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CONTROL AND INFORMATICS

Controllability of linear convex combination of linear discrete-time fractional systems

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Abstract. In this paper the controllability properties of the convex linear combination of fractional, linear, discrete-time systems are characterized and investigated. The notions of linear convex combination and controllability in the context of fractional-order systems are recalled. Then, the controllability property of such a linear combination of discrete-time, linear fractional systems is proven. Further, the reduction of an infinite problem of transition matrix derivation is reduced to a finite one, which greatly simplifies the numerical burden of the controllability issue. Examples of controllable and uncontrollable, single-input, linear systems are presented. The possibility of extension of the considerations to multi-input systems is shown.

Key words: fractional order systems; controllability; linear convex combination.

1. INTRODUCTION

The concept of the convex linear combination of dynamical control systems plays an important role in different areas of technology and physics. For example, this concept is widely used in quantum physics systems [1, 2] in the description of the convex linear combination of quantum Pauli channels. On the other hand, controllability is a fundamental concept in mathematical control theory (see e.g. [3–5] and references therein). This concept has been widely discussed in the literature [5, 6] and its direct connection with the minimum energy control [7] clearly indicates numerous possible applications. Recently, this idea has been also used in the context of fractional-order linear systems (see e.g. [8, 9]. However, there are still many open or unsolved problems in this area. Taking into account convex linear combinations and, on the other hand, the controllability concept, in the paper we shall consider the controllability of linear combination of finite dimensional, discrete-time, fractional control systems with constant coefficients. In the proof of the main result a well-known, purely algebraic controllability condition is used. The paper is organized as follows. In Section 2 linear convex combination for finite dimensional, linear control systems is defined [10]. Further, Section 2 contains the main result of the paper concerning controllability properties of convex linear combination for fractional control systems given in Theorem 2. It should be pointed out that Theorem 2 is only a sufficient but not a necessary condition for controllability of convex combination. Next, in Section 3 a new method for the determination of a transition matrix for fractional order discrete-time systems is introduced. This method reduces the infinite problem to a finite one. Further, in Sections 4 and 5, taking into account the results of Section 2, several examples of linear convex combinations of controllable or uncontrollable control systems are presented and their controllability is discussed. Finally, in Section 6 possible extensions for more general convex combination are proposed and discussed.

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2. CONTROLLABILITY OF LINEAR CONVEX COMBINATION OF FRACTIONAL LINEAR SYSTEMS

Let us consider the fractional discrete-time linear system with constant coefficients:

$$
\begin{aligned}
x_{i+1} &= Ax_i + \sum_{k=1}^{i+1} (-1)^{k+1} \binom{\alpha}{k} x_{i-k+1} + bu_i \\
&= A_\alpha x_i + \sum_{k=2}^{i+1} c_k x_{i-k+1} + bu_i, \\
&0 < \alpha \le 2, \quad i = 0, 1, 2, \ldots, \\
&x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m,
\end{aligned}
\tag{1}
$$

where

$$
A_\alpha = A + I_n \alpha, \quad c_k = (-1)^{k+1} \binom{\alpha}{k},
$$

$$
\binom{\alpha}{k} = \begin{cases} 1, & k = 0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}, & k = 1, 2, \ldots, \end{cases}
$$

The solution of (1) has the form:

$$
x_i = \Phi_i x_0 + \sum_{k=0}^{i-1} \Phi_{i-k-1} bu_k, \tag{2}
$$

where Φ_i is determined by:

$$\Phi_{i+1} = \Phi_i A_a + \sum_{k=2}^{i+1} c_k \phi_{i-k+1}, \quad \Phi_0 = I_n. \tag{3}$$

Definition 1. The fractional system (1) is controllable in p steps if there exists an input sequence $u_0, u_1, \dots, u_{p-1}$ which steers the state x_i of the system from the initial state x_0 to the final state x_f, i.e. $x_p = x_f$.

For completeness of considerations let us recall a well-known neccessary and sufficient condition for controllability:

Theorem 1. The fractional system (1) is controllable in p steps if and only if

1. $$\operatorname{rank}[b, A_\alpha b, \dots, A_\alpha^{p-1} b] = n \tag{4}$$

or equivalently

2. $$\operatorname{rank}[I_n z - A_\alpha, b] = n, \quad \forall z \in \mathbb{C}. \tag{5}$$

Proof is given in [11, p. 11]. □

Consider two fractional, discrete-time, controllable, linear systems with constant coefficients which by the same linear transformation matrix $P \in \mathbb{R}^{n\times n}$ can be transformed to the similar canonical form:

$$PA_1P^{-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_{10} & -a_{11} & -a_{12} & \dots & -a_{1,n-1} \end{bmatrix},$$

$$Pb_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$PA_2P^{-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_{20} & -a_{21} & -a_{22} & \dots & -a_{2,n-1} \end{bmatrix},$$

$$Pb_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{6}$$

Theorem 2. The linear convex combination [10] of the controllable systems:

$$A = qA_1 + (1-q)A_2, \quad b = qb_1 + (1-q)b_2, \tag{7}$$

is also controllable for all values of $q \in [0,1]$.

Proof. Using (6), and (7) we obtain:

$$\begin{aligned} A &= qA_1 + (1-q)A_2 \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} \end{bmatrix}, \\ b &= qb_1 + (1-q)b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \end{aligned} \tag{8}$$

where $a_k = a_{1k}q + a_{2k}(1-q), \quad k = 0, 1, \dots, n-1$. The pair (8) is controllable for all values of $q \in [0,1]$. □

Theorem 2 is only a sufficient but not necessary condition for the controllability of linear convex combination.

3. FRACTIONAL DISCRETE-TIME SYSTEMS TRANSITION MATRIX DETERMINATION

The solution (2) of linear fractional order equation (1) contains transition matrices Φ_i, which play important role in linear control systems. Determination of the transition matrices for discrete-time fractional systems is also essential from the point of view of controllability and observability analysis. However, in the case of these systems problem is much more difficult than for integer-order ones. This is caused by the fact that the transition matrix for fractional order systems is defined by an infinite series (on infinite interval). In this section an approach for the reduction of the problem on infinite interval to equivalent one defined on finite interval will be presented. This method is based on the results taken directly from linear matrix algebra. Theoretical considerations will be illustrated by a simple numerical example.

To simplify the notation it is assumed that the matrices $A \in \mathbb{R}^{n\times n}$ have only distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. The matrices

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \qquad 0 < \alpha < 1, \tag{9}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}, \qquad 0 < \alpha < 1 \tag{10}$$

will be expressed by

$$\Phi_0(t) = c_0 I_n + c_1 A + c_2 A^2 + \ldots + c_{n-1} A^{n-1}, \quad c_k = c_k(\lambda_1, \ldots, \lambda_n, t), \quad k = 0, 1, \ldots, n-1 \tag{11}$$

and

$$\Phi(t) = \bar{c}_0 I_n + \bar{c}_1 A + \bar{c}_2 A^2 + \ldots + \bar{c}_{n-1} A^{n-1}, \quad \bar{c}_k = \bar{c}_k(\lambda_1, \ldots, \lambda_n, t), \quad k = 0, 1, \ldots, n-1. \tag{12}$$

It is well-known [11] that (11) and (12) are satisfied for the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix A

$$\Phi_0(\lambda_k, t) = c_0 + c_1 \lambda_k + c_2 {\lambda_k}^2 + \ldots + c_{n-1} {\lambda_k}^{n-1}, \quad k = 1, \ldots, n \tag{13}$$

and

$$\Phi(\lambda_k, t) = \bar{c}_0 + \bar{c}_1 \lambda_k + \bar{c}_2 {\lambda_k}^2 + \ldots + \bar{c}_{n-1} {\lambda_k}^{n-1}, \quad k = 1, \ldots, n \tag{14}$$

or

$$\begin{bmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \Phi_0(\lambda_1, t) \\ \vdots \\ \Phi_0(\lambda_n, t) \end{bmatrix} \tag{15}$$

and

$$\begin{bmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} \bar{c}_0 \\ \vdots \\ \bar{c}_{n-1} \end{bmatrix} = \begin{bmatrix} \Phi(\lambda_1, t) \\ \vdots \\ \Phi(\lambda_n, t) \end{bmatrix}. \tag{16}$$

Note that if $\lambda_i \neq \lambda_j$ for $i \neq j$, $i, j = 1, \ldots, n$ then the matrix

$$\Lambda = \begin{bmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{bmatrix} \tag{17}$$

is nonsingular ($\det\Lambda \neq 0$) and from (15) and (16) we may find the coefficients $c_0, \ldots, c_{n-1}$ and $\bar{c}_0 \ldots, \bar{c}_{n-1}$.

The coefficients $c_0, \ldots, c_{n-1}$ and $\bar{c}_0 \ldots, \bar{c}_{n-1}$ can also be found using the Lagrange-Sylvester formula [12] which for $\lambda_i \neq \lambda_j$, $, i, j = 1, \ldots, n$ takes the form [12]

$$f(A) = \sum_{k=1}^{n} Z_k f(\lambda_k), \tag{18}$$

where

$$Z_k = \prod_{i=1, i\neq k}^{n} \frac{A - \lambda_i I_n}{\lambda_k - \lambda_i} = \frac{(A - \lambda_1 I_n) \cdots (A - \lambda_{k-1} I_n)(A - \lambda_{k+1} I_n) \cdots (A - \lambda_n I_n)}{(\lambda_k - \lambda_1) \cdots (\lambda_k - \lambda_{k-1})(\lambda_k - \lambda_{k+1}) \cdots (\lambda_k - \lambda_n)}. \tag{19}$$

The considerations can be extended to general case. The above procedure is illustrated by Example 1.

Example 1. Find the coefficients c_k and $\bar{c}_k$ for $k = 1, 2$ for the matrix

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

with the eigenvalues $\lambda_1 = -1$, $\lambda_2 = -4$ ($\det[Is - A] = \lambda^2 + 5\lambda + 4$)

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} \Phi_0(-1, t) \\ \Phi_0(-4, t) \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -4 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \Phi_0(-1, t) \\ \Phi_0(-4, t) \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}\Phi_0(-1, t) - \frac{1}{3}\Phi_0(-4, t) \\ \frac{1}{3}\Phi_0(-1, t) - \frac{1}{3}\Phi_0(-4, t) \end{bmatrix} \tag{20}$$

and

$$\begin{bmatrix} \bar{c}_0 \\ \bar{c}_1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} \Phi(-1, t) \\ \Phi(-4, t) \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -4 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \Phi(-1, t) \\ \Phi(-4, t) \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}\Phi(-1, t) - \frac{1}{3}\Phi(-4, t) \\ \frac{1}{3}\Phi(-1, t) - \frac{1}{3}\Phi(-4, t) \end{bmatrix}. \tag{21}$$

Therefore we have

$$\Phi_0(t) = c_0 I_2 + c_1 A$$

and

$$\Phi(t) = \bar{c}_0 I_2 + \bar{c}_1 A = \begin{bmatrix} \bar{c}_0 - 2\bar{c}_1 & \bar{c}_1 \\ 2\bar{c}_1 & \bar{c}_0 - 3\bar{c}_1 \end{bmatrix}.$$

Remark 1. Note, that by applying the above approach it is possible to reduce the investigation of the controllability and observability problems from infinite interval to a finite one.

4. LINEAR CONVEX COMBINATION OF UNCONTROLLABLE LINEAR SYSTEMS

In the previous sections it has been presented that a linear convex combination of controllable linear systems is also controllable. However, it is also possible to prove that such a linear convex combination of two uncontrollable linear systems is controllable. To show this let us consider the pair of uncontrollable linear systems:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_0^1 & -a_1^1 & -a_2^1 & -a_3^1 & \dots & -a_{n-1}^1 \end{bmatrix},$$

$$b_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_0^2 & -a_1^2 & -a_2^2 & -a_3^2 & \dots & -a_{n-1}^2 \end{bmatrix},$$

$$b_2 = b_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{22}$$

It should be pointed out that the above systems are uncontrollable, since in matrix A_1 the second row, and in matrix A_2 the third row are composed of zeros. Following [10] we define the linear convex combination of systems given by (22):

$$A = qA_1 + (1-q)A_2$$
$$= \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1-q & 0 & \dots & 0 \\ 0 & 0 & 0 & 1+q & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & a_3 & \dots & a_{n-1} \end{bmatrix}, \tag{23}$$
$$a_k = a_k^1 q + a_k^2(1-q),$$

$$b = qb_1 + (1-q)b_2 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Theorem 3. The linear convex combination (23) is controllable for any $q \neq 1$, if the systems (22) are uncontrollable.

Proof. Using the Hautus test we obtain:

$$\operatorname{rank}[\mathbb{I}z - A, b] =$$
$$\operatorname{rank} \begin{bmatrix} z & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & z & -1+q & 0 & \dots & 0 & 0 \\ 0 & 0 & z & -1-q & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ a_0 & a_1 & a_2 & a_3 & \dots & z+a_{n-1} & 1 \end{bmatrix} = n,$$
$$\text{for all} \quad q \neq 1. \tag{24}$$

Therefore, the linear convex combination (23) of uncontrollable systems is controllable for all $q \in [0,1)$. □

From this example it follows that Theorem 3 is the only sufficient condition for controllability of convex linear combination.

5. CONTROLLABILITY OF THE INVERSE SYSTEM

Using the Frobenius cannonical form it can be shown that the result of Theorem 2 may be extended for the linear convex combination of inverse control systems. Let us consider the following matrices in Frobenius forms:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} \end{bmatrix},$$

$$A_2 = A_1^T = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \dots & -a_1 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \tag{25}$$

$$A_4 = A_3^T = \begin{bmatrix} -a_{n-1} & 1 & 0 & \dots & 0 & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 & 0 \\ -a_{n-3} & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -a_1 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

It is well known that the inverse matrices of the Frobenius matrices (25) are also matrices in the canonical Frobenius forms [13]. For example, the inverse matrix of A_1 ($\det A_1 \neq 0$) has the form [13]

$$A_1^{-1} = \begin{bmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & -\frac{a_3}{a_0} & \dots & -\frac{a_{n-1}}{a_0} & -\frac{1}{a_0} \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{26}$$

It is also well known [13] that every controllable pair $\bar{A}_1 \in \mathbb{R}^{nxn}$, $\bar{b}_1 \in \mathbb{R}^{nx1}$ can be reduced by the similarity transformation to the canonical form ($\det P \neq 0$):

$$A_1 = P^{-1}\bar{A}_1 P = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} \end{bmatrix},$$

$$b_1 = P^{-1}\bar{b}_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{27}$$

Let us assume that $\det A_1 = \det \bar{A}_1 = a_0 \neq 0$. In this case for the matrix A_1 there exists its inverse matrix (26).

Theorem 4. If the pair (27) (equivalently $(\bar{A}_1, \bar{b}_1)$) is controllable then the pair (A^{-1}, b_1) is also controllable.

Proof. It is easy to check that the pair (27) is controllable for all values of the coefficients a_k, $k = 0, 1, \dots, n-1$. Note that

$$\operatorname{rank}[b_1, A_1^{-1}b_1, \dots, (A_1^{-1})^{n-1}b_1] = \operatorname{rank} \begin{bmatrix} 0 & -\frac{1}{a_0} & \frac{a_1}{a_0^2} & \dots & x \\ 0 & 0 & 1 & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} = n,$$

$$\text{if and only if } a_0 \neq 0. \tag{28}$$

Similar results can be obtained from Hautus theorem since

$$\operatorname{rank}[\mathbb{I}z - A, b] = \operatorname{rank} \begin{bmatrix} z & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & z & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & z & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ a_0 & a_1 & a_2 & a_3 & \dots & z + a_{n-1} & 1 \end{bmatrix} = n, \tag{29}$$

$\forall z \in \mathbb{C}$. □

Remark 2. The considerations can be extended to multiple input systems. Let $B \in \mathbb{R}^{nxm}$, $m > 1$ and $b = Bk$, $k \in \mathbb{R}^m$. The vector k is chosen so that the pair (A, b) is controllable. Then

$$\det[b, Ab, \dots, A^{n-1}b] = \det[Bk, ABk, \dots, A^{n-1}Bk] \neq 0. \tag{30}$$

Note that such vector k exists if the pair (A, B) is controllable since

$$\operatorname{rank}[Bk, ABk, \dots, A^{n-1}Bk] = \operatorname{rank}[B, AB, \dots, A^{n-1}B] \operatorname{rank} \begin{bmatrix} k & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k \end{bmatrix}. \tag{31}$$

6. CONCLUSIONS

The linear, convex combination (7) of two fractional, discrete-time, systems has been analyzed (Theorem 2). The fractional discrete-time, linear systems with distinct eigenvalues have been investigated. It has been shown that the linear, convex combination of uncontrollable, linear systems is controllable for $q \neq 1$ (Theorem 3). Also, it has been presented that if the pair (A_1, b_1) is controllable then the pair (A_1^{-1}, b_1) is also controllable (Theorem 4). The considerations can be extended to linear, fractional systems with different orders.

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