10.24425/acs.2022.142849

Archives of Control Sciences
Volume 32(LXVIII), 2022
No. 3*, pages 577–588*

Exponential decay of transient values in discrete-time positive nonlinear systems

Tadeusz KACZOREK and Andrzej RUSZEWSKI

Dedicated to Professor Andrzej Świerniak
on the occasion of his 70th birthday

The exponential decay of transient values in discrete-time nonlinear standard and fractional orders systems with linear positive linear part and positive feedbacks is investigated. Sufficient conditions for the exponential decay of transient values in this class of positive nonlinear systems are established. A procedure for computation of gains characterizing the class of nonlinear elements are given and illustrated on simple example.

Key words: exponential decay, transient value, discrete-time, fractional order, positive, nonlinear, feedback, system

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions [1, 2, 10, 13, 17, 21]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models and electrical circuits. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [1, 2, 11, 13].

Mathematical fundamentals of the fractional calculus are given in the monographs [11, 18, 19]. The positive fractional linear systems have been investigated

Authors are with Bialystok University of Technology, Faculty of Electrical Engineering, Wiejska 45D, 15-351 Białystok, Poland. Corresponding author is T. Kaczorek, e-mail: kaczorek@ee.pw.edu.pl

Received 07.05.2021.

in [3, 4, 6–13, 21]. Positive linear systems with different fractional orders have been addressed in [1, 9, 21]. Linear positive electrical circuits have been investigated in [13]. The global stability of nonlinear systems with positive feedbacks and positive stable linear parts has been investigated in [6–8, 14] and the stability of discrete-time systems with delays in [20]. The exponential decay of transient values in continuous-time positive nonlinear systems has been analyzed in [5].

In this paper the exponential decay of transient values in discrete-time nonlinear standard and fractional positive systems with positive feedbacks will be addressed.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning the positivity and stability of fractional orders linear systems are recalled. The asymptotic stability of interval fractional linear systems with interval matrices are considered in Section 3. The main results of the paper are given in Sections 4 and 5. In Section 4 sufficient conditions for the global stability and the exponential decay of transient values in the positive nonlinear systems are established and procedures for computation of the gains characterizing the class of characteristics of nonlinear elements are given. In Section 5 the results of Section 4 are extended to the fractional nonlinear positive systems. The considerations are illustrated by numerical examples. Concluding remarks are given in Section 6.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Fractional positive discrete-time linear systems

Consider the autonomous fractional discrete-time linear system

$$\Delta^\alpha x_{i+1} = Ax_i, \qquad 0<\alpha<1, \quad i\in Z_+, \tag{1}$$

where

$$\Delta^\alpha x_i = \sum_{j=1}^{i} c_j x_{i-j}, \tag{2a}$$

$$c_j = (-1)^j\binom{\alpha}{j}, \quad \binom{\alpha}{j} = \begin{cases} 1 & \text{for } j=0, \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for } j=1,2,\ldots \end{cases} \tag{2b}$$

is the fractional α-order difference of x_i and $x_i\in\Re^n$, $u_i\in\Re^m$ are the state and input vectors and $A\in\Re^{n\times n}$.

Substitution of (2) into (1) yields

$$x_{i+1} = A_\alpha x_i - \sum_{j=2}^{i+1} c_j x_{i-j+1}\,, \quad i \in Z_+\,, \tag{3a}$$

where

$$A_\alpha = A + I_n \alpha\,. \tag{3b}$$

Definition 1 [11, 20] *The fractional system (1) is called (internally) positive if* $x_i \in \Re_+^n$, $i \in Z_+$ *for any initial conditions* $x_0 \in \Re_+^n$.

Theorem 1 [11, 20] *The fractional system (1) is positive if and only if*

$$A_\alpha \in \Re_+^{n\times n}. \tag{4}$$

Proof is given in [11].

Definition 2 *The fractional positive system (1) is called asymptotically stable if*

$$\lim_{i\to\infty} x_i = 0 \quad \text{for all} \;\; x_0 \in \Re_+^n\,. \tag{5}$$

Theorem 2 [11] *The fractional positive system (1) is asymptotically stable if and only if one of the equivalent conditions is satisfied:*

1. *All coefficient of the characteristic polynomial*

$$p_A(z) = \det\left[I_n(z+1) - A\right] = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 \tag{6}$$

are positive, i.e. $a_k > 0$ *for* $k = 0, 1, \ldots, n-1$.

2. *There exists strictly positive vector* $\lambda^T = \begin{bmatrix}\lambda_1 & \cdots & \lambda_n\end{bmatrix}^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ *such that*

$$\left[A - I_n\right]\lambda < 0. \tag{7}$$

Proof is given in [11].

Theorem 3 *The positive system (1) is asymptotically stable if the sum of entries of each column (row) of the matrix A is less than one.*

Proof. The proof follows from condition (7) for $\lambda^T = \begin{bmatrix}1 & \cdots & 1\end{bmatrix}$ since $\begin{bmatrix}1 & \cdots & 1\end{bmatrix}^T A < \begin{bmatrix}1 & \cdots & 1\end{bmatrix}$ if the sum of entries of each column of the matrix A is less than 1. Proof for rows is similar. □

3. Fractional interval positive linear discrete-time systems

Consider the interval fractional positive discrete-time linear system (1) with the interval matrix $A \in \Re_+^{n\times n}$ defined by

$$A_1 \leqslant A \leqslant A_2 \quad \text{or equivalently} \quad A \in [A_1, A_2]. \tag{8}$$

Definition 3 *The interval fractional positive system with (8) is called asymptotically stable if the system is asymptotically stable for all matrices $A \in \Re_+^{n\times n}$ belonging to the interval $[A_1, A_2]$.*

Definition 4 *The matrix*

$$A = (1-k)A_1 + kA_2\,, \qquad 0 \leqslant k \leqslant 1, \quad A_1 \in \Re^{n\times n}, \quad A_2 \in \Re^{n\times n} \tag{9}$$

is called the convex linear combination of the matrices A_1 and A_2.

Theorem 4 *The convex linear combination (9) is asymptotically stable if and only if the matrices $A_1 \in \Re^{n\times n}$ and $A_2 \in \Re^{n\times n}$ are asymptotically stable.*

For two fractional positive linear systems

$$x_{1,i+1} = A_1 x_{1,i}\,, \qquad A_1 \in \Re_+^{n\times n} \tag{10a}$$

and

$$x_{2,i+1} = A_2 x_{2,i}\,, \qquad A_2 \in \Re_+^{n\times n} \tag{10b}$$

there exists a strictly positive vector $\lambda \in \Re_+^n$ such that

$$A_1\lambda < \lambda \quad \text{and} \quad A_2\lambda < \lambda \tag{11}$$

if and only if the systems (10) are asymptotically stable.

Theorem 5 *If the matrices A_1 and A_2 of fractional positive systems (10) are asymptotically stable then their convex linear combination*

$$A = (1-k)A_1 + kA_2 \quad \textit{for} \;\; 0 \leqslant k \leqslant 1 \tag{12}$$

is also asymptotically stable.

Proof. By condition (7) of Theorem 2 if the fractional positive linear systems (10) are asymptotically stable then there exists strictly positive vector $\lambda \in \Re_+^n$ such that

$$A_1\lambda < \lambda \quad \text{and} \quad A_2\lambda < \lambda. \tag{13}$$

Using (7) and (13) we obtain

$$A\lambda = \left[(1-k)A_1 + kA_2\right]\lambda = (1-k)A_1\lambda + kA_2\lambda < \lambda$$

for $0 \leqslant k \leqslant 1$. Therefore, if the positive linear systems (10) are asymptotically stable then their convex linear combination (12) is also asymptotically stable. □

Theorem 6 *The interval positive systems (9) are asymptotically stable if and only if the positive linear systems (10) are asymptotically stable.*

Proof. By condition (7) of Theorem 2 if the matrices $A_1 \in \Re_+^{n\times n}$, $A_2 \in \Re_+^{n\times n}$ are asymptotically stable then there exists a strictly positive vector $\lambda \in \Re_+^n$ such that (7) holds. The convex linear combination (12) satisfies the condition $A\lambda < \lambda$ if and only if (13) holds. Therefore, the interval system (9) is asymptotically stable if and only if the positive linear system is asymptotically stable. □

4. Exponential decay of transient values in feedback positive nonlinear systems with interval matrices

Consider the nonlinear multi-input multi-output feedback system shown in Fig. 1 which consists of the nonlinear element with matrix characteristic $u = f(e)$, positive linear part with interval matrices and feedback with gain matrix H.

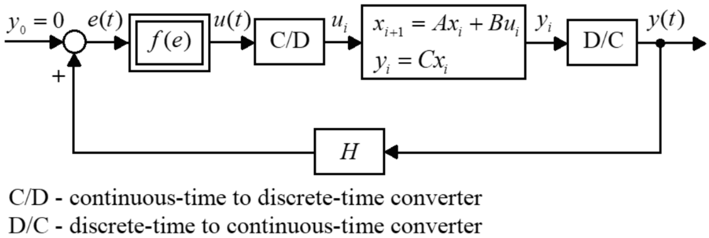

Figure 1: The positive nonlinear feedback system

The linear part is described by the equations

$$x_{i+1} = Ax_i + Bu_i\,, \qquad i \in Z_+ = \{0, 1, \ldots\}, \tag{14a}$$

$$y_i = Cx_i\,, \tag{14b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors of the system $A \in \Re^{n\times n}$, $B \in \Re^{n\times 1}$, $C \in \Re^{1\times n}$ are interval

$$A_1 \leqslant A \leqslant A_2\,, \quad B_1 \leqslant B \leqslant B_2\,, \quad C_1 \leqslant C \leqslant C_2\,. \tag{14c}$$

The matrix characteristic of the nonlinear element satisfies the condition

$$u \leqslant Ke, \quad u_i = f(e_i) \leqslant k_{i1}e_1 + \ldots + k_{ip}e_p\,, \quad i = 1, \ldots, m, \tag{15a}$$

where

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \qquad K = \begin{bmatrix} k_{11} & \cdots & k_{1p} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mp} \end{bmatrix}, \qquad e = \begin{bmatrix} e_1 \\ \vdots \\ e_p \end{bmatrix}, \tag{15b}$$

In general case the feedback matrix H is not square.

Theorem 7 *The multi-input multi-output nonlinear discrete-time system consisting of the positive linear part with interval matrices (14c), the matrix nonlinear element satisfying the condition (15) and the feedback with the matrix $H \in \Re_+^{m\times p}$ is globally stable if there exists a matrix K with positive entries such that the sum of entries of each column (row) of the matrix*

$$(1-q)A_1 + qA_2 + BKHC = \begin{cases} A_1 + B_1K_1HC_1 \in \Re_+^{n\times n} & \textit{for} \ \ q = 0, \\ A_2 + B_2K_2HC_2 \in \Re_+^{n\times n} & \textit{for} \ \ q = 1 \end{cases} \tag{16}$$

is less than one.

Proof. As the Lyapunov [15, 16] function $V(x_i)$ we choose

$$V(x_i) = \lambda^T x_i \geqslant 0 \quad \text{for} \quad x_i \in \Re_+^n, \quad i \in Z_+, \tag{17}$$

where $\lambda \in \Re_+^n$ is strictly positive vector.

Using (17) and (14) we obtain

$$\begin{aligned} \Delta V(x_i) &= V(x_{i+1}) - V(x_i) = \lambda^T(Ax_i + Bu_i) - \lambda^T x_i \\ &= \lambda^T\left(Ax_i + Bf(e_i)\right) - \lambda^T x_i \leqslant \lambda^T\left[(A - I_n) + BKHC\right]x_i \end{aligned} \tag{18}$$

since $u \leqslant Ke = KHCx_i$.

By Theorem 3 from (18) it follows that $\Delta V(x_i) < 0$ if the sum of entries of each column (row) of the matrix (16) is less than one. □

To find the maximal matrix K for which the nonlinear system shown in Fig. 1 is globally stable the following procedure can be used.

Procedure 1

Step 1. Find the matrix K_1 such that the sum of entries of each column of the matrix

$$A_1 + B_1K_1HC_1 \in \Re_+^{n\times n} \tag{19}$$

is less than one.

If $mp > n$ then we choose $mp - n$ nonnegative entries of the matrix K and the remaining its entries (components of vector k) we compute as the solution of the linear matrix equation

$$Gk = h, \tag{20}$$

where the matrix G and the column vector h are defined by the sum of entries of each column (row) of the matrix (19).

Step 2. In a similar way as in Step 1 find the matrix K_2 such that the sum of entries of each column of the matrix

$$A_2 + B_2K_2HC_2 \in \Re_+^{n\times n} \tag{21}$$

is less than one.

Step 3. Knowing K_1 and K_2 find the desired matrix K which satisfies the conditions (19) and (21).

Example 1 Consider the nonlinear system with the positive linear part with the interval matrices

$$A_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}, \\ C_1 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}, \tag{22}$$

the matrix nonlinear element satisfying the condition (15) and the matrix

$$H = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}. \tag{23}$$

Find the maximal matrix K for which the nonlinear system is globally stable.

Using Procedure 1 we obtain

Step 1. Using (19) and (22) we obtain

$$A_1 + B_1K_1HC_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.4 \end{bmatrix} + \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \end{bmatrix} \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix} \\ = \begin{bmatrix} 0.052k_{11} + 0.052k_{12} + 0.3 & 0.036k_{11} + 0.076k_{12} + 0.2 \\ 0.039k_{11} + 0.039k_{12} + 0.2 & 0.027k_{11} + 0.057k_{12} + 0.4 \end{bmatrix}. \tag{24}$$

The sum of entries of the first column of the matrix (24) is $0.091k_{11}+0.091k_{12}+0.5$ and the sum of entries of the second column of the matrix (24) is $0.063k_{11} + 0.133k_{12} + 0.6$.

Solving the system of linear inequalities

$$\begin{cases} 0.091k_{11} + 0.091k_{12} + 0.5 < 1, \\ 0.063k_{11} + 0.133k_{12} + 0.6 < 1 \end{cases}. \tag{25}$$

we obtain $k_{11} < 4.725$ and $k_{12} < 0.769$.

Therefore, the maximal K_1 for which the matrix (24) is Schur is $K_1 = \begin{bmatrix} 4.725 & 0.769 \end{bmatrix}$.

Step 2. Using (21) and (22) we obtain

$$A_2 + B_2K_2HC_2 = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix} \begin{bmatrix} k_{12} & k_{21} \end{bmatrix} \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} \\ = \begin{bmatrix} 0.085k_{11} + 0.1k_{12} + 0.5 & 0.065k_{11} + 0.13k_{12} + 0.3 \\ 0.068k_{11} + 0.08k_{12} + 0.3 & 0.052k_{11} + 0.104k_{12} + 0.5 \end{bmatrix}. \tag{26}$$

The sum of entries of the first column of the matrix (26) is $0.153k_{11}+0.18k_{12}+0.8$ and the sum of entries of the second column of the matrix (26) is $0.117k_{11} + 0.234k_{12} + 0.8$.

Solving the system of linear inequalities

$$\begin{cases} 0.153k_{11} + 0.18k_{12} + 0.8 < 1, \\ 0.117k_{11} + 0.234k_{12} + 0.8 < 1 \end{cases} \tag{27}$$

we obtain $k_{11} < 0.733$ and $k_{12} < 0.488$.

Therefore, the maximal K_2 for which the matrix (26) is Schur is $K_2 = \begin{bmatrix} 0.733 & 0.488 \end{bmatrix}$.

Step 3. Using the results obtained in Steps 1 and 2 we obtain that the maximal K for which the matrices (24) and (26) are Schur is $K = \begin{bmatrix} 0.733 & 0.488 \end{bmatrix}$.

Lemma 1 *If z_k, $k = 1, \ldots, n$ are the eigenvalues of the matrix $A \in \Re_+^{n\times n}$, then the matrix $A - I_n\gamma$ has the eigenvalues $z_k + \gamma$, $k = 1, \ldots, n$, $0 \leqslant \gamma < 1$.*

Proof. Let z_k, $k = 1, \ldots, n$ be the roots of the equation

$$\det\left[I_n z - A\right] = 0 \tag{28}$$

then

$$\det\left[I_n z - (A - I_n\gamma)\right] = \det\left[I_n(z+\gamma) - A\right] = 0. \tag{29}$$

Therefore, if z_k, $k = 1, \ldots, n$ are the eigenvalues of the matrix A then $z_k + \gamma$, $k = 1, \ldots, n$, are the eigenvalues of the matrix $A - I_n\gamma$. □

From Lemma 1 it follows that if the eigenvalues of the matrix A are located in the unit circle then the eigenvalues of the matrix $A - I_n\gamma$ are located in the circle with the radius $1 - \gamma$. In this case transient values of the linear system are decreasing faster than $(1-\gamma)^i$ for $i = 1, 2, \ldots$.

Example 2 The asymptotically stable matrix with positive entries

$$A = \begin{bmatrix} 0.6 & 0.2 \\ 0.1 & 0.7 \end{bmatrix} \tag{30}$$

with the characteristic polynomial

$$\det\left[I_2 z - A\right] = \begin{vmatrix} z-0.6 & -0.2 \\ -0.1 & z-0.7 \end{vmatrix} = z^2 - 1.3z + 0.4 \tag{31}$$

has the eigenvalues: $z_1 = 0.5$, $z_2 = 0.8$.

The matrix

$$A_\gamma = A - I_2\gamma = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.5 \end{bmatrix}, \qquad \gamma = 0.2 \tag{32}$$

has the characteristic polynomial

$$\det\left[I_2 z - A_\gamma\right] = \begin{vmatrix} z-0.4 & -0.2 \\ -0.1 & z-0.5 \end{vmatrix} = z^2 - 0.9z + 0.18 \tag{33}$$

and the eigenvalues $z_1 = 0.3$, $\quad z_2 = 0.6$.

This simple example confirms the Lemma 1.

From the above considerations and Theorem 7 we have the following theorem.

Theorem 8 *Transient values in the multi-input multi-output nonlinear system consisting of the positive linear part with interval matrices (14c), the matrix nonlinear element satisfying the condition (15) and the feedback with the matrix* H *are decreasing faster than* $(1-\gamma)^i$ *for* $i = 1, 2, \ldots$ *if the sum of entries of each column(row) of the matrix*

$$\begin{aligned}&(1-q)(A_1 - I_n\gamma) + q(A_2 - I_n\gamma) + BKHC \\ &\qquad = \begin{cases} A_1 - I_n\gamma + B_1K_1HC_1 \in \Re_+^{n\times n} & \textit{for} \;\; q = 0, \\ A_2 - I_n\gamma + B_2K_2HC_2 \in \Re_+^{n\times n} & \textit{for} \;\; q = 1 \end{cases}\end{aligned} \tag{34}$$

is less than one.

5. Fractional positive feedback nonlinear systems

Consider the nonlinear system shown in Fig. 1 which consists of the positive linear part, nonlinear element with characteristic $u = f(e)$ and the scalar gain feedback h. The linear part is described by the equations (3) with interval matrix A satisfying the condition (8). To simply the notion it is assumed that the scalar characteristic $u = f(e)$ satisfies the condition

$$0 < f(e) < ke, \qquad 0 \leqslant k < \infty. \tag{35}$$

The following theorem gives sufficient conditions for the decreasing of the transient values in the nonlinear system faster than $(1-\gamma)^i$ for $i = 1, 2, \ldots$

Theorem 9 *Transient values in the fractional nonlinear discrete-time system shown in Fig. 1 with the positive linear part (9), the nonlinear element satisfying the condition (35) and the scalar positive feedback are decreasing faster than* $(1-\gamma)^i$ *for* $i = 1, 2, \ldots$ *if the sum of entries of each column (row) of the matrix*

$$\begin{aligned}&(1-q)(A_1 - I_n\gamma) + q(A_2 - I_n\gamma) + khBC \\ &\qquad = \begin{cases} A_1 - I_n\gamma + khB_1C_1 \in \Re_+^{n\times n} & \textit{for} \;\; q = 0, \\ A_2 - I_n\gamma + khB_2C_2 \in \Re_+^{n\times n} & \textit{for} \;\; q = 1 \end{cases}\end{aligned} \tag{36}$$

is less than one.

Proof. As the Lyapunov function $V(x)$ for the system with not interval matrices we choose

$$V(x_i) = \lambda^T x_i \geqslant 0 \quad for \quad x_i \in \mathfrak{R}^n_+, \quad i \in Z_+, \tag{37}$$

where $\lambda \in \mathfrak{R}^n_+$ is strictly positive vector.

Using (37) and (35) we obtain

$$\begin{aligned} \Delta V(x_i) &= V(x_{i+1}) - V(x_i) = \lambda^T x_{i+1} - \lambda^T x_i \\ &= \lambda^T \left(A_\alpha x_i + \sum_{k=2}^{\infty} c_k x_{i-k+1} + Bu_i \right) - \lambda^T x_i \\ &= \lambda^T \left[(A_\alpha - I_n) x_i + \sum_{k=2}^{\infty} c_k x_{i-k+1} + hBf(e_i) \right] \\ &\leqslant \lambda^T \left[(A_\alpha - I_n + khBC) x_i + \sum_{k=2}^{\infty} c_k x_{i-k+1} \right] \end{aligned} \tag{38}$$

Taking into account that

$$\sum_{k=2}^{\infty} c_k x_{i-k+1} \leqslant \sum_{k=2}^{\infty} c_k x_i = (1-\alpha) x_i \tag{39}$$

we obtain

$$\Delta V(x_i) = \lambda^T [A_\alpha + I_n \alpha - I_n + khBC + (1-\alpha) I_n] x_i = \lambda^T [A + khBC] x_i < 0. \tag{40}$$

For the nonlinear system with interval matrices (9) we obtain (36). □

The considerations can be easily extended to the systems with multi-input multi-output nonlinear systems satisfying the condition (15).

6. Concluding remarks

The exponential decay of transient values in the discrete-time nonlinear standard and fractional orders with positive linear parts and linear positive feedbacks has been investigated. Sufficient conditions for the exponential decay of transient values in this class of positive nonlinear systems have been established (Theorems 7 and 9). Procedure for computation of gains characterizing the class of nonlinear elements are given and illustrated on simple example. The considerations have been extended to fractional order nonlinear systems. An open problem is an extension of the consideration to different fractional orders nonlinear systems.

References

[1] A. Berman and R.J. Plemmons: *Nonnegative Matrices in the Mathematical Sciences*. SIAM, 1994.

[2] L. Farina and S. Rinaldi: *Positive Linear Systems; Theory and Applications*. J. Wiley, New York, 2000.

[3] T. Kaczorek: Absolute stability of a class of fractional positive nonlinear systems. *International Journal of Applied Mathematics and Computer Science*, **29**(1), (2019), 93–98. DOI: 10.2478/amcs-2019-0007.

[4] T. Kaczorek: Analysis of positivity and stability of fractional discrete-time nonlinear systems. *Bulletin of the Polish Academy of Science. Technical Sciences*, **64**(3), (2016), 491–494. DOI: 10.1515/bpasts-2016-0054.

[5] T. Kaczorek: Exponential decay of transient values in positive nonlinear systems. *Bulletin of the Polish Academy of Science. Technical Sciences*, (in Press).

[6] T. Kaczorek: Global stability of nonlinear feedback systems with fractional positive linear parts. *International Journal of Applied Mathematics and Computer Science*, **30**(3), (2020), 493–500. DOI: 10.34768/amcs-2020-0036.

[7] T. Kaczorek: Global stability of positive standard and fractional nonlinear feedback systems. *Bulletin of the Polish Academy of Science. Technical Sciences*, **68**(2), (2020), 285–288. DOI: 10.24425/bpasts.2020.133112.

[8] T. Kaczorek: Global stability of nonlinear feedback systems with positive linear parts. *International Journal of Nonlinear Sciences and Numerical Simulation*, **20**(5), (2019), 575–579. DOI: 10.1515/ijnsns-2018-0189.

[9] T. Kaczorek: Positive linear systems with different fractional orders. *Bulletin of the Polish Academy of Science. Technical Sciences*, **58**(3), (2010), 453–458. DOI: 10.2478/v10175-010-0043-1.

[10] T. Kaczorek: Positive linear systems consisting of n subsystems with different fractional orders. *IEEE Transactions on Circuits and Systems*, **58**(7), (2011), 1203–1210. DOI: 10.1109/TCSI.2010.2096111.

[11] T. Kaczorek: *Selected Problems of Fractional Systems Theory*. Springer, Berlin 2011.

[12] T. Kaczorek: Stability of fractional positive nonlinear systems. *Archives of Control Sciences*, **25**(4), (2015), 491–496. DOI: 10.1515/acsc-2015-0031.

[13] T. Kaczorek and K. Rogowski: *Fractional Linear Systems and Electrical Circuits*. Springer, Cham 2015.

[14] T. Kaczorek and A. Ruszewski: Global stability of discrete-time nonlinear systems with descriptor standard and fractional positive linear parts and scalar feedbacks. *Archives of Control Sciences*, **30**(4), (2020), 667–681. DOI: 10.24425/acs.2020.135846.

[15] A.M. Lyapunov: *Obscaja zadaca ob ustoicivosti dvizenija*. Gostechizdat, Moskov, 1963. In Russian.

[16] H. Leipholz: *Stability Theory*. New York Academic Press, 1970.

[17] W. Mitkowski: Dynamical properties of Metzler systems. *Bulletin of the Polish Academy of Science. Technical Sciences*, **56**(4), (2008), 309–312.

[18] P. Ostalczyk: *Discrete Fractional Calculus*. World Scientific, River Edgle, 2016.

[19] I. Podlubny: *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[20] A. Ruszewski: Stability of discrete-time fractional linear systems with delays. *Archives of Control Sciences*, **29**(3), (2019), 549–567. DOI: 10.24425/acs.2019.130205.

[21] L. Sajewski: Stabilization of positive descriptor fractional discrete-time linear systems with two different fractional orders by decentralized controller. *Bulletin of the Polish Academy of Science. Technical Sciences*, **65**(5), (2017), 709–714. DOI: 10.1515/bpasts-2017-0076.