BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 71(4), 2023, Article number: e145675
DOI: 10.24425/bpasts.2023.145675

CONTROL, INFORMATICS AND ROBOTICS

The positivity of the fractional order model of a two-dimensional temperature field

Krzysztof OPRZĘDKIEWICZ*

AGH University of Science and Technology, al. A. Mickiewicza 30, 30-059 Kraków, Poland

Abstract. The paper presents analysis of the positivity for a two-dimensional temperature field. The process under consideration is described by the linear, infinite-dimensional, noninteger order state equation. It is derived from a two-dimensional parabolic equation with homogenous Neumann boundary conditions along all borders and homogenous initial condition. The form of control and observation operators is determined by the construction of a real system. The internal and external positivity of the model are associated to the localization of heater and measurement. It has been proven that the internal positivity of the considered system can be achieved by the proper selection of attachment of a heater and place of a measurement as well as the dimension of the finite-dimensional approximation of the considered model. Conditions of the internal positivity associated with construction of real experimental system are proposed. The postivity is analysed separately for control and output of the system. This allows one to analyse the positivity of thermal systems without explicit control. Theoretical considerations are numerically verified with the use of experimental data. The proposed results can be applied i.e. to point suitable places for measuring of a temperature using a thermal imaging camera.

Key words: noninteger order systems; heat transfer equation; fractional order state equation; Caputo operator; positivity; thermal camera.

1. INTRODUCTION

In reality there exist many processes and phenomena described by signals taking only positive values. Such phenomena are known in medicine, chemistry, biology, economy or different areas of engineering.

Theory of positive systems has been developed by many Researchers over the years. Fundamentals are presented e.g. in books: [1–5]. Presentation [6] is also interesting.

Control methods dedicated to positive systems allowing to keep their positivity are given e.g. in: [7, 8]. A specific class of positive parabolic problems has been analysed i.e. in [9]. An interesting "academic" example of a positive system is presented in the paper [10].

An issue important from point of view of practice is to assure of a positivity by a technical system we deal with. At first glance, the number of results in this range seems to be large(see e.g. [11–15]).

It is important to note that these theoretical results do not give guidelines on how to construct a positive system. It can be tried to apply the general positivity conditions, but this can be difficult or dangerous in practice.

Numerical methods can be also employed to testing of positivity, as it is mentioned e.g. in [15]. However such a numerical estimation is the NP hard problem. An alternative is delivered by an idea of an "eventual positivity". It permits an existence of nonpositive solutions too. This issue is presented e.g. in [16,17].

The fractional calculus allows one to describe many complex physical phenomena and processes. Examples of fractional order models are presented e.g. by [18–23]. Anomalous diffusion problem using fractional order approach and semigroup theory is given e.g. by [24]. An observability of fractional order systems is discussed e.g. in paper [25].

The use of the Kelvin scale allows one to describe thermal processes using positive approach. Processes of heat transfer and dissipation are analysed by researchers and engineers for years. This broad class of processes can be described using noninteger order approach also (see e.g. [26–31]).

Recently the study of positive fractional-order systems is caused by the fact that many fractional-order systems also describe nonnegative physical phenomena and technical systems. Fundamental results from area of the positivity of fractional order systems have been published by T. Kaczorek incl. in papers [32–35].

In this paper new analytical conditions of the internal positivity for real, experimental thermal system are proposed and verified. The system under consideration is described by the fractional order, two-dimensional state space model. The proposed results allow one to attach a positivity property to a construction of a real system.

The organization of the paper is following. It starts with recalling elementary ideas and definitions from fractional calculus and theory of positive systems. Next the experimental heat system and its model are presented. The main result presents analysis of the internal positivity of the considered system. Finally the numerical verification of presented results as well as the numerical tests of the external postivity using experimental data is given.

*e-mail: kop@agh.edu.pl

Manuscript submitted 2022-10-17, revised 2023-01-24, initially accepted for publication 2023-04-04, published in August 2023.

2. PRELIMINARIES

2.1. Basics of fractional calculus

The noninteger-order, integro-differential operator is defined as follows (see e.g. [3, 19, 22, 36]):

Definition 1. (The elementary noninteger-order operator) The noninteger-order integro-differential operator is defined as follows:

$$ {}_aD_t^\alpha f(t) = \begin{cases} \dfrac{d^\alpha f(t)}{dt^\alpha} & \alpha > 0, \\ f(t) & \alpha = 0, \\ \int\limits_a^t f(\tau)(\mathrm{d}\tau)^\alpha & \alpha < 0, \end{cases} \tag{1} $$

where a and t denote time limits for operator calculation, $\alpha \in \mathbb{R}$ denotes the noninteger order of the operation. If $\alpha \in \mathbb{Z}$, then the operator (1) turns to classic integer order operator.

The fractional-order, integro-differential operator can be described by definitions given by Grünwald and Letnikov, Riemann and Liouville (RL) and Caputo (C). In this paper the C definition is employed (see e.g. [3, 19, 22, 36]):

Definition 2. (The Caputo definition of the FO operator)

$$ {}_0^CD_t^\alpha f(t) = \frac{1}{\Gamma(M-\alpha)} \int\limits_0^\infty \frac{f^{(M)}(\tau)}{(t-\tau)^{\alpha+1-M}} \, \mathrm{d}\tau, \tag{2} $$

where $M-1 < \alpha < M$ is the fractional order of operation and $\Gamma(..)$ is the Gamma function.

The Laplace transform of the Caputo operator is defined as follows (e.g. [37]):

Definition 3. (The Laplace transform of Caputo operator)

$$ \begin{aligned} &\mathscr{L}({}_0^CD_t^\alpha f(t)) = s^\alpha F(s), \quad \alpha < 0, \\ &\mathscr{L}({}_0^CD_t^\alpha f(t)) = s^\alpha F(s) - \sum_{k=0}^{M-1} s^{\alpha-k-1} {}_0D_t^k f(0), \\ &\qquad \alpha > 0, \quad M-1 < \alpha \leq M \in \mathbb{Z}. \end{aligned} \tag{3} $$

The general form of a fractional-order linear state space equation is as follows:

$$ \begin{aligned} {}_0D_t^\alpha x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{4} $$

In (4) $\alpha \in (0,1)$ is the fractional order, $x(t) \in \mathbb{R}^N$, $u(t) \in \mathbb{R}^L$, $y(t) \in \mathbb{R}^P$ are the state, control and output vectors respectively, A, B, C are the state, control and output operators, respectively.

2.2. Positivity

Next ideas and conditions of internal and external positivity of the FO system should be remembered. They are given e.g. in [33, 34].

Definition 4. (The internal positivity)
The FO system (4) is called internally positive if $x(t) \in \mathbb{R}_+^N$, $y(t) \in \mathbb{R}_+^P$, $t \geq 0$ for any initial conditions $x_0 \in \mathbb{R}_+^N$ and all inputs $u(t) \in \mathbb{R}_+^M$.

Theorem 1. The FO system (4) is internally positive if and only if:

$$ A \in \mathbb{M}_N, \quad B \in \mathbb{R}_+^U, \quad C \in \mathbb{R}_+^P, \tag{5} $$

where $\mathbb{M}_N$ denotes the Metzler matrix.

Definition 5. (The external positivity)
The FO system (4) is called externally positive if and only if $y(t) \in \mathbb{R}_+^P$, $t \geq 0$ for homogenous initial condition $x_0 = 0$ and all inputs $u(t) \in \mathbb{R}_+^M$.

Theorem 2. The FO system (4) is externally positive if and only if its impulse response matrix $g(t)$ is nonnegative for $t \geq 0$, i.e:

$$ g(t) = \mathscr{L}^{-1}\{C(s^\alpha I - A)^{-1}B\} \in \mathbb{R}_+^{PxM}. \tag{6} $$

The internal positivity always implies the external positivity, but the reverse implication is not true. The prooving of the external positivity without internal positivity is not a trivial issue. The solution to this problem for a specified class of dynamic systems is given in the paper [38].

3. THE CONSIDERED HEATING SYSTEM

The considered heating system is shown simplified in Fig. 1. This is the PCB plate with the flat electric heater, denoted by H. Its coordinates are described by x_{h1}, x_{h2}, y_{h1} and y_{h2} respectively. The temperature of the whole PCB is monitored using an industrial thermal imaging camera, the location and size of measurement area are configurable. The size of camera sensor is $X \times Y$ pixels. The area of measurement is marked as S and its coordinates are denoted by x_{s1}, x_{s2}, y_{s1} and y_{s2} respectively. Detailed description of this experimental system is given in the section "Simulations and Experiments". The heater and sensor functions are expressed by the simple rectangular functions:

$$ b(x,y) = \begin{cases} 1, & x,y \in H, \\ 0, & x,y \notin H. \end{cases} \tag{7} $$

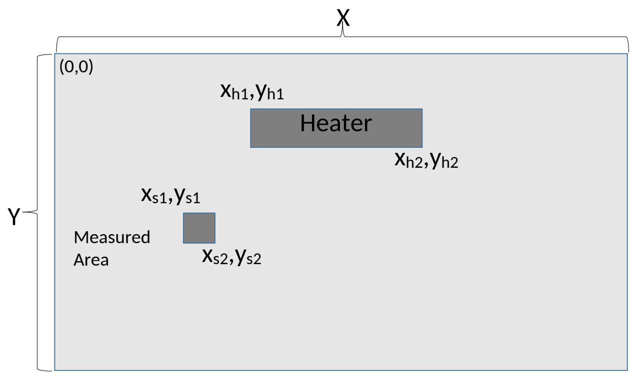

Fig. 1. The experimental system

$$c(x,y)=\begin{cases}1, & x,y\in S,\\ 0, & x,y\notin S.\end{cases} \tag{8}$$

4. THE FRACTIONAL MODEL OF THE SYSTEM

The two-dimensional, FO model presented in this section has been presented with details in papers [39, 40]. Here its crucial elements are recalled. The heat transfer equation is given by (9):

$$\begin{cases} {}_0^C D_t^{\alpha} Q(x,y,t) = a_w\left(\dfrac{\partial^{\beta} Q(x,y,t)}{\partial x^{\beta}}+\dfrac{\partial^{\beta} Q(x,y,t)}{\partial y^{\beta}}\right)\\ \qquad -R_a Q(x,y,t)+b(x,y)u(t),\\ \dfrac{\partial Q(0,y,t)}{\partial x}=0, \quad t\geq 0,\\ \dfrac{\partial Q(X,y,t)}{\partial x}=0, \quad t\geq 0,\\ \dfrac{\partial Q(x,0,t)}{\partial y}=0, \quad t\geq 0,\\ \dfrac{\partial Q(x,Y,t)}{\partial y}=0, \quad t\geq 0,\\ Q(x,y,0)=Q_0, \quad 0\leq x\leq X,\ 0\leq y\leq Y,\\ y(t)=k_0\displaystyle\int_0^X\int_0^Y Q(x,y,t)c(x,y)\,\mathrm{d}x\mathrm{d}y. \end{cases} \tag{9}$$

where α and β are fractional orders of derivatives of time and length, $a_w>0$, $R_a\geq 0$ are coefficients of heat conduction and heat exchange, k_0 is a steady-state gain of the model, $b(x,y)$ and $c(x,y)$ are heater and sensor functions, expressed by (7) and (8).

The heat equation (9) can be expressed as an infinite-dimensional state equation:

$$\begin{cases} {}_0^C D_t^{\alpha} Q(t) = AQ(t)+Bu(t),\\ y(t)=CQ(t), \end{cases} \tag{10}$$

where:

$$\begin{aligned} AQ &= a_w\left(\frac{\partial^{\beta} Q(x,y)}{\partial x^{\beta}}+\frac{\partial^{\beta} Q(x,y)}{\partial y^{\beta}}\right)-R_a Q(x,y),\\ D(A) &= \left\{Q\in H^2(0,1):\ Q'(0)=0,\right.\\ &\qquad \left. Q'(X)=0,\ Q'(Y)=0\right\},\\ &a_w,\ R_a>0,\\ CQ(t) &= \langle c, Q(t)\rangle,\ \ Bu(t)=bu(t). \end{aligned} \tag{11}$$

The state vector $Q(t)$ takes the following form:

$$Q(t)=\left[q_{0,0},\ q_{0,1},\ q_{0,2}\ldots,\ q_{1,1},\ q_{1,2},\ldots\right]^T. \tag{12}$$

The state operator A is as follows:

$$\begin{aligned} A=\operatorname{diag}\{&\lambda_{0,0},\ \lambda_{0,1},\ \lambda_{0,2},\ \ldots,\ \lambda_{1,1},\\ &\lambda_{1,2},\ \ldots,\ \lambda_{2,1},\ \lambda_{2,2}\ldots,\ \lambda_{m,n},\ \ldots\}. \end{aligned} \tag{13}$$

The shape of the heated plant (thin metallic surface) suggests assuming the homogenous Neumann boundary conditions. Consequently the eigenfunctions and the eigenvalues take the following form:

$$w_{m,n}(x,y)=\begin{cases} 1, & m,\ n=0,\\ \dfrac{2Y}{\pi n}\cos\dfrac{n\pi y}{Y}, & m=0,\ n=1,2,\ldots\\ \dfrac{2X}{\pi m}\cos\dfrac{m\pi x}{X}, & n=0,\ m=1,2,\ldots\\ \dfrac{2}{\pi}\dfrac{1}{\left(\dfrac{m^{\beta}}{X^{\beta}}+\dfrac{n^{\beta}}{Y^{\beta}}\right)^{\frac{1}{\beta}}}\cos\dfrac{m\pi x}{X}\cos\dfrac{n\pi y}{Y}, & \\ & m,\ n=1,2,\ldots \end{cases} \tag{14}$$

$$\lambda_{m,n}=-a_w\left[\frac{m^{\beta}}{X^{\beta}}+\frac{n^{\beta}}{Y^{\beta}}\right]\pi^{\beta}-R_a, \quad m,\ n=0,1,2,\ldots \tag{15}$$

The main difference to the one-dimensional heat transfer equation is that the eigenvalues (15) can be multiple. The analysis of existence of multiple eigenvalues is presented in paper [40].

The control operator takes the following form [40]:

$$B=\left[b_{0,0},\ b_{0,1},\ \ldots,\ b_{1,0},\ b_{1,1},\ \ldots\right]^T, \tag{16}$$

where:

$$b_{m,n}=\langle H, w_{m,n}\rangle=\int_0^X\int_0^Y b(x,y)w_{m,n}(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{17}$$

Taking into account (14) we obtain:

$$b_{m,n}=\begin{cases} (x_{h2}-x_{h1})(y_{h2}-y_{h1}), & m,\ n=0,\\ \dfrac{1}{h_{yn}}(x_{h2}-x_{h1})\,a_{nhy}, & m=0,\ \ n=1,2,3,\ldots,\\ \dfrac{1}{h_{xm}}(y_{h2}-y_{h1})\,a_{mhx}, & n=0,\ \ m=1,2,3,\ldots,\\ \dfrac{k_{m,n}}{h_{xm}h_{yn}}a_{mhx}a_{nhy}, & m,\ n=1,2,3,\ldots \end{cases} \tag{18}$$

where:

$$\begin{aligned} h_{xm}&=\frac{m\pi}{X},\\ h_{yn}&=\frac{n\pi}{Y}. \end{aligned} \tag{19}$$

$$k_{m,n}=\frac{2}{\pi}\frac{1}{\sqrt[\beta]{\dfrac{m^{\beta}}{X^{\beta}}+\dfrac{n^{\beta}}{Y^{\beta}}}}, \tag{20}$$

$$\begin{aligned} a_{mhx}&=\left(\sin(h_{xm}x_{h2})-\sin(h_{xm}x_{h1})\right),\\ a_{nhy}&=\left(\sin(h_{yn}y_{h2})-\sin(h_{yn}y_{h1})\right); \end{aligned} \tag{21}$$

The output operator is as beneath [40]:

$$C = \left[c_{0,0},\ c_{0,1},\ \ldots,\ c_{1,0},\ c_{1,1},\ \ldots\right], \tag{22}$$

where:

$$c_{m,n} = \langle S, w_{m,n}\rangle = \int_0^X \int_0^Y c(x,y) w_{m,n}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{23}$$

In (23) each element $c_{m,n}$ is expressed analogically, as (18):

$$c_{m,n} = \begin{cases} (x_{s2}-x_{s1})(y_{s2}-y_{s1}), & m,n=0, \\ \dfrac{1}{h_{yn}}(x_{s2}-x_{s1})a_{nsy}, & m=0,\ \ n=1,2,3,\ldots, \\ \dfrac{1}{h_{xm}}(y_{s2}-y_{s1})a_{msx}, & n=0,\ \ m=1,2,3,\ldots, \\ \dfrac{k_{m,n}}{h_{xm}h_{yn}} a_{msx}a_{nsy}, & m,\ n=1,2,3,\ldots \end{cases} \tag{24}$$

In (24) $h_{xm,yn}$ are expressed by (19) and:

$$\begin{aligned} a_{msx} &= \left(\sin(h_{xm}x_{s2}) - \sin(h_{xm}x_{s1})\right), \\ a_{nsy} &= \left(\sin(h_{yn}y_{s2}) - \sin(h_{yn}y_{s1})\right). \end{aligned} \tag{25}$$

The dynamic system expressed by (13)–(23) is infinte-dimensional and of course its explicit form cannot be employed to modeling. This requires applying a finite-dimensional approximation. It is obtained by truncation of further modes of decomposed model at $M \times N$-th place (see [40]). In such a situation the state vector has the dimension $M \times N$ and operators A, B and C turn to matrices of suitable size. The values of M and N assuring the satisfactory accuracy of the model can be estimated numerically or analytically.

4.1. The step and impulse responses of the model

The step response of the model we obtain using spectrum decomposition property. It takes the following form (see [40]):

$$y_\infty(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} y_{m,n}(t), \tag{26}$$

where m,n-th mode of response is as follows:

$$y_{m,n}(t) = \frac{E_\alpha(\lambda_{m,n}t^\alpha) - 1(t)}{\lambda_{m,n}} b_{m,n}c_{m,n}. \tag{27}$$

In (27) $E_\alpha(..)$ is the one parameter Mittag-Leffler function, $\lambda_{m,n}$, $b_{m,n}$ and $c_{m,n}$ are expressed by (15), (17) and (23) respectively.

During simulations the finite-dimensional sum needs to be employed:

$$y_{MN}(t) = \sum_{m=0}^{M}\sum_{n=0}^{N} y_{m,n}(t). \tag{28}$$

The analysis of the external positivity requires knowing an impulse response of a system. It can be computed analogically as the step response, using the decomposition of the spectrum of the system.

The impulse response for a single mode of the system (10)–(23) is as follows:

$$g_{m,n}(t) = t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_{m,n}t^\alpha\right) b_{m,n}c_{m,n}, \tag{29}$$

where $E_{\alpha,\alpha}(..)$ is the two-parameter Mittag-Leffler function, the rest of parameters are the same as in (27).

Consequently the impulse response and its finite-dimensional approximation are as below:

$$g_\infty(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{m,n}(t), \tag{30}$$

$$g_{MN}(t) = \sum_{m=0}^{M}\sum_{n=0}^{N} g_{m,n}(t), \tag{31}$$

where $g_{m,n}(t)$ are the single modes expressed by (29).

5. MAIN RESULTS

Necessary and sufficient condition of the internal positivity is given by (5).

The state operator A is described by (13) and it is the Metzler matrix for each positive value of coefficients a_w and R_a. Testing of the internal positivity requires checking the positivity of the control and observation operators B and C only. Signs of particular elements of these operators are determined by the size and location of control and observation as well as the order of the model. Next, the positivity can be considered separately for control and observation. This is discussed beneath.

Lemma 1. (The positivity of the control operator B) Consider the dynamic system described by (13)–(23). Assume that the size of its finite-dimensional approximation is $M \times N$. The control operator B is positive iff the coordinates of the heater $x_{h1,2}$ and $y_{h1,2}$ meet the following condition:

$$\begin{cases} x_{h1} + \dfrac{\Delta x_h}{2} < \dfrac{X}{2M}, \\ y_{h1} + \dfrac{\Delta y_h}{2} < \dfrac{Y}{2N}, \end{cases} \tag{32}$$

where X and Y are the dimensions of the whole area, $\Delta x_h = x_{h2} - x_{h1}$ and $\Delta y_h = y_{h2} - y_{h1}$ describe the size of the heater.

Proof. The proof will be presented only for the horizontal coordinate x. For the vertical coordinate y it is analogical.

At the beginning notice that the sign of each element $b_{m,n}$ is determined only by the sign of the factors a_{mhx} and a_{nsy}, expressed by (25), because h_{xm} and h_{yn} expressed by (19) are always positive.

Next transform a_{hmx} to its equivalent form:

$$a_{mhx} = 2\cos\left(h_{xm}x_{h1} + \frac{h_{xm}\Delta x_h}{2}\right)\sin\left(\frac{h_{xm}\Delta x_h}{2}\right), \tag{33}$$

where h_{xm} is expressed by (19). The factor (33) is positive iff its both components are simultaneously positive or simultaneously negative. With respect to (33) this is expressed as follows:

$$\begin{aligned} &\cos\left(h_{xm}x_{h1} + \frac{h_{xm}\Delta x_h}{2}\right) > 0 \\ &\wedge \\ &\sin\left(\frac{h_{xm}\Delta x_h}{2}\right) > 0 \end{aligned} \tag{34}$$

or

$$\begin{aligned} &\cos\left(h_{xm}x_{h1} + \frac{h_{xm}\Delta x_h}{2}\right) < 0 \\ &\wedge \\ &\sin\left(\frac{h_{xm}\Delta x_h}{2}\right) < 0, \end{aligned} \tag{35}$$

Using elementary properties of sine and cosine functions the relations (34)–(35) can be expressed as follows:

$$\begin{aligned} &0 < h_{xm}\left(x_{h1} + \frac{\Delta x_h}{2}\right) < \frac{\pi}{2} \\ &\wedge \\ &0 < h_{xm}\left(\frac{\Delta x_h}{2}\right) < \frac{\pi}{2}, \end{aligned} \tag{36}$$

or

$$\begin{aligned} &\pi < h_{xm}\left(x_{h1} + \frac{\Delta x_h}{2}\right) < \frac{3\pi}{2} \\ &\wedge \\ &\pi < h_{xm}\left(\frac{\Delta x_h}{2}\right) < \frac{3\pi}{2}. \end{aligned} \tag{37}$$

Recalling (19) yields:

$$\begin{aligned} &0 < \left(x_{h1} + \frac{\Delta x_h}{2}\right) < \frac{X}{2m} \\ &\wedge \\ &0 < \left(\frac{\Delta x_h}{2}\right) < \frac{X}{2m}, \end{aligned} \tag{38}$$

or

$$\begin{aligned} &\frac{X}{m} < \left(x_{h1} + \frac{\Delta x_h}{2}\right) < \frac{3X}{m} \\ &\wedge \\ &\frac{X}{m} < \left(\frac{\Delta x_h}{2}\right) < \frac{3X}{m}. \end{aligned} \tag{39}$$

The inequalities (38)–(39) must be met for $0 < x < X$ and $m = 1, \ldots, M$. The inequality (39) for $m = 1$ is not met for $0 < x < X$. In the inequality (38) the strongest limitation is met for $m = M$ and for $x_{h1} + \frac{\Delta x_h}{2}$. This condition gives directly the first inequality (32) and the proof for x is completed. The second dependence for vertical coordinate y can be proven analogically. □

From the condition (32) the condition describing the maximum orders M_B and N_B can be proven, assuring the postivity of the control operator B for fixed location of heater. It is given by the following Lemma:

Lemma 2. (The dimension of the model assuring the positivity of the control operator B) Consider the dynamic system described by (13)–(23). Assume that the heater is attached in points x_{h1}, y_{h1} and its size is Δ_{xh}, Δ_{yh}.

The control operator B is positive iff the dimensions M_B and N_B of the finite-dimensional approximation of the model meet the following condition:

$$\begin{cases} M_B < \dfrac{X}{2x_{h1} + \Delta x_h}, \\ N_B < \dfrac{Y}{2y_{h1} + \Delta y_h}. \end{cases} \tag{40}$$

The positivity of the output operator C can be proven analogically. The suitable conditions can be proved analogically as Lemma 2. They are given beneath.

Proposition 1. (The positivity of the output operator C) Consider the dynamic system described by (13)–(23). Assume that the size of its finite-dimensional approximation is $M \times N$. The output operator C is positive iff the coordinates of the measuring place $x_{s1,2}$ and $y_{s1,2}$ meet the following condition:

$$\begin{cases} x_{s1} + \dfrac{\Delta x_s}{2} < \dfrac{X}{2M}, \\ y_{s1} + \dfrac{\Delta y_s}{2} < \dfrac{Y}{2N}, \end{cases} \tag{41}$$

where X and Y are the dimensions of surface, $\Delta x_s = x_{s2} - x_{s1}$ and $\Delta y_h = y_{s2} - y_{s1}$ describe the size of the measuring place.

Proposition 2. (The dimension of the model assuring the positivity of the output operator C) Consider the dynamic system described by (13)–(23). Assume that the temperature is measured in the point: x_{s1}, y_{s1} and its size is Δ_{xs}, Δ_{ys}. The output operator C is positive iff the dimensions of the finite-dimensional approximation of the model M_C and N_C meet the following condition:

$$\begin{cases} M_C < \dfrac{X}{2x_{s1} + \Delta x_s}, \\ N_C < \dfrac{Y}{2y_{s1} + \Delta y_s}. \end{cases} \tag{42}$$

The main difference between testing the positivity of the output and control operators is that the field of measurement can be much smaller than the surface of the heater. This allows one to obtain the field of observation bigger than the surface of heating.

Finally the conditions of the internal postivity of the considered thermal system can be formulated.

Proposition 3. (The internal positivity of the system) Consider the dynamic system described by (13)–(23). Assume that the dimension of its finite-dimensional approximation is $M \times N$. The system is internally positive iff observation and control meet the conditions (32) and (40).

Proposition 4. (The dimensions of the finite-dimensional approximation assuring the internal positivity) Consider the finite-dimensional approximation of the dynamic system described by (13)–(23). Assume that its dimensions assuring the positivity of control and output operators are equal $M_{B,C}$ and $N_{B,C}$ respectively.

The dimensions $M_{\text{int}P}$, $N_{\text{int}P}$ of the finite-dimensional approximation assuring the internal positivity are expressed as follows:

$$\begin{aligned} M_{\text{int}P} &= \min(M_B, M_C), \\ N_{\text{int}P} &= \min(N_B, N_C). \end{aligned} \tag{43}$$

It is worth noting that the conditions of the internal positivity proposed above are a little bit easier to keep than the analogical conditions formulated for one-dimensional heat system discussed in the paper [38].

Lemma 2 allows one to prove the Proposition about the internal postivity of the infinite-dimensional system.

Proposition 5. (The internal positivity of the infinite-dimensional system) The infinite-dimensional system (13)–(23) cannot be internally positive.

Proof. From dependencies (32) and (41) it can be concluded that the size of the area of heating and measurement descrease in the function of dimensions M and N. For $M \to \infty$ and $N \to \infty$ this size goes to zero. □

6. SIMULATIONS AND EXPERIMENTS

6.1. Example 1

As the first example assume that the size of the sensor in the thermal camera equals 380×290 pixels. We are looking for locations of heater and measurement assuring the internal positivity of the system. The size of the heater is 100×20 pixels and the size of the measurement field is 2×2 pixels. The "positive" locations of heater and sensor for different values of M and N are given in Table 1 and illustrated by Fig. 2. They were estimated using conditions (32) and (41).

Table 1
Dimensions of heater and measurement field assuring the internal positivity for dimensions of the model equal: $M = N = 2, 3, 4$

M, N	x_{h1}	Δx_h	y_{h1}	Δy_h	x_{s1}	Δx_s	y_{s1}	Δy_s
2,2	45	100	62	20	94	2	71	2
3,3	13	100	38	20	62	2	47	2
4,4	0	100	26	20	46	2	35	2

6.2. Example 2

As the next example look for the dimensions of model, for which the internal positivity is kept for fixed locations of heater and measurement. To do it the conditions (40) and (42) will be employed. This job will be done for the experimental system discussed in the paper [40]. The heater was attached in the location given in Table 2. The measurements were realized in different places, described in Table 3. The orders M_B, N_B, M_C and N_C assuring the positivity of control and observation are given also in these tables.

In Table 3 values "0, 0" for place No 3 denote that this place cannot be positive for any value of M or N.

Table 2
The dimensions M_B, N_B assuring the internal positivity, computed using (40)

x_{h1}	x_{h2}	y_{h1}	y_{h2}	M_B, N_B
100	270	40	60	1,3

Table 3
The dimensions M_C, N_C assuring the internal positivity, computed using (42)

Place	x_{s1}	x_{s2}	y_{s1}	y_{s2}	M_C, N_C
1	50	52	75	77	3,2
2	200	202	100	102	1,1
3	300	302	200	202	0,0
4	130	230	40	60	1,3

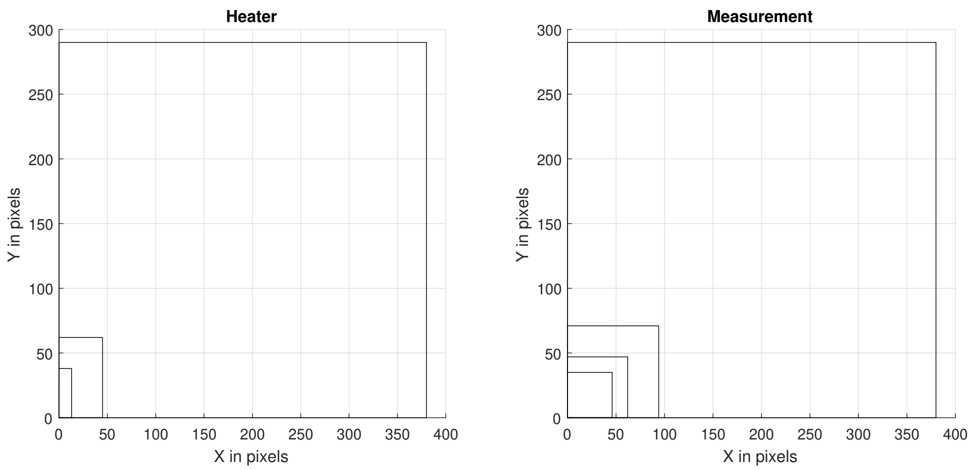

Fig. 2. The areas of the internal positivity for $M = N = 2, 3, 4$

6.3. Example 3

Finally the external positivity using the general condition (6) will be numerically examined. This is done using experimental data collected in Table 4. The model was identified for orders $M = N = 5$. For such orders the model is not internally positive (see Tables 2, 3), but the external positivity can be tested. Results of the external positivity tests are presented in right column of Table 4. Additionally tests are illustrated by the impulse responses obtained using (31) and shown in Fig. 3. This table and figure show that the external positivity of the proposed system can be obtained without internal one, analogically as in the one-dimensional system, discussed in the paper [38].

Table 4
Parameters of the model for $M = N = 5$

Place	α	β	a_w	R_a	Ext. Positivity?
1	1.0794	1.8641	0.0032	0.0032	Yes
2	0.9590	0.3959	0.0357	0.0057	Yes
3	1.4877	1.8712	0.0208	0.0003	No
4	0.8156	1.2400	0.0098	0.0234	Yes

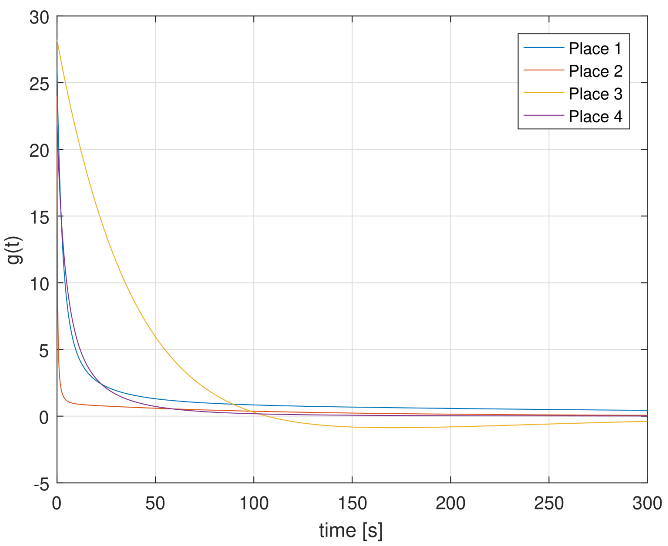

Fig. 3. The impulse responses in all considered places of measurement for $M = N = 5$

7. FINAL CONCLUSIONS

The first conclusion is that the positivity of the considered, two-dimensional heat system is associated to the dimensions and location of control and measurement as well as the size of model. This is an analogy to the one-dimensional case.

Next, the internal positivity is a little bit easier to obtain for two-dimensional system, than for one-dimensional case.

The internal positivity does not depend on both orders α neither β of the model. This makes possible to apply the proposed conditions to estimation of the positivity for both integer and noninteger order systems. This property has been also observed for one-dimensional thermal system (see [38]) and has been noticed by other researchers.

From the separated analysis of the positivity of control and output it can be concluded that the internal positivity of the output operator C is easier to obtain than the positivity of the control operator B. This is important during eventual application of the presented results in thermal imaging. If we deal with e.g. modeling of thermal traces, then only a response to initial condition is necessary to analyse.

The formulation of general conditions of positivity for general fractional system can be difficult due to the positivity determined by the form of operators A, B and C. Next, the form of these operators is determined by a construction of a real system we deal with. Surely, the formulating of general guidelines to construct of positive systems and their models is very interesting and it is planned to be further investigated.

Other areas of the further investigation covers the formulation of an analytical condition of the external positivity, analogically as in the one-dimensional case. Other issues to analyse there are e.g. propositions of control algorithms for thermal positive systems.

ACKNOWLEDGEMENTS

This paper was sponsored by AGH UST project no 16.16. 120.773.

REFERENCES

[1] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. New York, Chichester, Weinheim, Brisbane, Singapore, Toronto: John Wiley and sons, 2011.
[2] T. Kaczorek, *Positive 1D and 2D Systems*. New York: Springer, 2002.
[3] T. Kaczorek and K. Rogowski, *Fractional Linear Systems and Electrical Circuits*. Bialystok: Bialystok University of Technology, 2014.
[4] U. Krause, *Positive Dynamical Systems in Discrete Time. Theory, Models, and Applications*. De Gruyter, 2015.
[5] D. Luenberger, *Introduction to Dynamic Systems. Theory, Models and Applications*. USA: J. Wiley and Sons, 1979.
[6] A. Ratzer and M. Valcher, "A tutorial on positive systems and large scale control," in *2018 IEEE Conference on Decision and Control (CDC), Miami Beach, FL, USA, 17-19 Dec. 2018*, 2018.
[7] A. Benzaouia, F. Mesquine, M. Benhayoun, and A.B. Braim, "Stabilization of continuous-time fractional positive systems with delays and asymmetric control bounds," *J. Dyn. Syst. Meas. Control*, vol. 141, no. 5, p. 8, 2019.
[8] A. Rantzer, "Distributed control of positive systems," *Eur. J. Control*, vol. 24, no. 1, pp. 72–80, 2015.
[9] C. Walker, "Positive solutions of some parabolic system with cross-diffusion and nonlocal initial conditions," *NoDea-Nonlinear Differ. Equ. Appl.*, vol. 19, no. 1, p. 195–218, 2012.
[10] M. Zheng and Y. Ohta, "Positive fir system identification using maximum entropy prior," *IFAC PapersOnLine*, vol. 51, no. 15, pp. 7–12, 2018.
[11] R. Drummond, M.C. Turner, and S.R. Duncan, "External positivity of linear systems by weak majorisation," in *2019 American Control Conference (ACC)*, 2019, pp. 5191–5196.

[12] Y. Ebihara, “Construction of externally positive systems for discrete-time lti system analysis,” *IFAC PapersOnLine*, vol. 51, no. 25, p. 447–452, 2018.

[13] C. Grussler and A. Rantzer, “Modified balanced truncation preserving ellipsoidal cone-invariance,” in *53rd IEEE Conference on Decision and Control (CDC)*. United States: Institute of Electrical and Electronics Engineers Inc., 2014, p. 2365–2370.

[14] C. Grussler and A. Rantzer, “A tractable second-order cone certificate for external positivity with application to model order reduction,” *arXiv:1906.06139v1*, vol. 1, no. 1, pp. 1–10, 2019.

[15] C. Grussler, J. Umenberger, and I. Manchester, “Identification of externally positive systems,” in *2017 IEEE 56th Annual Conference on Decision and Control, CDC 2017*, vol. 2018-January. United States: Institute of Electrical and Electronics Engineers Inc., 1 2018, pp. 6549–6554.

[16] C. Altafini, “Representing externally positive systems through minimal eventually positive realizations,” in *54th IEEE Conference on Decision and Control (CDC)*, 2015, pp. 6385–6390.

[17] C. Altafini, “Minimal eventually positive realizations of externally positive systems,” *Automatica*, vol. 68, no. 6, pp. 140–147, 2016.

[18] R. Caponetto, G. Dongola, L. Fortuna, and I. Petras, “Fractional order systems: Modeling and Control Applications,” in *World Scientific Series on Nonlinear Science*, L.O. Chua, Ed. Berkeley: University of California, 2010, pp. 1–178.

[19] S. Das, *Functional Fractional Calculus for System Identyfication and Control*. Berlin: Springer, 2010.

[20] A. Dzieliński, D. Sierociuk, and G. Sarwas, “Some applications of fractional order calculus,” *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 58, no. 4, pp. 583–592, 2010.

[21] C. Gal and M. Warma, “Elliptic and parabolic equations with fractional diffusion and dynamic boundary conditions,” *Evol. Equ. Control Theory*, vol. 5, no. 1, pp. 61–103, 2016.

[22] I. Podlubny, *Fractional Differential Equations*. San Diego: Academic Press, 1999.

[23] D. Sierociuk, T. Skovranek, M. Macias, I. Podlubny, I. Petras, A. Dzielinski, and P. Ziubinski, “Diffusion process modeling by using fractional-order models,” *Appl. Math. Comput.*, vol. 257, no. 1, pp. 2–11, 2015.

[24] E. Popescu, “On the fractional cauchy problem associated with a feller semigroup,” *Math. Rep.*, vol. 12, no. 2, pp. 181–188, 2010.

[25] T. Kaczorek, “Reduced-order fractional descriptor observers for a class of fractional descriptor continuous-time nonlinear systems,” *Int. J. Appl. Math. Comput. Sci.*, vol. 26, no. 2, pp. 277–283, 2016.

[26] R. Almeida and D.F.M. Torres, “Necessary and sufficient conditions for the fractional calculus of variations with caputo derivatives,” *Commun. Nonlinear Sci. Numer. Simul.*, vol. 16, no. 3, pp. 1490–1500, 2011.

[27] B. Baeumer, S. Kurita, and M. Meerschaert, “Inhomogeneous fractional diffusion equations,” *Fract. Calc. Appl. Anal.*, vol. 8, no. 4, pp. 371–386, 2005.

[28] M. Dlugosz and P. Skruch, “The application of fractional-order models for thermal process modelling inside buildings,” *J. Build. Phys.*, vol. 1, no. 1, pp. 1–13, 2015.

[29] A. Kochubei, “Fractional-parabolic systems, preprint, arxiv: 1009.4996 [math.ap],” 2011.

[30] W. Mitkowski, “Approximation of fractional diffusion-wave equation,” *Acta Mech. Automatica*, vol. 5, no. 2, pp. 65–68, 2011.

[31] A. Rauh, L. Senkel, H. Aschemann, V.V. Saurin, and G. Kostin, “An integrodifferential approach to modeling, control, state estimation and optimization for heat transfer systems,” *Int. J. Appl. Math. Comput. Sci.*, vol. 26, no. 1, pp. 15–30, 2016.

[32] T. Kaczorek, “Practical stability of positive fractional discrete-time systems,,” *Bull. Pol. Acad. Sci. Tech. Sci.*, vol. 56, no. 4, pp. 313–317, 2008.

[33] T. Kaczorek, “Positive fractional linear systems,” *PAR magazine*, vol. 2011, no. 2, pp. 91–112, 2011.

[34] T. Kaczorek, “Positive stable realizations of fractional continuous-time linear systems,” *Int. J. Appl. Math. Comput. Sci.*, vol. 21, no. 4, pp. 697–702, 2011.

[35] T. Kaczorek, “Stability of interval positive fractional discrete-time linear systems,” *Int. J. Appl. Math. Comput. Sci.*, vol. 28, no. 3, pp. 451–456, 2018.

[36] T. Kaczorek, “Singular fractional linear systems and electrical circuits,” *Int. J. Appl. Math. Comput. Sci.*, vol. 21, no. 2, pp. 379–384, 2011.

[37] T. Kaczorek, *Selected Problems of Fractional Systems Theory*. Berlin: Springer, 2011.

[38] K. Oprzędkiewicz, “Positivity problem for the class of fractional order, distributed parameter systems.” *ISA Trans.*, vol. 112, no. 1, pp. 281–291, 2021.

[39] K. Oprzędkiewicz, K. Dziedzic, and W. Mitkowski, “Accuracy analysis of the fractional order, positive, state space model of heat transfer process,” in *MMAR 2021: 25th international conference on Methods and Models in Automation and Robotics*, Poland, 2021, pp. 325–330.

[40] K. Oprzędkiewicz, W. Mitkowski, and M. Rosol, “Fractional order model of the two dimensional heat transfer process.” *Energies*, vol. 14, no. 19, pp. 1–17, 2021.