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On Sequencing Fuzzy Interval Games

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Abstract
In sequencing situations, it may affect parameters used to determine an optimal order in the queue, and consequently the decision of whether (or not) to rearrange the queue by sharing the realized cost savings. In this paper, we extend one machine sequencing situations and their related cooperative games under fuzzy uncertainty. Here, the agents costs per unit of time and processing time in the system are fuzzy intervals. In the sequel, we define sequencing fuzzy interval games and show that these games are convex. Further, fuzzy equal gain splitting rule is given. Finally, a numerical example is illustrated priority based scheduling algorithm.

Keywords
Sequencing situations; Fuzzy intervals; Cooperative games; Convex games; Equal gain splitting rule.

Introduction

Sequencing situations arise in several instances of real life. Here, we refer to the classical one-machine sequencing situations that arise when a set of ordered jobs has to be processed sequentially on a machine. The use of an optimal ordering may reduce the cost connected with the time spent in the system and is particularly interesting in sequencing situations, where several agents are involved. In such situations, the optimal order increases the efficiency of the system for the agents as a whole, (because it increases the efficiency of the system), but since the agents are basically interested in their individual benefit, an agreement is equally important. The agreement includes how to compensate those agents that are required to spend more time in the system and how to share the joint cost savings. In the classical approach to the problem, the processing time of each job and the cost per unit of time associated with it are supposed to be known with certainty. It should be clear that the optimality of an ordering may be affected when the actual processing times and unitary costs are different from the forecasted ones. In (Curiel et al., 1989), the class of sequencing games are introduced. An updated survey on these games can be found in (Curiel et al., 2022). We also refer to the survey on Operations Research Games (Branzei et al., 2010; Borm et al., 2001).

This paper extends the analysis of cooperative sequencing games to a setting with fuzzy interval data. We consider sequencing situations in which a certain number of customers has to be served by one server under fuzzy uncertainty. Each of them has a fuzzy interval cost function which depends on his/her completion time, i.e., the time which he/she has to wait plus the time it takes to serve him/her (Alparslan et al., 2008; Calleja et al., 2006; Curiel et al., 1989).

In cooperative game theoretical models, fuzzy coalitions are used differently. In (Aubin, 1974; 1981; Butnariu, 1978), authors extend the domain of the characteristic function from subsets to fuzzy coalitions of the set of players. That is, the characteristic function assigns to each fuzzy coalition again to a real number. Recent developments show that new models in cooperative games have been introduced when the worth of the coalitions is fuzzy intervals. In contrast, Mares (Mareš, 1999; 2001; Mareš & Vlach, 2004) is concerned with the uncertainty in the values of characteristic functions. In (Mallozzi et al., 2011), the authors introduced a new core-like set for cooperative games under fuzzy uncertainty. The Shapley value of cooperative fuzzy games is introduced in (Yu & Zhang, 2010).

The interval Shapley function for interval fuzzy games based on the extended Hukuhara difference is

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studied by (Meng et al., 2016). In (Kong et al., 2018), the concepts of the general prenucleolus and the least square general prenucleolus over the pre-imputation set for cooperative fuzzy games are proposed. In (Alparslan & Özcan, 2023; Özcan & Gök, 2021a; 2021b; Özcan et al., 2022; 2023) some applications of fuzzy interval cooperative games are given. Different game theoretical models and uses of are presented in (Alparslan Gök et al., 2009a; 2011; 2009b; Hamidoğlu et al., 2021; Hamidoğlu, 2021; Savku & Weber, 2020).

The paper is organized as follows. In Section 2, we recall basic notions and results from one-machine sequencing situations and related games, fuzzy interval calculus and the theory of fuzzy interval cooperative games. In Section 3, we introduce sequencing fuzzy interval situations and show that these games are convex. Section 3 extends the classical sequencing games to the fuzzy interval setting. Finally, we illustrate a numerical example related to sequencing fuzzy interval situations in Section 4. We close with a conclusion in Section 5.

Materials and methods

In this section, we give some basic notations and results from one-machine sequencing problems and fuzzy intervals. For further information see (Alparslan Gök et al., 2008; Curiel et al., 1989; 2002; Dubois, 1980; Dubois & Prade, 1997; Dubois et al., 2000; Lawler et al., 1993).

Sequencing situations and related games

One-machine sequencing situations occur when a set of ordered jobs has to be processed sequentially on one machine. Formally, a one-machine sequencing situation is a 4-tuple $(N,\, \sigma_0,\, \lambda,\, p)$ where:

- $N = \{1, \ldots, n\}$ is the set of jobs;
- $\sigma_0 \colon : \; N \to N$ is a permutation that defines the initial order of the jobs;
- $\lambda = (\lambda_i)_{i \in N} \in \mathsf{R}^n_+$ is a non-negative real vector, where λ_i is the cost per unit of time of job i;
- $p = (p_i)_{i \in N} \in \mathsf{R}^n_+$ is a positive real vector, where p_i is the processing time of job i.

Given a sequencing situation and an ordering σ of the jobs, for each $i \in N$ we denote by $P(\sigma,\, i)$ the set of jobs preceding i, according to the order σ. The time spent in the system by job i is the sum between the waiting time that jobs in $P(\sigma,\, i)$ need to be processed and the processing time of job i yielding the related cost $\lambda_i \left(\sum_{j \in P(\sigma,\, i)} p_j + p_i\right)$. Then, the (total) cost associated with σ, namely C_σ, is given by $C_\sigma = \sum_{i \in N} \left(\sum_{j \in P(\sigma,\, i)} p_j + p_i\right)$.

The optimal order of the jobs σ^* produces the minimum cost $C_{\sigma^*} = \sum_{i \in N} \left(\sum_{j \in P(\sigma^*,\, i)} p_j + p_i\right)$ or the maximum cost saving $C_{\sigma_0} - C_{\sigma^*}$. In (Smith, 1956), they prove that an optimal order can be obtained by reordering the jobs according to decreasing urgency indices, where the urgency index of job $i \in N$ is defined by $u_i = \lambda_i / p_i$.

A sequencing game is a pair $\langle N,\, v \rangle$ where N is the set of players, that coincides with the set of jobs, and the characteristic function v assigns to each coalition S the maximal cost savings which the members of S can obtain by reordering only their jobs. We say that a set of jobs T is *connected according to an order* σ if for all $i,\; j \in T$ and $k \in N$, $\sigma(i) < \sigma(k) < \sigma(j)$ implies $k \in T$ (Curiel et al., 1989; 2002).

A switch of two connected jobs i and j, where i precedes j, generates a change in cost equal to $\lambda_j p_i - \lambda_i p_j$. This amount is positive if and only if the urgency indices verify $u_i < u_j$. Clearly, if $\lambda_j p_i - \lambda_i p_j$ is negative it is not beneficial for i and j to switch their positions. We denote the gain of the switch as

$$g_{ij} = (\lambda_j p_i - \lambda_i p_j)_+ = \max\{0,\;\; \lambda_j p_i - \lambda_i p_j\}$$

and, consequently, the gain of a connected coalition T according to an order σ is defined by

$$v(T) = \sum_{j \in T} \; \sum_{i \in P(\sigma,\, j) \cap T} g_{ij} \, .$$

If S is not a connected coalition, the order σ induces a partition in connected components, denoted by $S \backslash \sigma$. In view of this, the characteristic function v of the sequencing game can be defined as

$$v(S) = \sum_{T \in S \backslash \sigma} v(T)$$

for each $S \subset N$ or equivalently as

$$v = \sum_{i,\, j \in N \colon \; i < j} g_{ij} u_{[i,\, j]},$$

where $u_{[i,\, j]}$ is the unanimity game defined as:

$$u_{[i,\, j]}(S) = \begin{cases} 1, & \text{if } \{i,\, i+1, \ldots,\, j-1,\, j\} \subset S, \\ 0, & \text{otherwise.} \end{cases}$$

Note that sequencing games are convex (Curiel et al., 1989).

Equal Gain Splitting rule (*EGS*-rule) is defined by

$$EGS_i = \frac{1}{2} \sum_{k \in P(\sigma, i)} g_{ki} + \frac{1}{2} \sum_{j:\ i \in P(\sigma, j)} g_{ij}\,,$$

for each $i \in N$ (Curiel et al., 1989).

Fuzzy intervals

A fuzzy set (Zadeh, 1965) $\mathbf{F}$ in R is a function $\mu_{\mathbf{F}}\colon\ \mathsf{R} \to [0, 1]$ where $\mu_{\mathbf{F}}$ assigns to each point in R a degree of membership. For any $\alpha \in [0, 1]$, α – level set (α – cut) of $\mathbf{F}$ defined by as follows:

$$[\mu_{\mathbf{F}}]^{\alpha} = \{x \in \mathsf{R}\colon\ \mu_{\mathbf{F}}(x) \geq \alpha\} = \left[\mu_{\mathbf{F}}^{-},\, \mu_{\mathbf{F}}^{+}\right].$$

If $\alpha = 0$, then $[\mu_{\mathbf{F}}]^{0} = cl\,\{x \in \mathsf{R}\colon\ \mu_{\mathbf{F}}(x) > 0\}$. Here, $cl\,\{x \in \mathsf{R}\colon\ \mu_{\mathbf{F}}(x) > 0\}$ is the closure of $\{x \in \mathsf{R}\colon\ \mu_{\mathbf{F}}(x) > 0\}$.

A fuzzy set $\mathbf{F}$ in R is said to be a fuzzy interval, if the following conditions are satisfied (Dubois, 1980):

- $[\mu_{\mathbf{F}}]^{\alpha}$ is compact for any $\alpha \in [0, 1]$,
- $[\mu_{\mathbf{F}}]^{\alpha}$ is convex for any $\alpha \in [0, 1]$,
- $[\mu_{\mathbf{F}}]^{\alpha}$ is normal, i.e., there exist $x \in \mathsf{R}$ such that $\mu_{\mathbf{F}}(x) = 1$.

We denote the set of all fuzzy intervals by $\mathbf{F}(\mathsf{R})$. For any $\mathbf{F} \in \mathbf{F}(\mathsf{R})$ there exist $a, b, c, d \in \mathsf{R}$ and $L\colon\ [a, b] \to \mathsf{R}$ non-decreasing and $R\colon\ [c, d] \to \mathsf{R}$ non-increasing such that the membership function $\mu_{\mathbf{F}}$ is given as below (Dubois, 1980):

$$\mu_{\mathbf{F}}(x) = \begin{cases} L(x), & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ R(x), & c \leq x \leq d, \\ 0, & \text{otherwise}. \end{cases}$$

If L and R are linear, then $\mathbf{F}$ is called trapeziodal fuzzy interval and its membership function is given by (Dubois, 1980):

$$\mu_{\mathbf{F}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & b \leq x \leq c, \\ \dfrac{x-d}{c-d}, & c \leq x \leq d, \\ 0, & \text{otherwise}. \end{cases}$$

This trapeziodal fuzzy interval is denoted by (a, b, c, d). We denote the set of all trapeziodal fuzzy intervals by $\mathbf{F}_T(\mathsf{R})$. In this case if $a = b$ and $c = d$, then (a, b, c, d) is compact interval, if $a = b = c = d$, then (a, b, c, d) is a real number.

For any $\alpha \in [0, 1]$, the α – level set of a trapeziodal fuzzy interval of $\mathbf{F}$ with membership function $[\mu_{\mathbf{F}}]^{\alpha}$ is given by (Dubois, 1980):

$$\begin{aligned} [\mu_{\mathbf{F}}]^{\alpha} &= [a(1-\alpha) + \alpha b,\, (1-\alpha)\, d + \alpha c] \\ &= \left[\mu_{\mathbf{F}}^{-},\, \mu_{\mathbf{F}}^{+}\right]. \end{aligned}$$

Let $\mathbf{F}_1,\ \mathbf{F}_2 \in \mathbf{F}(\mathsf{R})$, then binary relation $\trianglerighteq$ is defined on $\mathbf{F}(\mathsf{R})$ as follows (Dubois, 1980; Dubois & Prade, 1997; Dubois et al., 2000). For all $\alpha \in [0, 1]$

$$\mathbf{F}_1 \trianglerighteq \mathbf{F}_2 \Leftrightarrow [\mu_{\mathbf{F}_1}]^{\alpha} \geq [\mu_{\mathbf{F}_2}]^{\alpha} \Leftrightarrow \mu_{\mathbf{F}_1}^{-} \geq \mu_{\mathbf{F}_2}^{-}$$

and $\mu_{\mathbf{F}_1}^{+} \geq \mu_{\mathbf{F}_2}^{+}$.

Let $\mathbf{F}_1 = (a_1, a_2, a_3, a_4)$ and $\mathbf{F}_2 = (b_1, b_2, b_3, b_4) \in \mathbf{F}_T(\mathsf{R})$ be two trapeziodal fuzzy intervals and $k \in \mathsf{R}^{+}$, then the following conditions holds:

- $\mathbf{F}_1 + \mathbf{F}_2 = (a_1 + b_1,\, a_2 + b_2,\, a_3 + b_3,\, a_4 + b_4)$,
- $k \cdot \mathbf{F}_1 = (ka_1,\, ka_2,\, ka_3,\, ka_4)$,
- $\mathbf{F}_1 \trianglerighteq \mathbf{F}_2 \Leftrightarrow a_1 \geq b_1,\ a_2 \geq b_2,\ a_3 \geq b_3$, and $a_4 \geq b_4$.

Fuzzy interval cooperative games

Fuzzy interval cooperative game is a pair $\langle N, \mathbf{U}\rangle$, where $N = \{1, 2, \ldots, n\}$ is the set of players and $\mathbf{U}\colon\ 2^N \to \mathbf{F}(\mathsf{R})$ maps the coalitions $S \in 2^N$ into fuzzy intervals $\mathbf{U}(S) \in \mathbf{F}(\mathsf{R})$ with $v(0) = 0$. Here, 0 is a fuzzy interval with membership function given by (Mallozzi et al., 2011):

$$\mu_0(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

It is obvious that, the definition above is extension of cooperative interval games in the sense of (Alparslan Gök et al., 2011; 2008) and the classial games. We denote by $\mathbf{F}(\mathsf{R})^N$ the set of all such fuzzy payoff vectors and $\mathbf{F}G^N$ the family of all fuzzy interval cooperative games.

We call a game $\langle N, \mathbf{U}\rangle$ is convex if $\mathbf{U}(S) + \mathbf{U}(T) \trianglelefteq \mathbf{U}(S \cap T) + \mathbf{U}(S \cup T)$ for all $S, T \in 2^N$.

Results

In this section, we introduce a one-machine sequencing fuzzy interval situation described by a 4-tuple $(N, \sigma_0, \lambda, p)$, where N and σ_0 are as in classical case whereas $[\mu_{\lambda_i}]^{\alpha} = \left[\mu_{\lambda_i}^{-},\, \mu_{\lambda_i}^{+}\right]_{i \in N} \in \mathbf{F}(\mathsf{R}_{+})^N$ and $[\mu_{p_i}]^{\alpha} = \left[\mu_{p_i}^{-},\, \mu_{p_i}^{+}\right]_{i \in N} \in \mathbf{F}(\mathsf{R}_{+})^N$ are vectors of fuzzy intervals with $\mu_{\lambda_i}^{-}$, $\mu_{\lambda_i}^{+}$ representing the minimal and maximal unitary cost of job i, respectively, and $\mu_{p_i}^{-}$, $\mu_{p_i}^{+}$ representing the minimal and maximal processing time of job i, respectively.

To handle sequencing situations in which all parameters are given by fuzzy intervals, we propose a trapezoidal fuzzy interval urgency approach and a trapezoidal fuzzy interval relaxation approach. The *trapezoidal fuzzy interval urgency index of job* is defined as

$$u_i = \frac{\lambda_i}{p_i} = \left(\frac{\lambda_{1i}}{p_{1i}}, \frac{\lambda_{2i}}{p_{2i}}, \frac{\lambda_{3i}}{p_{3i}}, \frac{\lambda_{4i}}{p_{4i}}\right),$$

where λ, $p \in \mathbf{F}_T(\mathsf{R})$ and $i \in N$. To extend Smith's result for finding the optimal order we need not only to compare u_i and u_j to check whether $u_i \lhd u_j$ for any two possible candidates i and j to a neighbor switches, but also that these fuzzy interval numbers are disjoint.

The *trapezoidal fuzzy interval relaxation index of job* is defined as

$$r_i = \frac{p_i}{\lambda_i} = \left(\frac{p_{1i}}{\lambda_{1i}}, \frac{p_{2i}}{\lambda_{2i}}, \frac{p_{3i}}{\lambda_{3i}}, \frac{p_{4i}}{\lambda_{4i}}\right),$$

where λ, $p \in \mathbf{F}_T(\mathsf{R})$ and $i \in N$.

Note that in the classical case the relaxation index is the inverse of the urgency index, so we may reformulate the rule of Smith saying that to obtain an optimal order, the jobs have to be ordered according to increasing fuzzy interval relaxation indices.

Two jobs i, $j \in N$ may be switched only if $r_i \rhd r_j$ and the fuzzy interval numbers are disjoint. Our setting corresponds to the maximal risk aversion of the agents that agree on a switch of their jobs only if it is surely profitable.

We notice that the domain of fuzzy interval sequencing situations under consideration is quite small, containing only situations where all fuzzy interval urgency indices exist and are pairwise disjoint and situations where all fuzzy interval relaxation indices exist and are pair-wise disjoint.

The following examples are inspired by (Alparslan Gök et al., 2008) and illustrate situations which cannot be handled by our approaches, whereas the following example allows application of our fuzzy interval urgency approach.

Example. Consider the two-agent situation with $p_1 = (1, 3, 5, 8)$, $p_2 = (1, 2, 3, 5)$, $\lambda_1 = (1, 2, 3, 4)$, $\lambda_2 = (1, 2, 4, 9)$. Now, r_1 is defined but r_2 is undefined; on the other hand u_1 is undefined and u_2 is defined. Hence no comparison is possible and, consequently, the reordering cannot take place.

Example. Consider the two-agent situation with $p_1 = (1, 3, 5, 8)$, $p_2 = (1, 3, 7, 11)$, $\lambda_1 = (1, 2, 3, 4)$, $\lambda_2 = (1, 2, 4, 5)$. Here, we can compute $r_1 = \left(1, \frac{3}{2}, \frac{5}{3}, 2\right)$ and $r_2 = \left(1, \frac{3}{2}, \frac{7}{4}, \frac{11}{5}\right)$, but we cannot reorder the jobs as the fuzzy intervals are not disjoint.

Example. Consider the four-agent situation with $N = \{1, 2, 3, 4\}$, $\sigma_0 = \{1, 2, 3, 4\}$, $\lambda \in \big((1, 4, 12, 20), (1, 4, 9, 20), (1, 3, 4, 5), (1, 2, 4, 5)\big)$ and $p \in \big((1, 2, 3, 4), (1, 2, 3, 5), (1, 6, 16, 30), (1, 4, 20, 40)\big)$. We can compute $u_1 = (1, 2, 3, 5)$, $u_2 = (1, 2, 3, 4)$, $r_3 = (1, 2, 4, 6)$ and $r_4 = (1, 2, 5, 8)$, while the other indices are undefined. Jobs 1 and 2 may be switched and also jobs 3 and 4 may be switched, but we can say nothing about jobs 1 and 4 that become adjacent after the first two switches, as there is no common index.

Example. Consider the two-agent situation with $p_1 = (1, 2, 3, 4)$, $p_2 = (2, 3, 4, 6)$, $\lambda_1 = (1, 4, 9, 16)$, $\lambda_2 = (12, 21, 32, 54)$. We can compute $u_1 = (1, 2, 3, 4)$ and $u_2 = (6, 7, 8, 9)$ and use them to reorder the jobs as the fuzzy intervals are disjoint.

Now, we introduce the fuzzy interval equal gain splitting rule by extending the equal gain splitting rule sequencing situations.

Let $(N, \sigma_0, \lambda, p)$ and (N, τ_0, λ, p) be sequencing fuzzy interval situations where either all fuzzy interval urgency indices exist and are pair-wise disjoint or all fuzzy interval relaxation indices exist and are pair-wise disjoint.

Let i, $j \in N$. We define the *fuzzy interval gain* of the switch of jobs i and j by

$$G_{ij} = \begin{cases} \lambda_j p_i - \lambda_i p_j\,, & \text{if jobs } i \text{ and } j \text{ switch,} \\ 0, & \text{otherwise.} \end{cases}$$

The *fuzzy interval equal gain splitting rule* is defined by

$$\mathbf{F}EGS_i(N, \sigma_0, \lambda, p) = \frac{1}{2} \sum_{j \in N:\, j > i} G_{ij} + \frac{1}{2} \sum_{j \in N:\, j < i} G_{ij} \in \mathbf{F}(R)$$

for each $i \in N$.

Example. Consider the sequencing trapezoidal fuzzy interval situation with $N = \{1, 2, 3\}$, $\sigma_0 = \{1, 2, 3\}$, $p = \big((1, 2, 3, 4), (2, 3, 4, 5), (1, 2, 3, 5)\big)$ and $\lambda = \big((1, 4, 9, 16), (10, 18, 32, 50), (11, 24, 42, 75)\big)$. The fuzzy interval urgency indices are $u_1 = (1, 2, 3, 4)$, $u_2 = (5, 6, 8, 10)$ and $u_3 = (11, 12, 14, 15)$, so all jobs may be switched. The fuzzy interval gains are obtained by:

$$\begin{aligned} G_{12}^1 &= \max\left\{0,\ \lambda_2^1 p_1^1 - \lambda_1^1 p_2^1\right\} \\ &= \max\left\{0,\ 10 \cdot 1 - 1 \cdot 2\right\} = 8, \\ G_{12}^2 &= \max\left\{0,\ \lambda_2^2 p_1^2 - \lambda_1^2 p_2^2\right\} \\ &= \max\left\{0,\ 18 \cdot 2 - 4 \cdot 3\right\} = 24, \end{aligned}$$

$$\begin{aligned} G_{12}^3 &= \max\left\{0,\ \lambda_2^3 p_1^3 - \lambda_1^3 p_2^3\right\} \\ &= \max\left\{0,\ 32\cdot 3 - 9\cdot 4\right\} = 60, \\ G_{12}^4 &= \max\left\{0,\ \lambda_2^4 p_1^4 - \lambda_1^4 p_2^4\right\} \\ &= \max\left\{0,\ 50\cdot 4 - 16\cdot 5\right\} = 120. \end{aligned}$$

Hence, $G_{12} = (8, 24, 60, 120)$. The other fuzzy interval gains are calculated similarly as follows: $G_{21} = (0, 0, 0, 0)$, $G_{13} = (10, 40, 99, 220)$, $G_{31} = (0, 0, 0, 0)$, $G_{23} = (12, 36, 72, 125)$, $G_{32} = (0, 0, 0, 0)$. Then, the *fuzzy interval equal gain splitting* ($\mathtt{F}EGS$) *rule is*

$$\begin{aligned} \mathtt{F}EGS_1 &= \frac{1}{2}\big(G_{12}^1 + G_{13}^1,\ G_{12}^2 + G_{13}^2,\ G_{12}^3 + G_{13}^3, \\ &\qquad G_{12}^4 + G_{13}^4\big) \\ &= \frac{1}{2}\left(8+10,\ 24+40,\ 60+99,\ 120+220\right) \\ &= (9,\ 32,\ 79.5,\ 170)\,, \\ \mathtt{F}EGS_2 &= \frac{1}{2}\big(G_{12}^1 + G_{23}^1,\ G_{12}^2 + G_{23}^2,\ G_{12}^3 + G_{23}^3, \\ &\qquad G_{12}^4 + G_{23}^4\big) \\ &= \frac{1}{2}\left(8+12,\ 24+36,\ 60+72,\ 120+125\right) \\ &= (10,\ 30,\ 66,\ 122.5)\,, \\ \mathtt{F}EGS_3 &= \frac{1}{2}\big(G_{13}^1 + G_{23}^1,\ G_{13}^2 + G_{23}^2,\ G_{13}^3 + G_{23}^3, \\ &\qquad G_{13}^4 + G_{23}^4\big) \\ &= \frac{1}{2}\left(10+12,\ 40+36,\ 99+72,\ 220+125\right) \\ &= (11,\ 38,\ 85.5,\ 172.5)\,. \end{aligned}$$

Consequently, the *fuzzy interval equal gain splitting* ($\mathtt{F}EGS$) *rule is obtained by*

$$\begin{aligned} \mathtt{F}EGS_i(N, \sigma_0, \lambda, p) &= \big((9,\ 32,\ 79.5,\ 170)\,, \\ &(10,\ 30,\ 66,\ 122.5)\,,\ (11,\ 38,\ 85.5,\ 172.5)\big). \end{aligned}$$

Now, we introduce the class of cooperative sequencing fuzzy interval games, and prove that the corresponding $\mathtt{F}EGS$ allocation belongs to the fuzzy interval core of the related sequencing fuzzy interval game.

The sequencing fuzzy interval game associated to a one-machine sequencing situation $(N, \sigma_0, \lambda, p)$, with $\lambda, p \in \mathtt{F}(\mathsf{R}_+)$, is defined by:

$$\mathtt{U} = \sum_{i,j \in N:\ i<j} G_{ij} u_{[i,j]}$$

provided that $G_{ij} \in \mathtt{F}(\mathsf{R})$ for all switching jobs $i, j \in N$. Here, $u_{[i,j]}$ is the unanimity game defined as:

$$u_{[i,j]} = \begin{cases} 1, & \text{if } \{i,\ i+1, \ldots,\ j-1,\ j\} \subset S, \\ 0, & \text{otherwise.} \end{cases}$$

In Proposition, we show that each sequencing fuzzy interval game is convex.

Proposition. Let $\langle N, \mathtt{U}\rangle$ be a sequencing fuzzy interval game. Then, $\langle N, \mathtt{U}\rangle$ is convex.

Proof. By definition $G_{ij} \rhd 0$ for all (i, j). It is well known that classical unanimity games are convex. Then, $|\mathtt{U}| = \sum_{i,j\in N:\ i<j} |G_{ij}|\ u_{[i,j]}$ is a convex game in the classical sense. So, $\langle N, \mathtt{U}\rangle$ is convex.

Discussion

In this section, we inspired by the example given in (Ergün et al., 2020).

Consider that we have three departments as Network and Systems Management (D1), Database Management (D2), and Energy Management (D3) in the factory. All departments are connected to Management Unit of Information Technology.

In each department, three basic jobs (i.e. processes) as Network I/O (J1), Disk IO (J2), and CPU (J3) are being run. Jobs are nonpreemptiable, which means that their execution on a processor cannot be suspended until completion. The properties of each jobs of departments are different and can be seen from Tables 1, 2, and 3. Consider the following set of jobs, assumed to have arrived at arrival time, in the order of J1, J2, J3 with the length of the execution time and service time in miliseconds. The running times of the processes may be uncertain depending on the variables like work intensity, capacity etc. These uncertainties are determined by fuzzy intervals.

Table 1
The properties of each jobs of D1

Job	Arrival time	Execute time	Priority	Service time
J1	$(1, 2, 3, 5)$	1	1	$(96, 102, 108, 120)$
J2	$(1, 2, 3, 4)$	3	2	$(181, 187, 193, 199)$
J3	$(2, 3, 4, 6)$	5	3	$(282, 288, 294, 301)$

Table 2
The properties of each jobs of D2

Job	Arrival time	Execute time	Priority	Service time
J1	$(2, 3, 4, 7)$	$(1, 2, 3, 5)$	2	$(152, 158, 169, 177)$
J2	$(1, 3, 5, 6)$	$(3, 4, 6, 7)$	1	$(126, 133, 140, 156)$
J3	$(3, 4, 6, 8)$	$(1, 3, 4, 5)$	3	$(188, 194, 201, 218)$

Table 3
The properties of each jobs of D3

Job	Arrival time	Execute time	Priority	Service time
J1	$(3,4,6,8)$	$(2,2,2,2)$	2	$(123,129,136,148)$
J2	$(1,2,3,4)$	$(3,3,3,3)$	3	$(151,157,163,169)$
J3	$(2,3,4,7)$	$(4,4,4,4)$	1	$(105,108,114,127)$

Waiting time (service time-arrival time) t of each job of D1, D2, and D3 is stated in Table 4.

Table 4
The waiting time t of each jobs of D1, D2 and D3

Job (Process)	Wait time
J1 of D1	$t_{11} = (95,\ 100,\ 105,\ 115)$
J2 of D1	$t_{12} = (180,\ 185,\ 190,\ 195)$
J3 of D1	$t_{13} = (280,\ 285,\ 290,\ 295)$
J1 of D2	$t_{21} = (150,\ 155,\ 165,\ 170)$
J2 of D2	$t_{22} = (125,\ 130,\ 135,\ 150)$
J3 of D2	$t_{23} = (185,\ 190,\ 195,\ 210)$
J1 of D3	$t_{31} = (120,\ 125,\ 130,\ 140)$
J2 of D3	$t_{32} = (150,\ 155,\ 160,\ 165)$
J3 of D3	$t_{33} = (103,\ 105,\ 110,\ 120)$

Let $c_{i,j} = (C,\ D,\ N)$ be the cost vector of running job i on department j, with C, D, N representing the cost of CPU, Disk I/O, and Network I/O, seperately.

Let $c_{i,j} = c \cdot C + d \cdot D + n \cdot N$, where c, d, n are the weights of C, D, N, respectively. These weights vary because of the type of the jobs. If the job is compute-intensive, then c be 3, d be 2, and n be 1. If it is a data parsing job, then d be 3, c be 2, and n be 1. If it is about network, then n be 3, d be 2, and c be 1. Hereby, we can say for D1, the priotry is $n > d > c$. This situation is different for D2 and D3. It can be seen in Table 5.

Let us take

$$C = (200,\ 205,\ 215,\ 230) \quad (\text{MHz}),$$
$$D = (100,\ 110,\ 115,\ 130) \quad (\text{TB}), \text{ and}$$
$$N = (50,\ 55,\ 65,\ 80) \quad (\text{Mbit}).$$

Then, the costs of D1 can be calculated as follows:

$$c_{11} = (550,\ 590,\ 640,\ 730)\,,$$
$$c_{12} = (750,\ 795,\ 840,\ 930)\,,$$
$$c_{13} = (850,\ 890,\ 940,\ 1030)\,.$$

Table 5
The weights of c, d, n of J1 for D1, D2, D3

Property of job	Compute intensity	Data parsing	Network
cost	c	d	n
J1D1	3	2	1
J2D1	2	3	1
J3D1	1	2	3
J1D2	3	2	1
J2D2	1	3	2
J3D2	1	2	3
J1D3	3	1	2
J2D3	2	3	1
J3D3	1	1	1

The urgency index a job can be obtained by dividing the cost by the waiting time:

$$u_{ij} = \frac{c_{ij}}{t_{ij}}\,.$$

Now, we can calculate and examine the priotries of each jobs of D1:

$$u_{11} = \frac{c_{11}}{t_{11}} = (5.789,\ 5.900,\ 6.095,\ 6.347)\,,$$
$$u_{12} = \frac{c_{12}}{t_{12}} = (4.116,\ 4.297,\ 4.421,\ 4.769)\,,$$
$$u_{13} = \frac{c_{13}}{t_{13}} = (3.035,\ 3.122,\ 3.241,\ 3.491)\,.$$

For D1, we see that $u_{11} \rhd u_{12} \rhd u_{13}$. We can calculate the costs of D2:

$$c_{21} = (550,\ 590,\ 640,\ 730)\,,$$
$$c_{22} = (600,\ 645,\ 690,\ 780)\,,$$
$$c_{23} = (850,\ 890,\ 940,\ 1030)\,.$$

Now, we can calculate and examine the priotries of each jobs of D2:

$$u_{21} = \frac{c_{21}}{t_{21}} = (4.333,\ 4.419,\ 4.484,\ 4.882)\,,$$
$$u_{22} = \frac{c_{22}}{t_{22}} = (4.800,\ 4.961,\ 5.111,\ 5.200)\,,$$
$$u_{23} = \frac{c_{23}}{t_{23}} = (4.594,\ 4.684,\ 4.820,\ 4.904)\,.$$

For D2, we see that $u_{22} \rhd u_{23} \rhd u_{21}$.

We can calculate the costs of D3:

$$c_{31} = (650,\ 685,\ 740,\ 830)\,,$$
$$c_{32} = (750,\ 795,\ 840,\ 930)\,,$$
$$c_{33} = (800,\ 835,\ 890,\ 980)\,.$$

Now, we can calculate and examine the priotries of each jobs of D3:

$$u_{31} = \frac{c_{31}}{t_{31}} = (5.416,\ 5.480,\ 5.692,\ 5.928)\,,$$
$$u_{32} = \frac{c_{32}}{t_{32}} = (5.000,\ 5.129,\ 5.250,\ 5.636)\,,$$
$$u_{33} = \frac{c_{33}}{t_{33}} = (7.767,\ 7.952,\ 8.090,\ 8.166)\,.$$

For D3, we see that $u_{33} \triangleright u_{31} \triangleright u_{32}$.

In this way, it is possible to determine which job is priority for each department. Let us construct the game now. We choose J1 job for D1, J2 for D2 and J3 for D3. Firstly, we need to calculate the fuzzy interval gains. Then,

$$\begin{aligned} c_{11} &= \lambda = \left(\lambda_1^1,\ \lambda_1^2,\ \lambda_1^3,\ \lambda_1^4\right) \\ &= (550,\ 590,\ 640,\ 730)\,, \\ t_{11} &= p_1 = \left(p_1^1,\ p_1^2,\ p_1^3,\ p_1^4\right) \\ &= (95,\ 100,\ 105,\ 115)\,, \\ u_{11} &= u_1 = \left(u_1^1,\ u_1^2,\ u_1^3,\ u_1^4\right) \\ &= (5.789,\ 5.900,\ 6.095,\ 6.347)\,, \end{aligned}$$

$$\begin{aligned} c_{22} &= \lambda = \left(\lambda_2^1,\ \lambda_2^2,\ \lambda_2^3,\ \lambda_2^4\right) \\ &= (600,\ 645,\ 690,\ 780)\,, \\ t_{22} &= p_2 = \left(p_2^1,\ p_2^2,\ p_2^3,\ p_2^4\right) \\ &= (125,\ 130,\ 135,\ 150)\,, \\ u_{22} &= u_2 = \left(u_2^1,\ u_2^2,\ u_2^3,\ u_2^4\right) \\ &= (4.800,\ 4.961,\ 5.111,\ 5.200)\,, \end{aligned}$$

$$\begin{aligned} c_{33} &= \lambda = \left(\lambda_3^1,\ \lambda_3^2,\ \lambda_3^3,\ \lambda_3^4\right) \\ &= (800,\ 835,\ 890,\ 980)\,, \\ t_{33} &= p_3 = \left(p_3^1,\ p_3^2,\ p_3^3,\ p_3^4\right) \\ &= (103,\ 105,\ 110,\ 120)\,, \\ u_{33} &= u_3 = \left(u_3^1,\ u_3^2,\ u_3^3,\ u_3^4\right) \\ &= (7.767,\ 7.952,\ 8.090,\ 8.166)\,. \end{aligned}$$

The fuzzy interval gains are calculated as follows:

$$\begin{aligned} G_{12}^1 &= \max\left\{0,\ \lambda_2^1 p_1^1 - \lambda_1^1 p_2^1\right\} \\ &= \max\{0,\ 600\cdot 95 - 550\cdot 125\} = 0, \\ G_{12}^2 &= \max\left\{0,\ \lambda_2^2 p_1^2 - \lambda_1^2 p_2^2\right\} \\ &= \max\{0,\ 645\cdot 100 - 590\cdot 130\} = 0, \\ G_{12}^3 &= \max\left\{0,\ \lambda_2^3 p_1^3 - \lambda_1^3 p_2^3\right\} \\ &= \max\{0,\ 690\cdot 105 - 640\cdot 135\} = 0, \\ G_{12}^4 &= \max\left\{0,\ \lambda_2^4 p_1^4 - \lambda_1^4 p_2^4\right\} \\ &= \max\{0,\ 780\cdot 110 - 730\cdot 150\} = 0. \end{aligned}$$

Hence, $G_{12} = \left(G_{12}^1,\ G_{12}^2,\ G_{12}^3,\ G_{12}^4\right) = (0,\ 0,\ 0,\ 0)$.

The other fuzzy interval gains can be calculated as follows:

$$\begin{aligned} G_{21} &= (11750,\ 12200,\ 13950,\ 19800), \\ G_{31} &= (0,\ 0,\ 0,\ 0), \\ G_{13} &= (19350,\ 21550,\ 23050,\ 25100), \\ G_{32} &= (0,\ 0,\ 0,\ 0), \\ G_{23} &= (38200,\ 40825,\ 44250,\ 53400). \end{aligned}$$

Example Let $N = \{1,\ 2,\ 3\}$ and the trapeziodal fuzzy interval coalitional values are

$$\begin{aligned} U(\{1\}) &= U(\{2\}) = U(\{3\}) \in (0,\ 0,\ 0,\ 0), \\ U(\{1,\ 2\}) &= \sum_{i\in 2}\sum_{k\in 1} G_{ki} = G_{12} = (0,\ 0,\ 0,\ 0), \\ U(\{2,\ 3\}) &= \sum_{i\in 3}\sum_{k\in 2} G_{ki} = G_{23} \\ &= (38200,\ 40825,\ 44250,\ 53400), \\ U(\{1,\ 3\}) &= \sum_{i\in 1}\sum_{k\in 3} G_{ki} = G_{31} = (0,\ 0,\ 0,\ 0), \\ U(N) &= (57550,\ 62375,\ 67300,\ 78500). \end{aligned}$$

Now, we compute the fuzzy interval equal splitting rule without constructing the game and by using cooperative game theory. We find the fuzzy interval equal splitting rule as follows:

$$\begin{aligned} \mathbf{F}EGS_1 &= (9675,\ 10775,\ 11525,\ 12550)\,, \\ \mathbf{F}EGS_2 &= (19100,\ 20412.5,\ 22125,\ 26700)\,, \\ \mathbf{F}EGS_3 &= (28775,\ 3187.5,\ 33650,\ 39250)\,. \end{aligned}$$

Thus, the fuzzy interval equal splitting rule calculated as:

$$\begin{aligned} \mathbf{F}EGS_i = \big(&(9675,\ 10775,\ 11525,\ 12550)\,, \\ &(19100,\ 20412.5,\ 22125,\ 26700)\,, \\ &(28775,\ 3187.5,\ 33650,\ 39250)\big). \end{aligned}$$

Conclusions

This paper is the first to study these problems for one-machine sequencing situations with fuzzy interval data. In the literature, two issues associated with sequencing situations have been investigated. The first one is to figure out the best sequence for agents in the waiting line. The second one is to suggest fair allocations of the savings that an optimum order might production, consequently operating agents to transfer places in accordance with that order. We presented two techniques to address these issues in such situations. Based on fuzzy interval urgency indices and fuzzy interval relaxation indices. However, these approaches have severe limits in applications as mainly illustrated in Section 3. For dealing with situations when neither strategy can be utilized alone, a combined approach using both types of fuzzy interval indices for determining an optimal order shows promising. Thus, one could find an optimal order by using, for example, generalized fuzzy interval indices corresponding to fuzzy interval relaxation indices based on the relation between urgency indices and relaxation indices determined by known fuzzy interval parameters.

In this paper, we study one-machine sequencing situations by using fuzzy intervals. We present different possible scenarios and extend classical results and well known rules and on sequencing games to fuzzy interval setting. Under the assumptions of one-machine sequencing situations the **F**EGS-rule is introduced as a solution concept. It is shown that fuzzy interval cooperativegames are convex. Finally, in addition to our theoretical results, a numerical example based on Priority Based Scheduling Algorithm is given.

This paper is novel and pioneering. For further research, other Operations Research situations can be extended by using fuzzy interval setting. We provide this offer to the reader, particularly in outlook section. Ellipsoidal calculus and probability theory for uncertainty modeling and handling, dynamical systems and stochastic calculus for modeling the time dependence of unfolded dynamic games, and strong corresponding instruments such as suitable implicit function theorems and novel algebraic geometries are among the mathematical instruments to be employed and partly even newly prepared. Our research should be the basis for such efforts.

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