i+1} - x_i)}{(s_{i+1})^3} = -Z_{4,i+1}^j \tag{2.18}$$

Expression (2.15) gives a system of n equations with respect to the unknown ε_i, γ_i.

The second set of equations can be derived from (2.12) in similar way:

$$W_{1,i}^j = \varepsilon_{i-1}\left(W_{4,i}^j\right) + \varepsilon_i\left(W_{4,i}^j - W_{5,i}^j\right) + \varepsilon_{i+1}\left(W_{5,i}^j\right) + \gamma_{i-1}\left(W_{2,i}^j\right) + \gamma_i\left(W_{2,i}^j - W_{3,i}^j\right) + \gamma_{i+1}\left(W_{3,i}^j\right) \tag{2.19}$$

where the following designations are used:

$$W_{1,i}^j = \left(\frac{1}{l_i} - \frac{1}{s_i}\right)(x_i - x_{i-1}) - \left(\frac{1}{l_{i+1}} - \frac{1}{s_{i+1}}\right)(x_{i+1} - x_i) - Q_i/k \tag{2.20}$$

(2.21)
$$W^j_{2,i} = \left(\left(\frac{1}{l_i} - \frac{1}{s_i}\right) + \frac{(x_i - x_{i-1})^2}{(s_i)^3}\right)$$
$$W^j_{3,i} = \left(\left(\frac{1}{l_{i+1}} - \frac{1}{s_{i+1}}\right) + \frac{(x_{i+1} - x_i)^2}{(s_{i+1})^3}\right) = -W^j_{2,i+1}$$

(2.22)
$$W^j_{4,i} = \frac{(x_i - x_{i-1})\,(y_i - y_{i-1})}{(s_i)^3} \qquad W^j_{5,i} = \frac{(x_{i+1} - x_i)\,(y_{i+1} - y_i)}{(s_{i+1})^3} = -W^j_{4,i+1}$$

Thus, expression (2.19) provides the second set of equations.

The following boundary conditions should be accounted for in solving equations (2.15) and (2.19).

(2.23)
$$\varepsilon^j_0 = \varepsilon^j_{n+1} = \gamma^j_0 = \gamma^j_{n+1} = 0$$

2.3. Algorithm and refinement of basic geometry

Numerating unknowns as X_m, where $1 \le m \le 2n$ and $X_{2i-1} = \varepsilon_i$ and $X_{2i} = \gamma_i$, and considering both systems (2.15) and (2.19) at consequent number of i we get a system, characterized by a diagonal matrix.

Two peculiarities of solution should be mentioned. First one is related with the "switching on" of procedure. If at the initial iteration all v_i and u_i are taken to be zero, the matrix gives no solution, because all coefficients of matrix would be zero. So, an initial approximation is needed. It is not a problem, and any logical geometry can be taken, provided the length of it is greater than the physical length of cable. In any case the influence of initial geometry is the object of investigation here.

The second peculiarity relates to refining the basic values of v^{j+1} and u^{j+1} when the $j+1$ iteration is performed. In spite that before the iteration the values of ultimate parameters are the sum of the previous set and of the correction according to formula (2.6), it is not expedient to use the found ultimate as new basic parameters at the next iteration. Introduce here the notion of so-called retardation coefficient, μ, according to which the basic solution at the next iteration will be refined:

(2.24)
$$x^{j+1}_i = x^j_i + \mu\gamma_i \quad y^{j+1}_i = y^{j+1}_i + \mu\varepsilon_i \quad s^{j+1}_i = \sqrt{\left(x^{j+1}_i - x^{j+1}_{i-1}\right)^2 + \left(y^{j+1}_i - y^{j+1}_{i-1}\right)^2}$$

In Section 3 a special procedure for the automatic calculation of μ, based on results of the actual iterations, will be proposed.

3. Shooting method

3.1. Designations and basic equation

Here slightly different designations and indexes will be used. In fact, consider that l_i, stands for the name of element as well as its length, so that on the whole we have $i + 1$

elements. In the shooting method we will characterize each element by 8 characteristic parameters. They are four parameters at the beginning of each element – $X_{i,b}$, $Y_{i,b}$ coordinates of the initial point of the element; angle of inclination $\alpha_{i,b}$ and axial tension force $T_{i,b}$. Also, we have four parameters at the end of element, they are – $X_{i,e}$, $Y_{i,e}$ coordinates of the end point of the element; angle $\alpha_{i,e}$ and tension force $T_{i,e}$.

Parameters at the beginning are related to those at the end by the following nonlinear expressions:

$$X_{i,e} = X_{i,b} + s_i\left(T_{i,b}\right) \cdot \cos\alpha_{i,b} \qquad Y_{i,e} = Y_{i,b} + s_i\left(T_{i,b}\right) \cdot \sin\alpha_{i,b} \tag{3.1}$$

$$\alpha_{i,e} = \alpha_{i,b} \qquad T_{i,e} = T_{i,b} \tag{3.2}$$

where $s_i\left(T_{i,b}\right)$ – is the actual length of the element on account of the deformation from axial force:

$$s_i = \frac{\left(1 + T_{i,b}\right)}{l_i} \tag{3.3}$$

These are Connection equations between the beginning and the end of each element.

Now consider the relation of the end point of previous element with the initial point of next one. The positions of the last point of previous element are the same as the first point of the successive one:

$$X_{i+1,b} = X_{i,e} \qquad Y_{i+1,b} = Y_{i,e}. \tag{3.4}$$

These two conjugation equations (relations between two neighboring elements) need to be supplemented by force equations. So, rewrite the equilibrium eqs. (2.3):

$$T_{i,e}\sin\alpha_{i,e} - F_{i,i+1} = T_{i+1,b}\sin\alpha_{i+1,b} \qquad T_{i,e}\cos\alpha_{i,e} - Q_{i,i+1} = T_{i+1,b}\cos\alpha_{i+1,b} \tag{3.5}$$

Here the indexes for the applied forces, $F_{i,i+1}$, $Q_{i,i+1}$, are changed (as compared with ACM) to underline that they are applied at the border between two elements. By squaring both equations (3.4) and (3.10) and adding them, we get:

$$T_{i+1,b} = \sqrt{\left(T_{i,e}\right)^2 - 2T_{i,e}F_{i,i+1}\sin\alpha_{i,e} - 2T_{i,e}Q_{i,i+1}\cos\alpha_{i,e} + \left(F_{i,i+1}\right)^2 + \left(Q_{i,i+1}\right)^2} \tag{3.6}$$

Then, we consequently obtain:

$$\cos\alpha_{i+1,b} = \frac{T_{i,e}\cos\alpha_{i,e} - Q_{i,i+1}}{T_{i+1,b}} \qquad \sin\alpha_{i+1,b} = \frac{T_{i,e}\sin\alpha_{i,e} - F_{i,i+1}}{T_{i+1,b}} \tag{3.7}$$

Values of angle and force at the end of previous element allows us to find them at beginning of the next one. So, equations (3.4) with (3.6)–(3.7) allow to determine all four parameters at the beginning of the next element on the basis of those at the end of the previous one.

This analysis shows that knowledge of the correct four values at the beginning of first section allows us to find the correct ones at any other point of any other section. Now we are ready to formulate an algorithm of solution. Yet we need to rewrite four Connection and four Conjugation equations in a linearized (differential) form.

3.2. Linearized governing equations

Suppose that we know some approximate solution (name it as a Probe one) for the whole cable at the j iteration. It means that we know the four initial parameters at the beginning of the first element, and also the actual length of the element according to (3.3). Then, by applying the Connection and Conjugation equations, we are able to get all four parameters at the end and beginning of each element, including for the end of the last element. For this solution we introduce the special upper index j. The positions of the last points, i.e., $X^j_{n+1,b}$, $Y^j_{n+1,e}$ in the probe solution may differ from the prescribed boundary conditions, which we designate as X_L and Y_L. Thus, it is necessary to set up a procedure of correction of the Probe solution.

Suppose that the values of all four parameters at the beginning of element i have been changed by small values, namely: $\Delta X_{i,e}$, $\Delta Y_{i,e}$, $\Delta\theta_{i,e}$, $\Delta T_{i,e}$ and determine how this influences the changes of them at the end of this element. In other words, write the Connection equations in a differential linearized form. Note that according to (3.3), the change of axial force leads to a change in length, so write:

$$\Delta s_i = \left(\Delta T_{i,e}/k_i\right) l_i \tag{3.8}$$

Then from (3.1)–(3.2) we can get the equations for their change

$$\Delta X_{i,e} = \Delta X_{i,b} + \left(\Delta T_{i,e}/k_i\right)\cos\alpha_{i,b} - \Delta\alpha_{i,b} s_i \left(T_{i,b}\right)\sin\alpha_{i,b} \tag{3.9}$$

$$\Delta Y_{i,e} = \Delta Y_{i,b} + \left(\Delta T_{i,e}/k_i\right)\sin\alpha_{i,b} + \Delta\alpha_{i,b} s_i \left(T_{i,b}\right)\cos\alpha_{i,b} \tag{3.10}$$

$$\Delta\alpha_{i,e} = \Delta\alpha_{i,b} \qquad \Delta T_{i,e} = \Delta T_{i,b} \tag{3.11}$$

Or in matrix form

$$\Delta\vec{W}_{i,e} = \lceil A_i \rceil \Delta\vec{W}_{i,b} \tag{3.12}$$

where $\Delta\vec{W}_i$ = column $(\Delta X_i, \Delta Y_i, \Delta\theta_i, \Delta N_i)$, and elements of matrix $\lceil A_i \rceil$ are given by equations (3.9)–(3.11). Thus, the matrix (3.12) is one of two key matrixes which is needed for refining the probe solution.

Another key linearized matrix stems from the Conjugation equations. Assume that we know all four parameters at the end of i element, $\Delta\vec{W}_{i,e}$ and find them at the beginning of the next one. According to equations (3.4) we can write:

$$\Delta X_{i+1,b} = \Delta X_{i,e} \qquad \Delta Y_{i+1,b} = \Delta Y_{i,e} \tag{3.13}$$

Equations for the derivatives of forces parameters (force and angle) are obtained from (3.9)–(3.10):

$$T_{i+1,b} + \Delta T_{i+1,b} \approx \sqrt{Y}_{0,0}\left(1 + \frac{\Delta T_{i,e} Y_{1,0} + \Delta\alpha_{i,e} Y_{2,0}}{Y_{0,0}}\right) \tag{3.14}$$

where

$$Y_{0,0} = \left(T_{i,e}\right)^2 + \left(F_{i,i+1}\right)^2 + \left(Q_{i,i+1}\right)^2 - 2T_{i,e}\left(F_{i,i+1}\sin\alpha_{i,e} + Q_{i,i+1}\cos\alpha_{i,e}\right) \tag{3.15}$$

$$(3.16)\qquad \begin{aligned} Y_{1,0} &= T_{i,e} - \left(F_{i,i+1}\sin\alpha_{i,e} + Q_{i,i+1}\cos\alpha_{i,e}\right)\\ Y_{2,0} &= -T_{i,e}\left(F_{i,i+1}\cos\alpha_{i,e} - Q_{i,i+1}\sin\alpha_{i,e}\right)\end{aligned}$$

Keeping in mind that $T_{i+1,b} = \sqrt{Y}_{0,0}$, according to (3.6), we can rewrite (3.14), which give us the following first force equation:

$$(3.17)\qquad \Delta T_{i+1,b} = \left(\frac{\Delta T_{i,e}Y_{1,0} + \Delta\alpha_{i,e}Y_{2,0}}{T_{i+1,b}}\right) = \Delta T_{i,e}Y_1 + \Delta\alpha_{i,e}Y_2$$

The second force equation for gains can be derived as follows. From (3.7) we get:

$$(3.18)\qquad \tan\left(\alpha_{i+1,b} + \Delta\alpha_{i+1,b}\right) = \frac{T_{i,e}\sin\alpha_{i,e} - F_{i,i+1} + \Delta T_{i,e}\cdot\sin\alpha_{i,e} + \Delta\alpha_{i,e}\cdot T_{i,e}\cos\alpha_{i,e}}{T_{i,e}\cos\alpha_{i,e} - Q_{i,i+1} + \Delta T_{i,e}\cdot\cos\alpha_{i,e} - \Delta\alpha_{i,e}\cdot T_{i,e}\sin\alpha_{i,e}}$$

Linearization of (3.18) with accounting for (3.6) gives the second governing equations as:

$$(3.19)\qquad \Delta\alpha_{i+1,b} = \Delta T_{i,e}Y_3 + \Delta\alpha_{i,e}\cdot Y_4$$

where:

$$(3.20)\qquad \begin{aligned} Y_3 &= \cos^2\alpha_{i+1,b}\frac{-\cos\alpha_{i,e}\tan\alpha_{i+1,b} + \sin\alpha_{i,e}}{T_{i,e}\cos\alpha_{i,e} - Q_{i,i+1}}\\ Y_4 &= \cos^2\alpha_{i+1,b}\frac{\sin\alpha_{i,e}\tan\alpha_{i+1,b} + \cos\alpha_{i,e}}{\cos\alpha_{i,e} - Q_{i,i+1}/T_{i,e}}\end{aligned}$$

Thus, equations (3.14), and (3.17) and (3.19) give the Conjugation equations for the changes (increments). In matrix form they can be presented as follows:

$$(3.21)\qquad \overrightarrow{\Delta W}_{i+1,b} = \lceil B_{i+1}\rceil\overrightarrow{\Delta W}_{i,e}$$

In conclusion of this section note, that we are able to determine all the four main parameters at any point provided that we know them at the first point of the first element. Thus, the state at the last point of last element is given by the following matrix expression:

$$(3.22)\qquad \overrightarrow{\Delta W}_{n+1,e} = \lceil A_{n+1}\rceil\left([B_{n+1}]\right)\lceil A_n\rceil\left([B_n]\right)\ldots\lceil A_2\rceil\left([B_2]\right)\lceil A_1\rceil\overrightarrow{\Delta W}_{1,b}$$

In particular, which is important for the implementation of the method, we can establish a relation between the increments at the beginning point of the first element and the increment of the position of the end point of the last element. Furthermore, the initial change is only the change of angle $\Delta\theta_{1,e}$ and on the change of force $\Delta N_{1,e}$. So, the main result of calculations (3.25) can be presented in form:

$$(3.23)\qquad \Delta X_{n+1,e} = f_{11}\Delta\theta_{1,e} + f_{12}\Delta N_{1,e} \qquad \Delta Y_{n+1,e} = f_{21}\Delta\theta_{1,e} + f_{22}\Delta N_{1,e}$$

where f_{km} are known from (3.22) coefficients.

3.3. Algorithm of solution

Suppose that for the $j-1$ iteration we know the basic length of each element, the four basic parameters at the beginning and at end of each element, and of course the basic positions of each element and point. Thus, the position of the last point $X^{j-1}_{n+1,b}$, $Y^{j-1}_{n+1,e}$ is known. This position may not to coincide with the prescribed boundary position of the last point. So, we need to change the initial angle and force at the beginning of first element on the small values $\Delta\theta_{1,e}$ and $\Delta N_{1,e}$, which can be found from the boundary conditions at the end point of last section. Thus, we get:

$$X^{j-1}_{n+1,b} + f_{11}\Delta\theta_{1,e} + f_{12}\Delta N_{1,e} = X_L \qquad Y^{j-1}_{n+1,b} + f_{21}\Delta\theta_{1,e} + f_{22}\Delta N_{1,e} = Y_L \tag{3.24}$$

So, two unknowns correction parameters $\Delta\theta_{1,e}$, $\Delta N_{1,e}$, can be found from eqs (3.24).

Next iteration requires prescribing all the basic parameters at the beginning of j iteration. In order to guarantee the convergence of the results, it is desirable to refine the solution slowly. So, the retardation coefficient μ will be used again (as in ACM) for refining the basic initial parameters:

$$\theta^{j}_{1,e} = \theta^{j-1}_{1,e} + \mu\,\Delta\theta_{1,e} \qquad N^{j}_{1,e} = N^{j-1}_{1,e} + \mu\,\Delta N_{1,e} \tag{3.25}$$

Availability of new $\theta^{j}_{1,e}$ and $N^{j}_{1,e}$ allows us to construct new basic geometry, and we can proceed with this iteration as described above.

4. Adjustment procedure for the retardation coefficient

In the following one of principal novelties of the present study will be outlined. In fact, this procedure: a) restricts the maximum allowable variation of main parameters; b) controls the process of convergence, by sharply decreasing (by a factor of 2) the allowable change of basic parameters when Correction parameters change their signs; and slowly increasing it (by a factor of 1.3), when Correction parameters keep the same signs. Both peculiarities are accounted for in determination of the above retardation coefficient μ.

Consider the choice of step for ACM. We have a solution at iteration j. Consider that from previous iteration we have the maximal axial force $T^{j}_{\max}$, the number of corresponding section, where it was attained i^{j}, and the generalized sign of its change, be it either "+", or "–", i.e. SIGN^{j} and the value of Rate parameter R^{j}. Determine at the given iteration $j+1$, the value of T^{j+1}_{i} (at the section of previous maxima), the maximum value $T^{j+1}_{\max}$ and the corresponding section number, where it is attained, say i^{j+1} . Then determine the value of SIGN^{j+1}

$$\mathrm{SIGN}^{j+1} = \begin{cases} +, & \text{if } T^{j+1}_{i} \geq T^{j}_{\max} \\ -, & \text{if } T^{j+1}_{i} < T^{j}_{\max} \end{cases} \tag{4.1}$$

and introduce the Rate Acceleration parameter at iteration $j+1$, i.e AR^{j+1}

$$\mathrm{AR}^{j+1} = \begin{cases} 1.3, & \text{if } \mathrm{SIGN}^{j} = \mathrm{SIGN}^{j+1} \\ 0.5, & \text{if } \mathrm{SIGN}^{j} \neq \mathrm{SIGN}^{j+1} \end{cases} \tag{4.2}$$

Now introduce the rate parameter:

$$R^{j+1} = \min \begin{cases} R^j \cdot \mathrm{AR}^{j+1} \\ 1 \end{cases} \tag{4.3}$$

At first iteration we consider that the previous $T^0_{\max}$ is and its sign is +, $R^0 = 1$. Then compare $T^{j+1}_{\max}$ and $T^j_{\max}$. The following relations between these quantities are imposed:

$$-0.4R^{j+1} \le \frac{T^{j+1}_{\max} - T^j_{\max}}{T^j_{\max}} \le 0.9R^{j+1} \tag{4.4}$$

If this requirement holds true, we take that Retardation coefficient μ to be $\mu = 1$. If $T^{j+1}_{\max} < T^j_{\max}$ and the left part of (4.4) is violated, then:

$$\mu = 0.4R^{j+1} \Bigg/ \left| \frac{T^{j+1}_{\max} - T^j_{\max}}{T^j_{\max}} \right| \tag{4.5}$$

If $T^{j+1}_{\max} > T^j_{\max}$ and the right part of (4.4) is violated, then

$$\mu = 0.9R^{j+1} \Bigg/ \left(\frac{T^{j+1}_{\max} - T^j_{\max}}{T^j_{\max}} \right) \tag{4.6}$$

Another requirement is that at any circumstance the axial force should not be negative. Introduce the notion $T^j_{\min}$ at previous iteration. So, make the requirement:

$$\frac{T^j_{\min} - T^{j+1}_{\min}}{T^j_{\min}} > 0.5 \tag{4.7}$$

If it is violated, then introduce an alternative value, μ_1, of retardation coefficient:

$$\mu_1 = \frac{0.5 \cdot T^j_{\min}}{T^j_{\min} - T^{j+1}_{\min}} \tag{4.8}$$

And we take as μ the min of μ and μ_1.

Similarly, the coefficient μ is found for SM. The difference is that we operate by $T_{1,b}$ and $\alpha_{1,b}$ only.

5. Examples of calculation

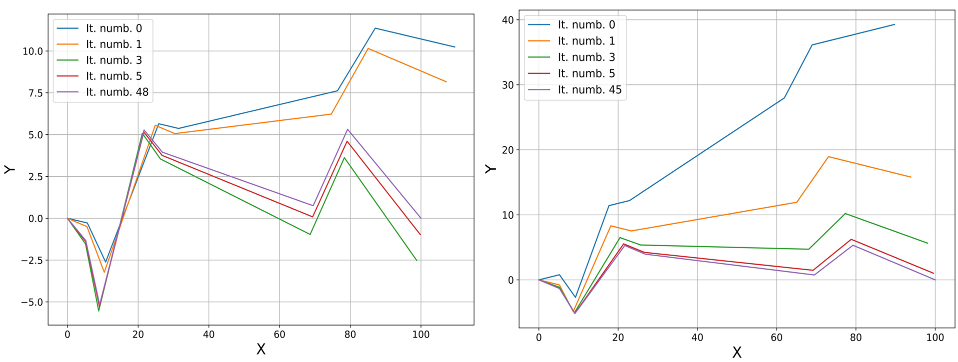
Here two examples for slack cable are considered: the first one with a gradual change of geometry (external forces act predominantly in the same direction); the second one with an abrupt change of geometry. The influence of the choice of initial position (geometry) on the convergence of results according to the proposed accelerated/decelerated procedure is investigated. Different initial positions are postulated, the only requirement for them being that the initial deformed length is larger than the length of cable. Of course, this restriction is not a very limiting one. Note, that all lengths and coordinates are given in meters, m, all forces are in Newtons, N, and stiffness is in Newtons, too.

5.1. Simple example 1

The length of cable is 100 m. Coordinates of the fixed points are: $B(0,0)$ and $E(X_e = 100,\ Y_e = 0)$. This means that physical length is equal to the geometrical distance between the fixed end points. The points where the vertical forces are applied are: $l_1 = 5$; $l_2 = 10$; $l_3 = 25$; $l_4 = 30$; $l_5 = 70$; $l_5 = 80$. The corresponding values of the vertical (only vertical ones are considered here) forces are: (–2000, 5000, –3000, 500, 1500, –2000). The sagging of the cable is given from application of the relatively small value of stiffness k, which is taken to be $k = 4 \cdot 10^4$ N.

Application of SM

Take the trial initial force and initial angle at left boundary equal to 5000 N, and 0.05 radians, respectively. Note that the actual exact values are about 2847 and 0.25. The development of geometry is shown on Fig. 2a. As it is shown, when the initial geometry isnot very far from the correct one the process of refinement of geometry proceeds very quickly, and at 4$^{\text{th}}$ iteration already resembles the correct one, but 28 iterations are needed to get the exact configuration. This means that the SM approach converges in a relatively slow manner.



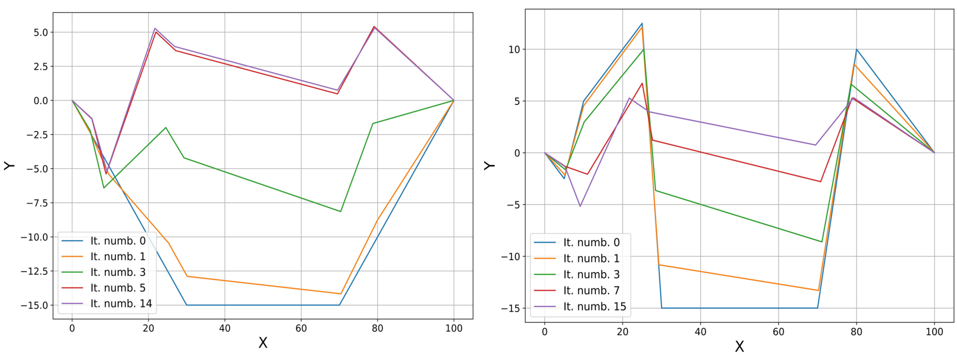
(a) (b)

Fig. 2. Development by SM of the trial geometry: a) initial force 5000 N, and an initial angle 0.05 rad, (b) initial force 2000 N, and an initial angle –0.15 rad

Another example is shown in Fig. 2b. Here the initial left boundary force and angle are equal to 2000 N and “minus” 0.15 radians, respectively. As in the former example they are not very far from the actual solution, but the initial angle is opposite to the correct one. Nevertheless, the results are very similar to the previous example, and starting from 4$^{\text{th}}$ iteration the geometry is already close to exact one. On the whole, the refinement process takes 30 iterations.

Application of ACM

The input data are the initial positions of the points. They are chosen randomly with only requirement that the geometrical length of cable is larger than its physical length. Here this condition is always fulfilled, because the physical length and distance between fixation points are the same, and the length of broken line is always longer than the length of straight line, which connect the first to the last point. Assume: $y_1 = 2.5$; $y_2 = 5$; $y_3 = 12$; $y_4 = 15$; $y_5 = 15$; $y_6 = 10$. This geometry seems almost specular to the actual solution. Nevertheless, the process of convergence is very fast, see Fig. 3a. The fifth iteration is rather close to the actual solution and the process is terminated at the 14$^{\text{th}}$ iteration.



(a) (b)

Fig. 3. Development by ACM of the geometry with iteration number: (a) first trial geometry, (b) "distant" initial geometry

Next set of input points are derived from the first one, with the positions of the third and of the sixth point presenting opposite sign. This geometry essentially differs from the actual one and seems rather "illogical". Nevertheless, the overall number of iterations remains the same and equal to 14, while at the 6$^{\text{th}}$ iteration the geometry is already close to exact one, Fig. 3b. This again proves the effectiveness of the method and its superiority with respect to the SM.

Application of both methods confirms the preliminary judgement that, if the method converges, it gives the correct result. To prove this, Table 1 shows the results attained at the termination of the iterative processes.

Table 1. Accuracy comparison between SM and ACM

Number of point i	Position x	Position y	Alfa	Tension, N
0	0 / 0	0 / 0	0.25090683 / 0.25090683	2847.101383 / 2847.101382
1	5.18818235 / 5.18818235	1.32977315 / 1.32977315	0.77605386 / 0.77605386	3864.392321 / 3864.392321
2	9.10134252 / 9.10134252	5.17047676 / 5.17047677	–0.69362907 / –0.69362907	3586.735800 / 3586.735798
3	21.66953811 / 21.66953811	–5.27941951 / –5.27941951	0.25090683 / 0.25090683	2847.101383 / 2847.101382
4	26.85772046 / 26.85772046	–3.94964636 / –3.94964636	0.07487392 / 0.07487392	2765.700789 / 2765.700788
5	69.50360276 / 69.50360276	–0.75060166 / –0.75060164	–0.43843304 / –0.43843304	3046.053932 / 3046.053931
6	79.24727059 / 79.24727059	–5.31909263 / –5.31909262	0.25090683 / 0.25090683	2847.101383 / 2847.101382
7	100 / 100	–0.00000004 / 0		

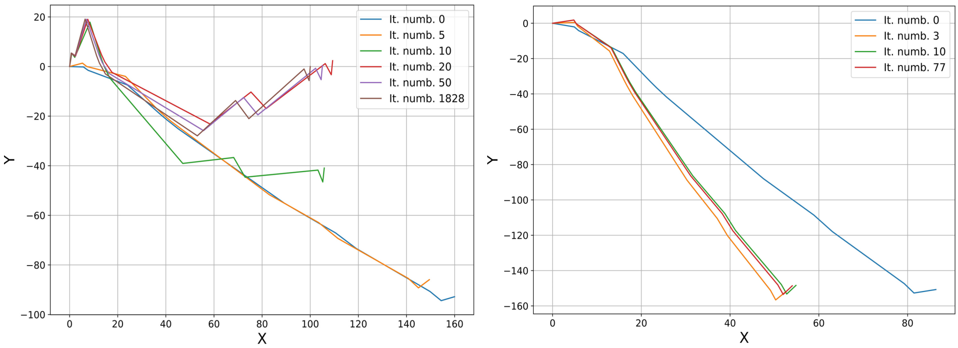
5.2. A more complex case: example

This case is more complicated. The physical length of cable is 160m. Coordinates of the fixed end points are: point B $(0, 0)$ and E $(100, 0)$. Here the physical length of the cable is much larger than the distance between the fixed end points. The eleven points of vertical forces application are (5, 7, 22, 37, 40, 45, 90, 111, 120, 150, 155). The corresponding values of the vertical forces are: (–4000, 2000, –3000, 250, 350, 500, 700, –1000, 990, –1400, 6500). The cable stiffness is $k = 4 \cdot 10^4$N.

SM application

The trial initial force and initial angle at left boundary are taken to be equal to 6000 N, and 0.03 radian, respectively. Note that exact ones are about 3633 N and –1,44 radians. Thus, the trial angle is rather different from the correct one. The development of geometry with iterations is shown in Fig. 4a. As the initial geometry isvery far from the correct one, the process of refinement develops slowly. The changes in the geometry are rather sharp and in right direction (towards the correct one) at the beginning, but successively develops very slowly. Therefore, although the 50th iteration was relatively close to the correct geometry, it took 858 iterations to get the correct one. This fact may cast some doubts as to the efficiency of the SM approach.

The results of next tests may appear somewhat unexpected. The initial trial angle is chosen to be 0.4, which is rather different from correct one, and an initial tension force is set to be equal to 2500. In this case the change of geometry proceeds very slowly due to the



(a) (b)

Fig. 4. Development of the geometry of example 2 by SM for initial values: (a) force equal to 6000 H, and angle to 0.03, b) force equal to 2500 N, and angle 0.4 – first 58 iterations

retardation coefficient and after 58$^{\text{th}}$ iteration it ceases to change at all. The reason is that the new calculated geometry at each new iteration deviates from basic one in a different direction. As a consequence, the Rate Acceleration coefficient is taken to be 0.5 according to Eq. (4.2) and this fact prevents the change of geometry.

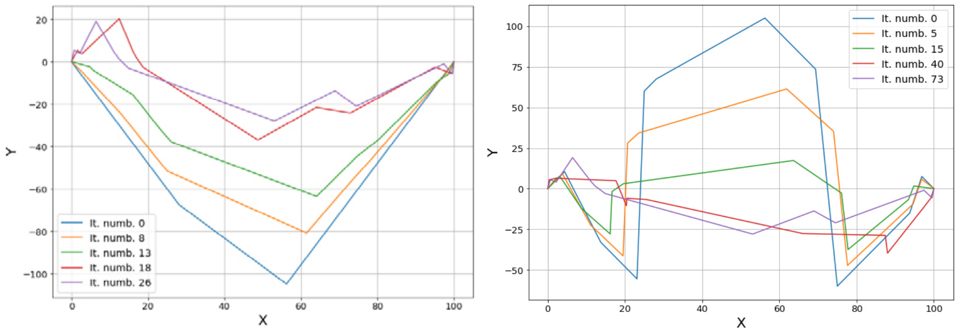
In our opinion, the reason in this divergence of results lays in the nature itself of the Shooting Method. Its simplicity and the small number of unknowns results in poor stability of complex tasks. Reference can be made to [17, 18], where the study of bending and elongation of a pipe due to interaction with ground shows a similar behavior.

ACM application

Input initial positions of points are chosen rather randomly, provided that overall length of the cable is larger than its physical length. They are $y_1 = 7.5$; $y_2 = 10.5$; $y_3 = 33$; $y_4 = 55.5$; $y_5 = 60$; $y_6 = 67.5$; $y_7 = 105$; $y_8 = 73.5$; $y_9 = 60$; $y_{10} = 15$; $y_{11} = 7.5$. In spite that initial geometry differs significantly from the correct one, the results obtained are very encouraging. They are shown is Fig. 5a. The convergence is very fast and the correct result is attained at the 26$^{\text{th}}$ iteration only.

Another set of initial points is derived from the first set by changing the signs at some points $y_1 = -7.5$; $y_2 = -10.5$; $y_3 = 33$; $y_4 = 55.5$; $y_5 = -60$; $y_6 = -67.5$; $y_7 = -105$; $y_8 = -73.5$; $y_9 = 60$; $y_{10} = 15$; $y_{11} = -7.5$. Evidently, this trial geometry is quite "illogical" and some convergence problems may be expected. As a result, the whole number of iterations is 73 – not an exceedingly high number, given that the cable is very slack, the number of forces is large, the initial geometry is "strange" and the axial stiffness of the cable is quite small.

As to comparison of the accuracy of both methods for the example 2. Table 2 gives the whole set of calculated results for this example. The all numbers were actually the same in case of convergence, so in Table 2. we do not mention the particular method or initial



(a) (b)

Fig. 5. Development of geometry of example 2 by ACM: (a) first trial geometry, (b) second trial geometry

geometry. The data confirms once again that both procedures converge to the exact solution and that the accuracy depends only on the termination condition.

Table 2. Calculated results for example 2 in case of convergence

Point num., i	x, m	y, m	angle	force, N	Point num., i	x, m	y, m	angle	force, N
0	0	0	–1.4461	3633.04	7	53.0760	27.9434	–0.7306	606.673
1	0.6783	5.4117	0.71859	600.273	8	68.9540	13.7177	0.92144	747.259
2	2.2064	4.0753	–1.2963	1667.22	9	74.4977	21.019	–0.7182	600.046
3	6.4411	19.115	1.25759	1466.52	10	97.4273	0.9839	1.14834	1102.06
4	11.232	4.3223	1.19498	1231.09	11	99.5338	5.6700	–1.4887	5513.37
5	12.367	1.44584	1.05406	914.585	12	100	0	–	–
6	14.893	–3.0007	0.57866	539.716	–	–	–	–	–

6. Conclusions

A new algorithm for the form finding of cables under concentrated forces has been presented with reference to two different methods: the shooting method, SM, which is the differential implementation of the Irvine's method [1] and the absolute coordinate method, ACM. The emphasis of paper is on investigation of influence of the initial trial form of cable on the convergence of results. Numerical examples with random initial configurations for a very slack cable have been analysed.

The goal of the present study is not the solution of any particular task, because any method would be able to cope with any problem, provided that the investigator can single out the convergency problem and change the initial geometry or the rules and coefficients of refinement of the geometry. The scope of the work has been to establish a user independent convergent procedure for a slack cable under concentrated forces, which can be extended to complex structures. The proposed ASM method combined with the original acceleration-retardation procedure of form refinement represents the result of the study. Limitations of the shooting method, on the contrary, have been pointed out even in presence of sophisticated form finding procedures. Overall:

1. Linearized equations in the framework of ACM, where positions of force application points were chosen as a set of governing unknowns, were derived.
2. The exact and linearized equations in the framework of SM, whose goal is to relate the positions and inner forces at each point with those at the previous one, have been derived analytically. This restricts the problem at each iteration to only 2 unknown values at the beginning of the cable.
3. An original iteration procedure of form finding has been implemented, which is based on: a) notions of maximal allowable changes for chosen parameters; and b) an acceleration-retardation procedure which takes into account whether the process converges (the solution moves in the same direction at two consecutive iterations) or diverges (maximums of parameters become smaller at next iteration). This procedure is alternative to the commonly applied Newton method.
4. ACM operates by a number of unknowns which is proportional to number of elements (outer loads). So, it is very flexible in accounting (reacting) for any deviances of initial form. Combined with the proposed retardation/acceleration procedure it exhibits an excellent convergence.

References

[1] H.M. Irvine, G.B. Sinclair, "The suspended elastic cable under the action of concentrated vertical loads", *International Journal of Solids and Structures*, 1975, vol. 12, no. 4, pp. 309–317, DOI: 10.1016/0020-7683(76)90080-9.

[2] H.M. Irvine, *Cable structures*. Cambridge, MIT Press, 1981, DOI: 10.1002/eqe.4290100213.

[3] W.T. O'Brien, "General solution of suspended cable Problems", *Journal of Structural Division* , 1967, vol. 94, no. 1, pp. 1–26, DOI: 10.1061/JSDEAG.0001574.

[4] H.B. Jayaraman, W.C. Knudson, "A Curved Element for the Analysis of Cable Structures", *Computers and Structures*, 1981, vol. 14, no. 3–4, pp. 325–333, DOI: 10.1016/0045-7949(81)90016-X.

[5] N. Impollonia, G. Ricciardi, F. Saitta, "Statics of elastic cables under 3D point forces", *International Journal of Solids and Structures*, 2011, vol. 48, no. 9, pp. 1268-1276, DOI: 10.1016/j.ijsolstr.2011.01.007.

[6] M.S.A. Abad, A. Shooshtari, V. Esmaeili, A.N. Riabi, "Nonlinear analysis of cable structures under general loadings", *Finite Elements in Analysis and Design*, 2013, vol. 73, pp. 11-19, DOI: 10.1016/j.finel.2013.05.002.

[7] Z. Chen, H. Cao, H. Zhu, "An Iterative Calculation Method for Suspension Bridge's Cable System Based on Exact Catenary Theory", *Baltic Journal of Road and Bridge Engineering*, 2013, vol. 8, no. 3, pp. 196–204, DOI: 10.3846/bjrbe.2013.25.

[8] D. Papini, "On shape control of cables under vertical static loads", Master's thesis, Lund University, 2010.

[9] J.H. Argyris, T. Angelopoulos, B. Bichat, "A general method for the shape finding of lightweight tension structures", *Computer Methods in Applied Mechanics and Engineering*, 1974, vol. 3, no. 1, pp. 135–149, DOI: 10.1016/0045-7825(74)90046-2.

[10] M. Crusells-Girona, F.C. Filippou, R.L. Taylor, "A mixed formulation for nonlinear analysis of cable structures", *Computers & Structures*, 2017, vol. 186, pp. 50-61, DOI: 10.1016/j.compstruc.2017.03.011.

[11] M. Ahmadizadeh, "Three-dimensional geometrically nonlinear analysis of slack cable structures", *Computers & Structures*, 2013, vol. 128, pp. 160-169, DOI: 10.1016/j.compstruc.2013.06.005.

[12] I. Orynyak, Z. Yaskovets, R. Mazuryk, "A novel numerical approach to the analysis of axial stress accumulation in pipelines subjected to mine subsidence", *Journal of Pipeline Systems Engineering and Practice*, 2019, vol. 10, no. 4, DOI: 10.1061/(ASCE)PS.1949-1204.0000405.

[13] I. Orynyak, R. Mazuryk, A. Orynyak, "Basic (discontinuous) and smoothing up (conjugated) solutions in transfer matrix method for static geometrically nonlinear beam and cable in plane", *Journal of Engineering Mechanics*, 2020, vol. 46, no. 5, DOI: 10.1061/(ASCE)EM.1943-7889.0001753.

[14] I. Orynyak, R. Mazuryk, "Application of method of discontinuous basic and enhanced smoothing solutions for 3D multibranched cable", *Engineering Structures*, 2022, vol. 251, Part B, DOI: 10.1016/j.engstruct.2021.113582.

[15] A.A. Shabana, Y.Y. Refaat, "Three Dimensional Absolute Nodal Coordinate Formulation for Beam Elements: Theory", *Journal of Mechanical Design,* 2001, vol. 123, no. 4, pp. 606-613, DOI: 10.1115/1.1410100.

[16] M. Tur, E. García, L. Baeza, F.J. Fuenmayor, "A 3D absolute nodal coordinate finite element model to compute the initial configuration of a railway catenary", *Engineering Structures,* 2014, vol. 71, pp. 234-243, DOI: 10.1016/j.engstruct.2014.04.015.

[17] F. Guarracino, V. Mallardo, "A refined analytical analysis of submerged pipelines in seabed laying", *Applied Ocean Research*, 1999, vol. 21, no. 6, pp. 281-293, DOI: 10.1016/S0141-1187(99)00020-6.

[18] Z.S. Yaskovets, I.V. Orynyak, "Application of the Method of Shooting for the Rapid Determination of the Stressed State of Underground Parts of Pipelines in the Zone Of Mine Works", *Materials Science*, 2019, vol. 54, no. 6, pp. 889–898, DOI: 10.1007/s11003-019-00277-0.

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