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Conditions for asymptotic stability of first order scalar differential-difference equation with complex coefficients

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We investigate a scalar characteristic exponential polynomial with complex coefficients associated with a first order scalar differential-difference equation. Our analysis provides necessary and sufficient conditions for allocation of the roots in the complex open left half-plane what guarantees asymptotic stability of the differential-difference equation. The conditions are expressed explicitly in terms of complex coefficients of the characteristic exponential polynomial, what makes them easy to use in applications. We show examples including those for retarded PDEs in an abstract formulation.

Key words: first order differential-difference equation with complex coefficients, stability of differential-difference equation, characteristic exponential polynomial of differential-difference equation, retarded differential-difference equation (DDE).

1. Introduction

In this article we study the asymptotic stability of a scalar linear differential-difference equation (DDE)

$$x'(t) = \lambda x(t) + \gamma x(t - \tau), \quad t \geq 0, \quad (1)$$

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where $\lambda, \gamma \in \mathbb{C}$ and $0 < \tau$, through the analysis of the corresponding characteristic equation

$$s - \lambda - \gamma e^{-s\tau} = 0. \quad (2)$$

This problem is frequently related to stability analysis of

$$x'(t) = f(x(t), x(t - \tau)), \quad t \geq 0, \quad (3)$$

where $f: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth (nonlinear) function about $x_0 \in \mathbb{R}^n$, via its linearization given by

$$x'(t) = Ax(t) + Bx(t - \tau), \quad (4)$$

where $A = J_1 f(x_0, x_0)$ and $B = J_2 f(x_0, x_0)$ are partial Jacobian matrices of f at (x_0, x_0) . To be more precise we use the following:

Definition 1 *An equilibrium solution $x^*(t) \equiv x_0 \in \mathbb{R}^n$ of (3) is exponentially stable if there exist $M, \omega, \delta > 0$ such that $\|x(t) - x_0\| \leq M e^{-\omega t}$ ($t \geq 0$) holds for every solution x of (3) satisfying an initial condition $\|x(t) - x_0\| < \delta$ ($t \in [-\tau, 0]$) with the Euclidean norm $\|\cdot\|$.*

By the principle of linearized stability [5] for the case at hand we have [10]

Fact 1 *Let the linearization of (3) about an equilibrium solution $x^*(t) \equiv x_0$ be expressed by (4) and let the corresponding characteristic equation be given by*

$$\det [sI - A - B e^{-s\tau}] = 0. \quad (5)$$

Then the following statements hold:

- (i) x^* is exponentially stable if $\operatorname{Re} s < 0$ for all characteristic roots s of (5),
- (ii) x^* is unstable if $\operatorname{Re} s > 0$ for some characteristic root s of (5).

Our motivation to investigate (1) with complex coefficients comes from the following infinite-dimensional setting. Consider (4) on a Hilbert space X where A is a diagonal generator of a strongly continuous semigroup and B is a linear diagonal operator on X (see Example 5.4 below, [7] or [13]). Then (1) describes dynamics of (4) about a single coordinate (sometimes referred to as a mode) that corresponds to a given eigenvalue λ of A . As in infinite dimensions surprisingly many dynamical systems are represented by diagonal generators [15], we thus want to describe the behavior of the whole infinite-dimensional system based on some form of “component-wise addition” of infinitely many modes. Such approach requires a readily usable condition for determining stability of each mode, and it is this condition that we strive for in this article.

In a finite-dimensional setting and under appropriate, though restrictive conditions, for example if A and B in (4) commute – see [9] or [14] – the problem of

asymptotic stability of (3) is equivalent to finding conditions on the coefficients of (1) which will guarantee that every root s of (2) is such that $\operatorname{Re}(s) < 0$. Note that in this setting coefficients λ and γ in (2) are again eigenvalues of A and B , respectively. It is, however, something rather special, even in finite-dimensions, that linearization of (3) produces commuting A and B .

In the remaining part whenever mentioning stability we refer to a situation when all the roots of (2) have negative real parts. The literature contains two intertwined approaches to stability problem – one based on analysis of some form of (4) in time domain and one based on analysis of (5). In the latter approach the case with $\lambda, \gamma \in \mathbb{R}$ is well understood – see [6], where the author obtained necessary and sufficient conditions for stability of $s - a - c e^{-s} = 0$ with $a, c \in \mathbb{R}$. In a more general form $A(s) + B(s) e^{-s\tau} = 0$, with $A(s)$ and $B(s)$ real polynomials see [12] and references therein. For a thorough exposition of other methods of analysis of the real λ and γ case see [18] and references therein.

The case $\lambda, \gamma \in \mathbb{C}$ is less analyzed. In particular, some sufficiency results can be found in [1], where the author presents a numerical analysis of (1) and stipulates asymptotic stability for every $\tau > 0$ if $-\operatorname{Re} \lambda > |\gamma|$. The authors of [3] present, based on algorithmic criteria, some sufficiency result for specific values of complex λ and γ , proving also the result in [1] for some cases. Authors of [17] built on [3] and provide additional sufficient conditions for stability. In [8] the author uses a continuous dependence of the roots of (1) on τ and manages to obtain stability conditions for some values of $\lambda \in \mathbb{R}$ and $\gamma \in \mathbb{C}$. Recently in [10] the author provides necessary and sufficient conditions for the zeros of (2) to be in the left complex half-plane. The argument there is based on analysis of the Lambert W function, what complicates applications of obtained conditions to an infinite set of equations of type (2). In particular, the condition from [10, Theorem 1.2] uses a nested trigonometric functions of $\operatorname{Im} \lambda$ and $\operatorname{Arg} \gamma$, what makes it hard to visualize a coefficients-plane region that ensures stability of a single equation, and is even more difficult for a series of such equations.

To the best of authors' knowledge [11] is the first work providing necessary and sufficient conditions for stability of (2) for $\lambda, \gamma \in \mathbb{C}$ and $\tau = 1$. Results of [11] are, however, based on specific analysis of roots of (2) which is uneasy to trace for different values of τ , even after change of parameters $a \mapsto a\tau$ and $\eta \mapsto \eta\tau$ (see (12) below). This may explain why, although it precedes many of the works mentioned above, [11] did not receive much recognition.

Our approach here combines analysis of roots placement depending on τ , as shown in [12, Proposition 6.2.3], with arguments of algebraic nature in the complex plane. This allowed us to obtain necessary and sufficient conditions for stability of (2) based explicitly on a relation between $\lambda, \gamma \in \mathbb{C}$ and $\tau > 0$. The conditions do not require to calculate any specific roots of a transcendental equation and allow to visualize how the “stability” region changes with parameters

in the coefficients-plane. Thus, we not only provide a different formulation of stability conditions, but our results are based on a new, different proof.

2. Preliminaries

The following observation, which can be found e.g. in [3] or [11], is crucial to simplify the problem of analysis of (1).

Lemma 1 *Let $a, b, c, d, \tau \in \mathbb{R}$ and let $\{s_0\}$ be the set of roots of*

$$s - (a + ib) - (c + id) e^{-s\tau} = 0 \quad (6)$$

and $\{z_0\}$ be the set of roots of

$$z - a - e^{-ib\tau}(c + id) e^{-z\tau} = 0. \quad (7)$$

Then $\operatorname{Re}(s_0) < 0$ for all s_0 if and only if $\operatorname{Re}(z_0) < 0$ for all z_0 .

Proof. Let z_0 be a root of (7). Then $s_0 = z_0 + ib$ is a root of (6). Conversely, let s_0 be a root of (6). Then $z_0 = s_0 - ib$ is a root of (7). As $\operatorname{Re}(s_0) = \operatorname{Re}(z_0)$ the result follows. \square

Remark 1 *It is worth to mention that a similar, yet different simplification is possible. In [2] the author's approach is based on a version of Lemma 1 where the non-delayed coefficient in (7) is complex and the one corresponding to the delay is real. In our approach, however, the current version of Lemma 1 is more convenient.*

We will also use the following result concerning parameter $\beta \in \mathbb{R}$ and real functions:

$$L, R: [0, \infty) \rightarrow \mathbb{R}, \quad L(r) := \frac{r}{r^2 + 1}, \quad R(r) := \arctan(r) + \beta. \quad (8)$$

Lemma 2 *Let $\beta \in \mathbb{R}$ and put:*

$$A = \{r \in [0, \infty) : L(r) \leq R(r)\},$$

where real functions L and R are given by (8). Then:

- (i) $A = [0, \infty)$ if and only if $\beta \geq 0$,
- (ii) $A = [r_0, \infty)$ with $r_0 > 0$ if and only if $\beta \in (-\frac{\pi}{2}, 0)$, wherein the correspondence $(0, \infty) \ni r_0 \longleftrightarrow \beta \in (-\frac{\pi}{2}, 0)$ is one-to-one,
- (iii) set A is empty if and only if $\beta \leq -\frac{\pi}{2}$.

Proof. Let us consider function $\varphi: [0, \infty) \rightarrow \mathbb{R}$ given by $\varphi = R - L$. Clearly, we have

$$\varphi'(r) = \frac{2r^2}{(r^2 + 1)^2} > 0, \quad \varphi(0) = \beta, \quad \varphi(r) \longrightarrow \beta + \frac{\pi}{2} \quad \text{as } r \rightarrow \infty.$$

In particular φ is increasing and $\varphi([0, \infty)) = [\beta, \beta + \frac{\pi}{2})$.

If $\beta \geq 0$, then $\varphi(r) \geq 0$ for $r \in [0, \infty)$, i.e. $A = [0, \infty)$. On the other hand if $L(0) \leq R(0)$, then $\beta \geq 0$. This gives assertion (i).

Suppose $\beta \in (-\frac{\pi}{2}, 0)$. Hence $\varphi(0) < 0$ and $\varphi(r) > 0$ for large enough $r > 0$. Then there exists a unique $r_0 > 0$ such that $\varphi(r_0) = 0$. This shows that $A = [r_0, \infty)$. If now $A = [r_0, \infty)$ for some $r_0 > 0$, then $\beta < 0$ by (i). To finish the proof, it is enough to notice that for $\beta \leq \frac{\pi}{2}$ we have $\varphi(r) < 0$ for $r \in [0, \infty)$, i.e. $A = \emptyset$. \square

Corollary 1 Equation $L(r) = R(r)$ has exactly one solution if and only if $\beta \in (-\frac{\pi}{2}, 0]$.

We also make use of the following half-planes:

$$\begin{aligned} \mathbb{C}_+ &:= \{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}, & \mathbb{C}_- &:= \{s \in \mathbb{C} : \operatorname{Re}(s) < 0\}, \\ \Pi_+ &:= \{s \in \mathbb{C} : \operatorname{Im}(s) > 0\}, & \Pi_- &:= \{s \in \mathbb{C} : \operatorname{Im}(s) < 0\}. \end{aligned}$$

3. Main results

By Lemma 1 we restrict our attention to (7). Taking $\eta = u + iv = e^{-ib\tau}(c + id)$ the conditions for stability of (2) are given on an (u, iv) -complex plane in terms of regions that depend on a and τ .

Remark 2 We take the principal argument of λ to be $\operatorname{Arg} \lambda \in (-\pi, \pi]$.

Let $\mathbb{D}_r \subset \mathbb{C}$ be an open disc centered at 0 with radius $r > 0$. We shall require the following subset of the complex plane, depending on $\tau > 0$ and $a \in (-\infty, \frac{1}{\tau}]$, namely: $\Lambda_{\tau, a}$

- for $a < 0$:

$$\begin{aligned} \Lambda_{\tau, a} &:= \left\{ \eta \in \mathbb{C} \setminus \mathbb{D}_{|a|} : \operatorname{Re} \eta + a < 0, |\eta| < |\eta_\pi|, \right. \\ &\quad \left. |\operatorname{Arg} \eta| > \tau \sqrt{|\eta|^2 - a^2} + \arctan \left(-\frac{1}{a} \sqrt{|\eta|^2 - a^2} \right) \right\} \cup \mathbb{D}_{|a|}, \quad (9) \end{aligned}$$

where η_π is such that

$$\sqrt{|\eta_\pi|^2 - a^2}\tau + \arctan\left(-\frac{1}{a}\sqrt{|\eta_\pi|^2 - a^2}\right) = \pi;$$

- for $a = 0$:

$$\Lambda_{\tau,a} := \left\{ \eta \in \mathbb{C} \setminus \{0\} : \operatorname{Re} \eta < 0, |\eta| < \frac{\pi}{2\tau}, |\operatorname{Arg} \eta| > \tau|\eta| + \frac{\pi}{2} \right\}; \quad (10)$$

- for $0 < a \leq \frac{1}{\tau}$

$$\Lambda_{\tau,a} := \left\{ \eta \in \mathbb{C} : \operatorname{Re} \eta + a < 0, |\eta| < |\eta_\pi|, \right. \\ \left. |\operatorname{Arg} \eta| > \tau\sqrt{|\eta|^2 - a^2} + \arctan\left(-\frac{1}{a}\sqrt{|\eta|^2 - a^2}\right) + \pi \right\}, \quad (11)$$

where η_π is such that $|\eta_\pi| > a$ and

$$\sqrt{|\eta_\pi|^2 - a^2}\tau + \arctan\left(-\frac{1}{a}\sqrt{|\eta_\pi|^2 - a^2}\right) = 0.$$

Figures 1, 2 and 3 show $\Lambda_{\tau,a}$ for fixed values of a and varying τ , while Figure 4 shows $\Lambda_{\tau,a}$ for fixed τ and varying a .

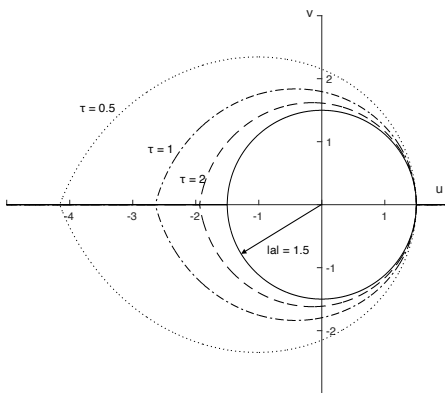


Figure 1: Outer boundaries of the $\Lambda_{\tau,a}$, defined in (9) with $\eta = u + iv$, for $a = -1.5$ and different values of τ : dotted for $\tau = 0.5$, dash-dotted for $\tau = 1$, dashed for $\tau = 2$. The solid line shows a circle with radius $|a| = 1.5$

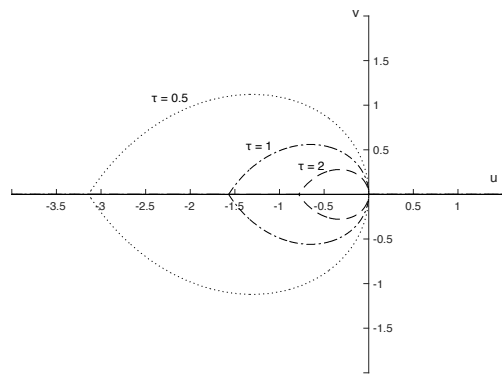


Figure 2: Outer boundaries of the $\Lambda_{\tau,a}$, defined in (10) with $\eta = u + iv$, for $a = 0$ and different values of τ : dotted for $\tau = 0.5$, dash-dotted for $\tau = 1$, dashed for $\tau = 2$

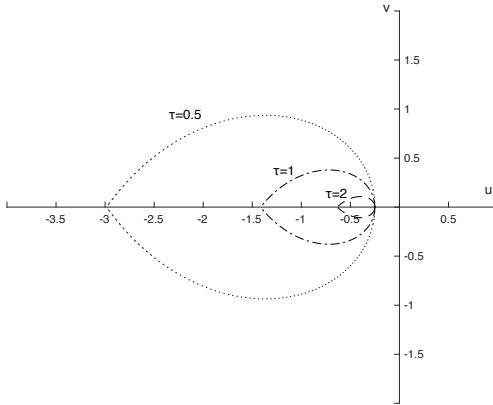


Figure 3: Outer boundaries of the $\Lambda_{\tau,a}$, defined in (11) with $\eta = u + iv$, for $a = 0.25$ and different values of τ : dotted for $\tau = 0.5$, dash-dotted for $\tau = 1$, dashed for $\tau = 2$

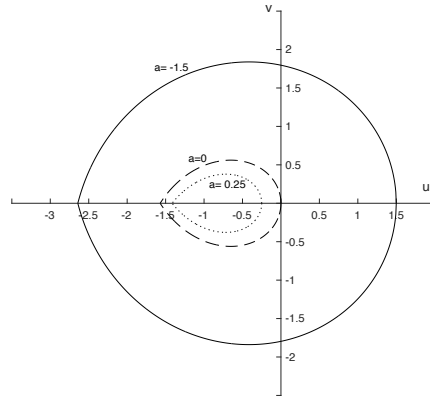


Figure 4: Outer boundaries of $\Lambda_{\tau,a}$, defined in (9)–(11) with $\eta = u + iv$, for $\tau = 1$ and different values of a : solid for $a = -1.5$, dashed for $a = 0$ and dotted for $a = 0.25$

The zeros of (2) are in the left half plane \mathbb{C}_- , according to Lemma 1, if and only if the roots of (7) belong to \mathbb{C}_- . Thus for $\lambda = a+ib, \gamma = c+id$ and $\eta := e^{-ib\tau} \gamma$ we have the following

Theorem 1 *Let $\tau > 0, a \in \mathbb{R}$ and $\eta \in \mathbb{C}$. Then every solution of the equation*

$$s - a - \eta e^{-s\tau} = 0 \tag{12}$$

belongs to $\mathbb{C} \setminus \mathbb{C}_+$ if and only if $a \leq \frac{1}{\tau}$ and η belongs to the closure of $\Lambda_{\tau,a}$ given by (9), (10) or (11).

As above stated, Theorem 1 gives a necessary and sufficient condition for zeros of (12) to not be in \mathbb{C}_+ . Although this is a weaker result than (asymptotic) stability, we formulated Theorem 1 in this way as it reflexes our approach to the problem and makes its (already long) proof more approachable. The immediately following Corollary 2 gives the necessary and sufficient condition of the zeros of (12) to be in \mathbb{C}_- i.e. asymptotic stability of (2).

The proof of Theorem 1 is composed of three parts. Part I contains points 2–5 and describes how infinitely many zeros of (2) move in the complex plane with changing the delay τ , until for some τ_0 they reach the imaginary axis i.e. a boundary line in stability analysis. Part II contains points 6–15 and shows how the relations between parameters of (2) for the boundary case can be used to determine stability regions in the parameter-plane. This is done with: $\tau = \tau_0$, particular values of a and varying η . Part III is composed of point 16 and shows the sufficiency of necessity results established in the previous parts.

Proof. [Proof of Theorem 1]

1. Denote the closure of $\Lambda_{\tau,a}$ by $\overline{\Lambda_{\tau,a}}$. For any $\tau > 0$ and $a \leq 0$ there is $0 \in \overline{\Lambda_{\tau,a}}$ and taking $\eta = 0$ the statement of the proposition obviously holds true, while for $0 < a \leq \frac{1}{\tau}$ we have $0 \notin \overline{\Lambda_{\tau,a}}$. Thus, for the remainder of the proof assume that $\eta \neq 0$.

Part I

2. It is known that for $\tau > 0$ equation (12) has infinitely many solutions. By the Rouché's theorem (see e.g. [18, Prop.1.14]) solutions of (12) vary continuously with τ , except at $\tau = 0$ where only one remains. Let $\eta = u + iv$ and let $s = x + iy$. In the limit as $\tau \rightarrow 0$ in (12) we obtain

$$x = a + u, \quad y = v,$$

and the solutions start in $\mathbb{C} \setminus \mathbb{C}_+$ i.e. with $x \leq 0$ if and only if $a + u \leq 0$.

Let us establish when at least one of the solutions crosses the imaginary axis for the first time as τ increases from zero upwards. At the crossing of the imaginary axis there is $s = i\omega$ for some $\omega \in \mathbb{R}$. In view of (12) we can treat s as an implicit function of τ and check the direction in which zeros of it cross the imaginary axis by analyzing the $\text{sgn Re } \frac{ds}{d\tau}$ if $s = i\omega$. By calculating the implicit function derivative we have:

$$\frac{ds}{d\tau} = -\frac{s^2 - as}{1 - a\tau + s\tau}.$$

As $\text{sgn Re } z = \text{sgn Re } z^{-1}$ we have if $s = i\omega$ that

$$\text{sgn Re } \frac{ds}{d\tau} = \text{sgn } \frac{1}{\omega^2 + a^2} > 0$$

and the zeros cross from the left to the right half-plane. As the sign of the above does not depend on τ , the direction of the crossing remains the same for every value of τ . Thus with $\eta = u + iv$ a necessary condition for the solutions of (12) to be in $\mathbb{C} \setminus \mathbb{C}_+$ is

$$a + u \leq 0. \tag{13}$$

3. Consider again (12) with fixed $\tau > 0$ and $a \in \mathbb{R}$ and take such $\eta \in \mathbb{C}$ that (13) holds. Let us focus on the crossing point i.e. let $s = i\omega$ for some $\omega \in \mathbb{R}$. Taking the complex conjugate of (12) at the crossing we obtain

$$-i\omega - a - \bar{\eta} e^{i\omega\tau} = 0. \tag{14}$$

Using now (12) for $s = i\omega$ and (14) to eliminate the exponential part we have $\omega^2 = |\eta|^2 - a^2$. From here we see that for a given $a \in \mathbb{R}$ and every $\eta = u + iv$ satisfying both, (13) and $|\eta| < |a|$, the crossing does not exist, regardless of τ , and all the solutions of (12) are in \mathbb{C}_- . Note, that all the solutions of (12) are in \mathbb{C}_- also when the inequality in (13) is sharp and $|\eta| \leq |a|$.

4. Let us focus on the case when the first crossing happens. To that end consider (12), fix $a \in \mathbb{R}$ and take η such that (13) and

$$|\eta| \geq |a| \quad (15)$$

hold. By point 2 as τ increases the roots of (12) move continuously to the right. Denote by τ_0 the smallest τ for which the crossing happens. By assumptions we know that such τ_0 exists. By point 3 the crossing takes place at $s = \pm i\sqrt{|\eta|^2 - a^2}$. Putting $s = i\sqrt{|\eta|^2 - a^2}$ into (12) with $\tau = \tau_0$ gives

$$\begin{aligned} \eta &= -a e^{i\sqrt{|\eta|^2 - a^2}\tau_0} + \sqrt{|\eta|^2 - a^2} e^{i(\frac{\pi}{2} + \sqrt{|\eta|^2 - a^2}\tau_0)} \\ &= e^{i\sqrt{|\eta|^2 - a^2}\tau_0} \left(-a + i\sqrt{|\eta|^2 - a^2} \right). \end{aligned} \quad (16)$$

Putting $s = -i\sqrt{|\eta|^2 - a^2}$ into (12) gives an equation corresponding to (16), namely

$$\eta = e^{-i\sqrt{|\eta|^2 - a^2}\tau_0} \left(-a - i\sqrt{|\eta|^2 - a^2} \right). \quad (17)$$

Equations (16) and (17) show the relation between all coefficients (or parameters) of (12) in the boundary case of transition between asymptotic stability and instability. Thus we focus on the triple (τ_0, a, η) and how changes within it influence stability of

$$s - a - \eta e^{-s\tau_0} = 0. \quad (18)$$

By point 2, for a and η as in (16) and with every $\tau > \tau_0$ equation (18) is unstable, while for $\tau < \tau_0$ it is stable. And so, while keeping τ_0 and a fixed, we turn our attention to varying η .

5. Let η be a solution of (16). Then $\bar{\eta}$ is a solution of (17) and these solutions are obviously symmetric about the real axis. It will be more convenient to use different notation that the one in (16) or (17). Define $\gamma_+ : [|a|, \infty) \rightarrow \mathbb{C}$ and $\gamma_- : [|a|, \infty) \rightarrow \mathbb{C}$ as the right side of (16) and (17), respectively, i.e.

$$\gamma_+(w) := e^{i\sqrt{w^2 - a^2}\tau_0} \left(-a + i\sqrt{w^2 - a^2} \right), \quad (19)$$

$$\gamma_-(w) := e^{-i\sqrt{w^2 - a^2}\tau_0} \left(-a - i\sqrt{w^2 - a^2} \right). \quad (20)$$

Let $\Gamma_+ := \gamma_+([|a|, \infty))$ be the image of (19) and $\Gamma_- := \gamma_-([|a|, \infty))$ be the image of (20). We easily see that $\gamma_+(w) = \overline{\gamma_-(w)}$ for every $w \geq |a|$ and so Γ_+ is symmetric to Γ_- about the real axis.

Up to this moment all considerations in points 2–5 were done regardless of the sign of parameter a . In the reminder of the proof, along with (13) and (15), we will consider additional assumptions on a , namely $a < 0$, $a = 0$, $a \in \left(0, \frac{1}{\tau_0}\right]$ and $a > \frac{1}{\tau_0}$.

Part II

6. Assume additionally that $a < 0$ and let a function describing a continuous argument increment of (19) be given by $\Delta\gamma_+ : [|a|, \infty) \rightarrow [0, \infty)$,

$$\Delta\gamma_+(w) = \sqrt{w^2 - a^2}\tau_0 + \arctan\left(-\frac{1}{a}\sqrt{w^2 - a^2}\right). \quad (21)$$

We easily see that it is a strictly increasing, non-negative function. We also define $\Delta\gamma_- : [|a|, \infty) \rightarrow (-\infty, 0]$ and $\Delta\gamma_-(w) = -\Delta\gamma_+(w)$ for every $w \in [|a|, \infty)$.

Looking at (19) note that the first component has modulus 1 and introduces counter-clockwise rotation, while the second component is always in the first quadrant, with a positive real part equal to $-a$, and its modulus is strictly increasing and tends to infinity as $w \rightarrow \infty$. Thus Γ_+ is a curve that is a counter-clockwise outward spiral that begins in $-a \in \mathbb{C}$. An exemplary pair of Γ_+ and Γ_- curves is shown in Fig. 5.

7. Let a set $\{\eta_{(2k-1)\pi}\}_{k \in \mathbb{N}}$ be such that the argument increment along Γ_+ as w changes from $|a|$ to $|\eta_{(2k-1)\pi}|$ is equal to $(2k-1)\pi$, that is

$$\Delta\gamma_+(|\eta_{(2k-1)\pi}|) = (2k-1)\pi. \quad (22)$$

Due to constraint (13) we take into account only these parts of Γ_+ (or Γ_-) that lie to the left of $u = -a$ line, as depicted in Figs. 5 and 6. Let us now focus on the closure of the first part of Γ_+ that lies in Π_+ i.e. $\gamma_+([|a|, |\eta_\pi|])$. By (21) and (22) for every $w \in [|a|, |\eta_\pi|]$ we have $\Delta\gamma_+(w) \in [0, \pi]$. For the case of the part of Γ_- equal to $\gamma_-([|a|, |\eta_\pi|])$ the argument expression gives $\Delta\gamma_-(w) = -\Delta\gamma_+(w)$. Expressing both cases with one equation, with the notation of (16) and (17), we arrive at

$$|\text{Arg } \eta| = \sqrt{|\eta|^2 - a^2}\tau_0 + \arctan\left(-\frac{1}{a}\sqrt{|\eta|^2 - a^2}\right), \quad |\eta| \leq |\eta_\pi|, \quad (23)$$

where η_π is such that:

$$\Delta\gamma_+(|\eta_\pi|) = \sqrt{|\eta_\pi|^2 - a^2}\tau_0 + \arctan\left(-\frac{1}{a}\sqrt{|\eta_\pi|^2 - a^2}\right) = \pi.$$

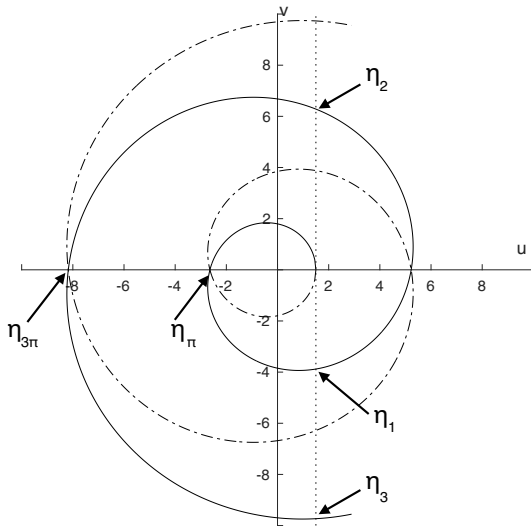


Figure 5: Curves Γ_+ (solid line) and Γ_- (dash-dotted line) drawn for $\tau_0 = 1$ and $a = -1.5$ with $|\eta| = w \in (|a|, 10)$. The constraint related to a and expressed by (13) is marked with a dotted line. The crossings of the real negative semi-axis by Γ_+ (and Γ_-) are at η_π and $\eta_{3\pi}$. The crossings of $u = -a$ by Γ_+ , as w increases, are at η_1, η_2 and η_3

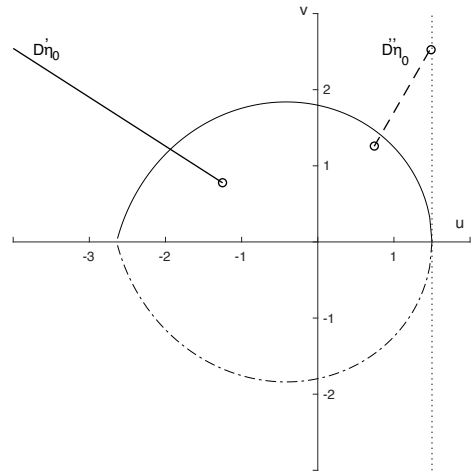


Figure 6: Enlargement of the central part of Fig. 5 with $\gamma_+([|a|, |\eta_\pi|]) \cup \gamma_-([|a|, |\eta_\pi|])$ with two cases D'_{η_0} (solid line) and D''_{η_0} (dashed line). Note that D''_{η_0} is bounded due to (13)

8. The set of all $\eta \in \mathbb{C}$ that satisfy (23) is the boundary of the $\Lambda_{\tau_0, a}$ region – see Fig. 1 for its shape. To show that for every $\eta \in \overline{\Lambda_{\tau_0, a}} \setminus \{\eta \in \mathbb{C} : |\eta| \leq |a|\}$ the roots of (18) are in $\mathbb{C} \setminus \mathbb{C}_+$ consider the following. For every η in the half-plane $\{u + iv \in \mathbb{C} : a + u < 0\}$ simple geometric considerations show that there exists exactly one η_0 fulfilling (23) and such that $\text{Arg } \eta = \text{Arg } \eta_0$. Conversely, let us fix η_0 fulfilling (23) and consider a function $\tau = \tau(|\eta|)$ defined on a ray from the origin and passing through η_0 . More precisely, define:

$$D_{\eta_0} := \{\eta = u + iv \in \mathbb{C} : |\eta| > |a| \text{ and } a + u < 0 \text{ and } \text{Arg } \eta = \text{Arg } \eta_0\}$$

and let $D^t_{\eta_0} := \{t \geq 0 : t = |\eta|, \eta \in D_{\eta_0}\}$. Now reformulate the equality in (23) to express τ as a function $\tau : D^t_{\eta_0} \rightarrow (0, \infty)$,

$$\tau(t) = \frac{\arctan\left(\frac{1}{a}\sqrt{t^2 - a^2}\right) + |\text{Arg } \eta_0|}{\sqrt{t^2 - a^2}}. \tag{24}$$

This is a well-defined positive continuous function. Indeed, for positivity note that for $u \leq 0$ there is $|\text{Arg } \eta_0| \geq \frac{\pi}{2}$, while for $u \in (0, -a)$ consider the following trigonometric identity

$$\begin{aligned} \arctan\left(\frac{1}{a}\sqrt{t^2 - a^2}\right) + |\text{Arg } \eta_0| &= \arctan\left(\frac{1}{a}\sqrt{t^2 - a^2}\right) + \arctan\left(\left|\frac{v}{u}\right|\right) \\ &= \arctan\left(\frac{\frac{u}{a}\sqrt{t^2 - a^2} + |v|}{u - \frac{1}{a}\sqrt{t^2 - a^2}|v|}\right) \end{aligned}$$

and the estimation $\frac{u}{a}\sqrt{t^2 - a^2} > -\sqrt{u^2 + v^2 - a^2} > -|v|$. The derivative of (24) is given by

$$\frac{d\tau}{dt}(t) = \frac{t}{t^2 - a^2} \left(\frac{a}{t^2} - \tau(t) \right). \quad (25)$$

As $a < 0$ we have $\frac{d\tau}{dt} < 0$ for every $t \in D_{\eta_0}^t$ and τ is a decreasing function. Thus for every $\eta \in D_{\eta_0}$ such that $|\eta| \leq |\eta_0|$ we have $\tau(|\eta|) \geq \tau(|\eta_0|) = \tau_0$, that is

$$|\text{Arg } \eta| \geq \sqrt{|\eta|^2 - a^2} \tau_0 - \arctan\left(\frac{1}{a}\sqrt{|\eta|^2 - a^2}\right), \quad |\eta| \leq |\eta_\pi|. \quad (26)$$

As the above is true for every η_0 fulfilling (23), condition (26) is true for every $\eta \in \overline{\Lambda_{\tau_0, a}} \setminus \{\eta \in \mathbb{C} : |\eta| < |a|\}$. Using now strict monotonicity of function τ in (24) we see that for a given $\eta' \in \Lambda_{\tau_0, a} \setminus \{\eta \in \mathbb{C} : |\eta| < |a|\}$ the delay time τ' allowed by this η' to be such that the first root of (18) reaches the imaginary axis is greater than τ_0 . This also gives that $\tau' > \tau_0$ implies $\Lambda_{\tau', a} \subset \Lambda_{\tau_0, a}$, as shown in Fig. 1.

9. Results of the previous point show that the only parts of Γ_+ and Γ_- that we need to consider are the ones already discussed i.e. $\gamma_+([|a|, |\eta_\pi|]) \cup \gamma_-([|a|, |\eta_\pi|])$. Indeed, let η_k , $k = 1, 2, \dots$ be consecutive points where Γ_+ crosses the constraint line $u = -a$, as depicted in Figs. 5 and 6. Then for every

$$\eta_+ \in \gamma_+(|\eta_\pi|, |\eta_1|) \cup \gamma_+(||\eta_{2k}|, |\eta_{(2k+1)}|), \quad k \in \mathbb{N}$$

there exists

$$\eta_0 \in \gamma_+([|a|, |\eta_\pi|]) \cup \gamma_-([|a|, |\eta_\pi|])$$

such that:

$$\text{Arg } \eta_0 = \text{Arg } \eta_+ \quad \text{and} \quad |\eta_0| < |\eta_+|.$$

The result of point 8 now gives a contradiction as τ_0 cannot be the smallest delay for which the first crossing happens. In fact, although (16) still describes (18) with a root corresponding to η_+ at the imaginary axis, at least one root of (18) – the one corresponding to η_0 – is already in \mathbb{C}_+ . The same argument holds for Γ_- .

10. It is easy to see that for $|\eta| = |a|$ estimation (26) is true and the closed disc $\{\eta \in \mathbb{C} : |\eta| \leq |a|\} \subset \overline{\Lambda_{\tau_0, a}}$. Taking into account that for the interior of this disc the roots of (18) are in \mathbb{C}_- (see point 3), we reach the necessity of the condition $\eta \in \overline{\Lambda_{\tau_0, a}}$ for $a < 0$.
11. Let now $a = 0$ and let $\tau_0 > 0$ be as before (considerations in points 1 and 2 remain the same). The crossing takes place at $s = \pm i|\eta|$. Equation

$$|\text{Arg } \eta| = |\eta|\tau_0 + \frac{\pi}{2} \quad (27)$$

comes now directly from (16). The analysis of points 5–9 simplifies greatly resulting in a necessity condition of the form:

$$|\text{Arg } \eta| \geq |\eta|\tau_0 + \frac{\pi}{2}, \quad |\eta| < \frac{\pi}{2\tau_0}. \quad (28)$$

12. Assume now $0 < a$. Equations (19) and (20) have the same form. The difference now is that the second product term in (19) is constantly in the second quadrant, with a negative real part $-a$ and imaginary part tending to $+\infty$ as $w \rightarrow \infty$. This changes e.g. the behavior of the continuous argument increment function $\Delta\gamma_+$, as it is in general no longer strictly increasing.

In fact for $0 < a$ we have $\Delta\gamma_+ : [a, \infty) \rightarrow [0, \infty)$,

$$\Delta\gamma_+(w) = \sqrt{w^2 - a^2}\tau_0 + \arctan\left(-\frac{1}{a}\sqrt{w^2 - a^2}\right) + \pi \quad (29)$$

and $\Delta\gamma_- : [a, \infty) \rightarrow [0, \infty)$, $\Delta\gamma_-(w) = -\Delta\gamma_+(w)$ for every $w \geq a$. As (29) is a differentiable function its derivative is

$$\frac{d\Delta\gamma_+}{dw}(w) = \frac{w}{\sqrt{w^2 - a^2}} \left(\tau_0 - \frac{a}{w^2}\right). \quad (30)$$

We have

$$\frac{d\Delta\gamma_+}{dw}(w) < 0 \quad \text{if and only if} \quad w < w_m := \sqrt{\frac{a}{\tau_0}}, \quad (31)$$

Taking into account the domain of (29) i.e. $a \leq w$, we see that for $a \in (0, \frac{1}{\tau_0})$ function $\Delta\gamma_+$ is firstly decreasing, reaching a local minimum $\Delta\gamma_+(w_m) > \frac{\pi}{2}$, and then it is increasing to infinity. For $a > \frac{1}{\tau_0}$ function $\Delta\gamma_+$ is increasing. These two cases are analyzed separately.

13. Fix $0 < a \leq \frac{1}{\tau_0}$. Similarly as in points 7 and 8 we focus initially on a part of Γ_+ given by $\gamma_+([a, \eta_\pi])$, as indicated in Fig. 7. Take η_1 that fulfils (19) and with $|\eta_1| = w_1 < w_m$. For such η_1 we have

$$\Delta\gamma_+(w_m) < \text{Arg } \eta_1 = \Delta\gamma_+(w_1) \leq \pi.$$

Define a ray from the origin and passing through η_1 by

$$D_{\eta_1} := \{\eta = u + iv \in \mathbb{C} : |\eta| > |a| \text{ and } a + u < 0 \text{ and } \text{Arg } \eta = \Delta\gamma_+(w_1)\}$$

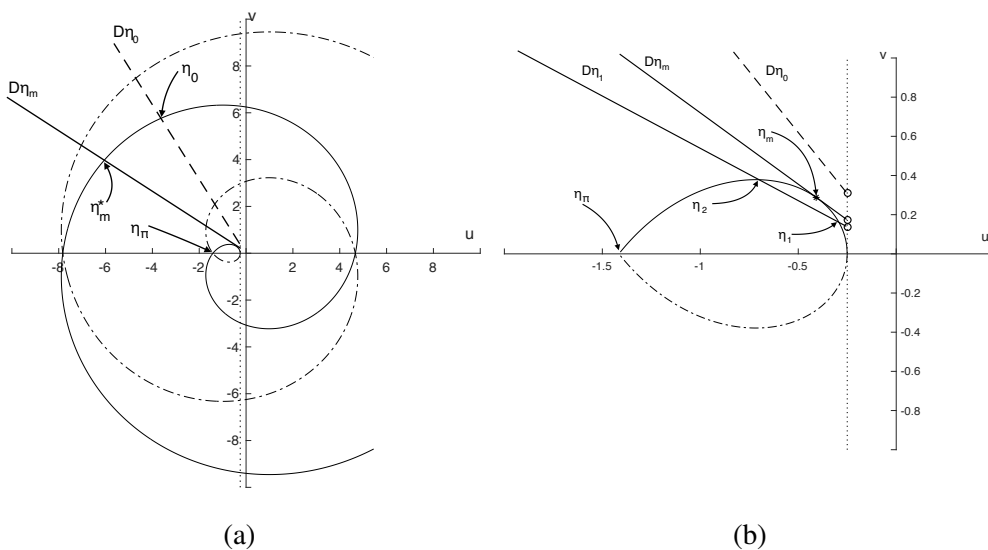


Figure 7: (a) Curves Γ_+ (solid line) and Γ_- (dash-dotted line) drawn for $\tau_0 = 1$ and $a = 0.25$ with $w \in (|a|, 10)$. The constraint related to a and expressed by (13) is marked with a dotted line. The first crossings of the real negative semi-axis by Γ_+ (and Γ_-) is at η_π . Auxiliary rays D_{η_m} and D_{η_0} are indicated in solid and dashed lines, respectively; (b) enlargement of the central part of (a) with $\gamma_+([|a|, |\eta_\pi|]) \cup \gamma_-([|a|, |\eta_\pi|])$ with D_{η_1} (solid line) and D_{η_0} (dashed line), $|\eta_1| = w_1$, $|\eta_2| = w_2$. The ray D_{η_0} is based on η_0 such that $\text{Arg } \eta_0 < \Delta\gamma_+(w_m)$; point $\eta_m = \gamma_+(w_m)$ is indicated with an arrow and a star * symbol, $\text{Arg } \eta_m = \text{Arg } \eta_m^*$

and let $D_{\eta_1}^t := \{t \geq 0 : t = |\eta|, \eta \in D_{\eta_1}\}$. To express τ as a function on this ray, i.e. $\tau : D_{\eta_1}^t \rightarrow (0, \infty)$ we now reformulate (29) to obtain:

$$\tau(t) := \frac{\arctan\left(\frac{1}{a}\sqrt{t^2 - a^2}\right) + \Delta\gamma_+(w_1) - \pi}{\sqrt{t^2 - a^2}}, \quad (32)$$

where $\Delta\gamma_+(w_1) = \text{Arg } \eta_1$. Note also that as $w_1 < w_m$ there exists η_2 , with $|\eta_2| = w_2$, such that $\eta_2 \in D_{\eta_1} \cap \gamma_+([a, \eta_\pi])$ and $w_m < w_2 \leq |\eta_\pi|$.

The derivative of (32) is again expressed by (25), namely:

$$\frac{d\tau}{dt}(t) = \frac{t}{t^2 - a^2} \left(\frac{a}{t^2} - \tau(t) \right),$$

but, unlike in point 8, this derivative is in general not negative due to $a > 0$. In fact, at the intersections $\{\eta_1, \eta_2\} = D_{\eta_1} \cap \gamma_+([a, \eta_\pi])$ we find

$$\begin{aligned} \frac{d\tau}{dt}(w_1) &= \frac{w_1}{w_1^2 - a^2} \left(\frac{a}{w_1^2} - \tau(w_1) \right) = \frac{w_1}{w_1^2 - a^2} \left(\frac{a}{w_1^2} - \tau_0 \right) \\ &= \frac{1}{\sqrt{w_1^2 - a^2}} \left(-\frac{d\Delta\gamma_+}{dw}(w_1) \right) > 0, \end{aligned} \quad (33)$$

where the last inequality comes from (31); similarly:

$$\frac{d\tau}{dt}(w_2) = \frac{1}{\sqrt{w_2^2 - a^2}} \left(-\frac{d\Delta\gamma_+}{dw}(w_2) \right) < 0. \quad (34)$$

We see that τ is an increasing function in a neighborhood of $t_1 = w_1$ and a decreasing one in a neighborhood of $t_2 = w_2$ i.e. at the boundaries of the $\Lambda_{\tau_0, a}$ region shown in Fig. 7. If we show that τ has only one extreme value – a local maximum – inside $\Lambda_{\tau_0, a}$, that is for some $t \in (w_1, w_2)$, then with the reasoning of point 8 we will show that for every η inside $\Lambda_{\tau_0, a}$ region the roots of (2) are in $\mathbb{C} \setminus \mathbb{C}_+$.

We are interested in the number of solutions of $\frac{d\tau}{dt}(t) = 0$, what is equivalent to the number of solutions of

$$\frac{a}{t^2} = \frac{\arctan\left(\frac{1}{a}\sqrt{t^2 - a^2}\right) + \beta}{\sqrt{t^2 - a^2}}, \quad (35)$$

where $\beta = \Delta\gamma_+(w_1) - \pi$. Define $r := \frac{1}{a}\sqrt{t^2 - a^2}$. Then $r > 0$ is a bijective image of $t > a$ and (35) can be rearranged to

$$\frac{r}{r^2 + 1} = \arctan(r) + \beta. \quad (36)$$

As $\frac{\pi}{2} < \Delta\gamma_+(w_m) < \text{Arg } \eta_1 \leq \pi$ we have $\beta \in (-\frac{\pi}{2}, 0]$ and by Corollary 1 we infer that there is only one local extremum i.e., local maximum of τ for $t \in (w_1, w_2)$. Hence for every $\eta \in D_{\eta_1}$, $w_1 \leq |\eta| \leq w_2$ we have $\tau(|\eta|) \geq \tau_0$ i.e.

$$\text{Arg } \eta_1 \geq \sqrt{|\eta|^2 - a^2}\tau_0 - \arctan\left(\frac{1}{a}\sqrt{|\eta|^2 - a^2}\right) + \pi.$$

Thus by the definition of D_{η_1} and symmetry about the real axis we obtain that for every η with $|\eta| \leq |\eta_\pi|$ such that

$$|\text{Arg } \eta| \geq \sqrt{|\eta|^2 - a^2}\tau_0 - \arctan\left(\frac{1}{a}\sqrt{|\eta|^2 - a^2}\right) + \pi \quad (37)$$

the time τ for this η to be such that the first root of (12) reaches the imaginary axis is bigger than or equal to τ_0 . Argument similar to the one in point 8 shows that if $|\text{Arg } \eta| \geq \Delta\gamma_+(w_m)$ then the only region we need to consider is the one given by (37). Thus we distinguish a ray

$$D_{\eta_m} = \{\eta = u + iv \in \mathbb{C} : |\eta| > |a|, \quad a + u < 0, \quad \text{Arg } \eta = \Delta\gamma_+(w_m)\}$$

together with a delay time function based on it, namely $\tau_m : D_{\eta_m}^t \rightarrow (0, \infty)$,

$$\tau_m(t) = \frac{\arctan\left(\frac{1}{a}\sqrt{t^2 - a^2}\right) + \Delta\gamma_+(w_m) - \pi}{\sqrt{t^2 - a^2}}, \quad (38)$$

where $\Delta\gamma_+(w_m) = \text{Arg } \eta_m$, see Fig. 7. The above analysis shows that for τ_m we have $\tau_m(t) \leq \tau_0$ for every $t \in D_{\eta_m}^t$, where the equality holds only for $t = w_m$.

14. Take now, without loss of generality due to symmetry, $\eta \in \mathbb{C}_- \cap \Pi_+$ such that $\text{Re } \eta < -a$ and $\frac{\pi}{2} < \text{Arg } \eta < \Delta\gamma_+(w_m) = \text{Arg } \eta_m$. We claim that for every such η , there is $\tau(|\eta|) < \tau_0$, where τ is defined on a ray containing η . Indeed, let us fix η as above and assume otherwise i.e. $\tau(|\eta|) \geq \tau_0$. Then there exists η_0 that fulfils (16), $\text{Arg } \eta_0 = \text{Arg } \eta$ and $\Delta\gamma_+(w_0) = \text{Arg } \eta + 2\pi$,

where $w_0 = |\eta_0|$ (see Fig. 7). As $\eta \in D_{\eta_0}$ we have $\tau: D_{\eta_0}^t \rightarrow (0, \infty)$ defined as in (32) but on the ray D_{η_0} , and such that for $t = |\eta|$ it takes the value:

$$\tau(t) = \frac{\arctan\left(\frac{1}{a}\sqrt{t^2 - a^2}\right) + \text{Arg } \eta_0 + \pi}{\sqrt{t^2 - a^2}}, \quad (39)$$

where we used a fact that $\Delta\gamma_+(w_0) = \text{Arg } \eta_0 + 2\pi$. Note that for a fixed t the above is a continuous function of $\text{Arg } \eta_0 \in (\frac{\pi}{2}, \pi)$. Let us take a sequence $\{\eta_0^k\}_{k \in \mathbb{N}}$ such that η_0^k fulfils (16), $|\eta_0^k| < |\eta_0^{k+1}|$ for every $k \in \mathbb{N}$ and $\eta_0^k \rightarrow \eta_m^*$ as $k \rightarrow \infty$, where $\text{Arg } \eta_m^* = \text{Arg } \eta_m$. Geometry of the problem shows that for every $k \in \mathbb{N}$ we have

$$\text{Arg } \eta_0^k < \text{Arg } \eta_0^{k+1} < \text{Arg } \eta_m \quad \text{and} \quad D_{\eta_0^k}^t \subset D_{\eta_0^{k+1}}^t \subset D_{\eta_m}^t.$$

For the fixed t from (39) consider a continuous, strictly increasing function $\tau_t: [\text{Arg } \eta_0, \pi] \rightarrow (0, \infty)$,

$$\tau_t(\text{Arg } \xi) = \frac{\arctan\left(\frac{1}{a}\sqrt{t^2 - a^2}\right) + \text{Arg } \xi + \pi}{\sqrt{t^2 - a^2}}.$$

Our hypothesis now gives:

$$\tau_0 \leq \tau(t) < \lim_{k \rightarrow \infty} \tau_t(\text{Arg } \eta_0^k) = \tau_t(\text{Arg } \eta_m^*) = \tau_m(t) \leq \tau_0,$$

where we used strict monotonicity and continuity of τ_t , continuity of γ_+ , definition of D_{η_m} and boundedness of τ_m given by (38). The above contradiction proves our claim.

Thus with $0 < a \leq \frac{1}{\tau_0}$ for the roots of (18) to be in $\mathbb{C} \setminus \mathbb{C}_+$ the region given by (37) is the only allowable one for η .

15. Fix $a > \frac{1}{\tau_0}$. By (31) and a comment directly below it the continuous argument increment function $\Delta\gamma_+$ given by (29) is now strictly increasing with range $\Delta\gamma_+([a, \infty)) = [\pi, \infty)$. The minimal value of $\Delta\gamma_+(w) = \pi$ for $w = |a|$ and point 14 shows that if the roots of (18) are in $\mathbb{C} \setminus \mathbb{C}_+$ then $\eta = -a$; there is no such η that the roots of (18) are in \mathbb{C}_- .

This finishes the necessity proof for (18) and, by the same argument, for (12).

Part III

16. To be able to use previous notation and ease referencing we show sufficiency for (18). Let $\tau_0 > 0$ be given and $a \leq \frac{1}{\tau_0}$. The behavior of the roots described in points 1 and 2 does not change. Every $\eta \in \overline{\Lambda_{\tau_0, a}}$, where $\Lambda_{\tau_0, a}$ is defined accordingly to a , is either inside $\mathbb{D}_{|a|}$ or satisfies (26), (28) or (37). Following backwards the reasoning in points 5–13 we reach the boundary condition (23), (27) or equality in (37), for which the roots of (18) are on the imaginary axis, what happens exactly when η is at the boundary of $\overline{\Lambda_{\tau_0, a}}$.

Corollary 2 *Let a delay $\tau > 0$, coefficients $\lambda, \gamma, \eta \in \mathbb{C}$ be such that $\lambda = a + ib$ with $a \leq \frac{1}{\tau}$, $b \in \mathbb{R}$, and let the corresponding $\Lambda_{\tau, a} \subset \mathbb{C}$ be given by (9)–(11). Then*

(i) *every solution of the equation $s - a - \eta e^{-s\tau} = 0$ belongs to \mathbb{C}_- if and only if $\eta \in \Lambda_{\tau, a}$;*

(ii) *every solution of*

$$s - \lambda - \gamma e^{-s\tau} = 0 \quad (40)$$

and its version with conjugate coefficients

$$s - \bar{\lambda} - \bar{\gamma} e^{-s\tau} = 0 \quad (41)$$

belongs to \mathbb{C}_- if and only if $\gamma e^{-ib\tau} \in \Lambda_{\tau, a}$.

Proof. Part (i) follows from the analysis of continuity and monotonicity of (24) or (32) given in the proof of Theorem 1. Part (ii) follows from (i) and Lemma 1 by defining $\eta = \gamma e^{-ib\tau}$ for the case of (40), while for the case of (41) by the real-axis symmetry of $\Lambda_{\tau, a}$ we have $\eta \in \Lambda_{\tau, a}$ if and only if $\bar{\eta} = \bar{\gamma} e^{ib\tau} \in \Lambda_{\tau, a}$. \square

4. Discussion

Before going to examples we make some comments concerning previous work of other authors with respect to the proof of Theorem 1. We also comment on practicality of results obtained in this paper.

Theorem 1 relies on subsets $\Lambda_{\tau, a}$ of the complex plane that are defined before the theorem itself. Their origin, however, becomes clear after going through points 6–7 of the proof of Theorem 1. The remainder of the proof is in fact an analysis of what happens inside those regions. It is worth to mention that inequalities in (9)–(11) can be obtained from the result in [10] after suitable simplifications.

As noted in the introduction an analysis of τ as a function of coefficients is present also in [8]. The author obtains there an inequality similar to (24), but does so in the context where γ of (1) is a 2×2 real matrix of a special form (and with $\lambda \in \mathbb{R}$).

The necessary and sufficient condition for stability of first order scalar differential-difference equations with complex coefficients characterized by (2) are given by Corollary 2. The condition is based on mutual implicit relation between coefficients of the characteristic equation (2) and a subset of \mathbb{C} given by (9)-(11). As the latter is defined by non-linear inequalities there arises a question whether numerical approximation of the rightmost root isn't a more practical approach than finding numerically (i.e. approximately) the region given by (9)-(11), especially given the abundance of literature of computational techniques to approximate characteristic roots.

Analysis of dependence between τ and the crossing of the imaginary axis by the first root in points 2-3 of the proof is well-known. In one of the early works [4] authors discuss (2) with $\lambda, \gamma \in \mathbb{R}$, in [16] the authors show a general approach for real polynomial case of (4) with multiple delays, what is also shown in [12]. More recently such analysis is also used in [8]. A good exposition of such techniques is in [18, Chapter 5.3.2]. Our calculations in point 2-3 are in fact based on [16] and we decided to include all of its steps for the reader's convenience.

The answer to the above question depends, in the authors' opinion, on the purpose of approaching that problem. If the purpose is an analysis of a given differential-difference equation, considered as a delayed dynamical system, fulfilling assumptions of Corollary 2, than a numerical check, up to a given accuracy, of at most one of inequalities (9)-(11) is usually a straightforward procedure.

If, on the other hand, the purpose is a synthesis of a delayed dynamical system that has some *a priori* specified properties, as may be the case of a controller design for such system, then a numerical search for the rightmost root may carry more relevant information.

5. Examples

With the above discussion in mind, we present examples concerning only analysis of given differential-difference equations. These examples illustrate how the necessary and sufficient conditions of Theorem 1 can be compared with and improve known literature results. Note initially that the stability condition discussed in [1] and later proved in [3], that is $-\operatorname{Re} \lambda > |\gamma|$, follows immediately from $|\eta| < |a|$ (point 3 in the proof of Theorem 1). Note also that as Corollary 2 concerns the placement of roots of the characteristic equation (2), it gives also a necessary and sufficient condition for stability of (1). With that in mind we give the following examples.

5.1. Example 1

Consider a differential-difference equation:

$$x'(t) = i20x(t) + \gamma x(t - 0.1), \quad (42)$$

where $\mathbb{R} \ni \gamma > 0$. Equation (42) is a special case of (1), for which necessary and sufficient conditions of stability were found in [3]. By Corollary 2 equation (42) is stable if and only if $\gamma e^{-i^2} \in \Lambda_{0.1,0}$, where $\Lambda_{0.1,0}$ is given by (10). We thus obtain that (42) is stable if and only if $\gamma < 20 - 5\pi$, what is equivalent to the condition given by [3, Theorem 3.1].

5.2. Example 2

Consider the differential-difference equation

$$x'(t) = \left(\frac{1}{4} + i\frac{\pi}{4}\right)x(t) - \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)x(t - 1), \quad (43)$$

for which the corresponding characteristic equation takes the form

$$s - \left(\frac{1}{4} + i\frac{\pi}{4}\right) - \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)e^{-s} = 0. \quad (44)$$

By Corollary 2 and (11) (or, in fact, by investigating Fig. 4) we see that (44) is stable.

5.3. Example 3

In [11] the author considers a semi-linear system version of (3) and – due to the approach method – states results only for a fixed delay $\tau = 1$. The exemplary system analyzed in [11] is transformed to the form of (4) with $\tau = 1$, $A = 0$ i.e.

$$x'(t) = Bx(t - 1), \quad B = \begin{pmatrix} -1 & \frac{1}{8} \\ -1 & -1 \end{pmatrix}. \quad (45)$$

As $A = 0$, and thus $\lambda = 0$, we are interested only in eigenvalues of B , which are $-1 \pm i\frac{1}{\sqrt{8}}$. The author concludes that the system is stable.

With conditions (10) we can improve results for the exemplary system in [11] by finding a maximal delay τ for which such system remains stable. Let $\eta = -1 + i\frac{1}{\sqrt{8}}$. Then $|\eta| = \frac{3\sqrt{2}}{4}$ and $\text{Arg } \eta = \pi - \arctan \frac{1}{\sqrt{8}}$ and by (10) we obtain that (45) is stable if and only if

$$0 < \tau < \frac{1}{|\eta|} \left(\text{Arg } \eta - \frac{\pi}{2} \right) = \frac{2\sqrt{2}}{3} \left(\frac{\pi}{2} - \arctan \frac{1}{\sqrt{8}} \right).$$

Note that we do not need to consider $\bar{\eta}$ due to the symmetry of $\Lambda_{\tau,0}$ about the real axis.

5.4. Example 4

Previous examples relate current results to the ones known from the literature and thus demonstrate the technique. The following example shows how the current results can be used in the case of a retarded partial differential equation in an abstract formulation.

Let the representation of our system be:

$$\begin{cases} \dot{z}(t) = Az(t) + A_1z(t - \tau) + Bu(t), \\ z(0) = x, \end{cases} \quad (46)$$

where the state space X is a Hilbert space, $A: D(A) \subset X \rightarrow X$ is a closed, densely defined diagonal generator of a C_0 -semigroup $(T(t))_{t \geq 0}$ on X , $A_1 \in \mathcal{L}(X)$ is also a diagonal operator and $0 < \tau < \infty$ is a fixed delay. The input function is $u \in L^2(0, \infty; \mathbb{C})$ and B is the control operator. We assume that X possesses a Riesz basis $(\phi_k)_{k \in \mathbb{N}}$ consisting of eigenvectors of A , which has a corresponding sequence of eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$.

A simplified form of (46) is analyzed in [13] from the perspective of admissibility which, roughly speaking, asserts whether a solution z of (46) follows a required type of trajectory. One of the key elements in the approach to admissibility analysis presented in [13] is to establish when a differential equation associated with the k -th component of (46), namely:

$$\begin{cases} \dot{z}_k(t) = \lambda_k z_k(t) + \gamma_k z_k(t - \tau), \\ z_k(0) = x_k, \end{cases} \quad (47)$$

is stable, where $\lambda_k \in \mathbb{C}$ is an eigenvalue of A , $\gamma_k \in \mathbb{C}$ is an eigenvalue of A_1 and $x_k \in \mathbb{C}$ is an initial condition for the k -th component of X . Then, having stability conditions for every $k \in \mathbb{N}$, one may proceed with analysis for the whole X . Based on Corollary 2 we immediately obtain a genuine approach method of obtaining these stability conditions, namely:

Proposition 1 For a given delay $\tau \in (0, \infty)$ and sequences $(\lambda_k)_{k \in \mathbb{N}}$ and $(\gamma_k)_{k \in \mathbb{N}}$ consider a corresponding set of Cauchy problems of the form (47). For every $k \in \mathbb{N}$ system (47) is stable if and only if

$$\lambda_k = a_k + i\beta_k \in \left\{ z \in \mathbb{C} : \operatorname{Re}(z) < \frac{1}{\tau} \right\} \quad \text{and} \quad \gamma_k e^{-i\beta_k \tau} \in \Lambda_{\tau, a_k} \quad \forall k \in \mathbb{N},$$

with Λ_{τ, a_k} defined in (9)–(11).

Notice that Proposition 1 not only extends [13, Proposition 3.5] by adding the necessary condition, but it also allows for analysis of unbounded A , as it includes e.g. the case when $a_k \rightarrow -\infty$ as $k \rightarrow \infty$. This is in fact exactly the case presented in [7].

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