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Examining the possibility of short-term prediction of traffic volume in smart city control systems with the use of regression models

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***Abstract* —This article deals with issues related to the optimization of traffic management in modern cities, the so-called Smart City. In particular, the article presents the process of evolution of the traffic flow prediction model at a selected crossroads in a selected city in Poland - the city of Rzeszów. Rzeszow is an example of a smart city equipped with an extensive system of real-time data collection and processing from multiple road points in the city. The research was aimed at a detailed analysis of the feasibility and degree of fit of different variants of the regression model: linear, polynomial, trigonometric, polynomial-trigonometric, and regression-based Random Forest algorithm. Several studies were carried out evaluating different generations of models, in particular, an analysis was carried out based on which the superiority of the trigonometric model was demonstrated. This model had the best fit and the lowest error rate, which could be a good conclusion for widespread use and implementation in Smart City supervisory systems.**

***Keywords*—Smart City; traffic flow; prediction; regression; polynomial-trigonometric model; trigonometric model; Random Forest model**

I. Introduction

DUE to the high density of population and various types of motor vehicles on the roads of modern cities around the world, the problem of traffic flow is becoming very important. This is a current challenge for most urban traffic management units as well as an important indicator in assessing the quality of life in a given city. There are many rankings of the quality of infrastructure of cities, and recently also indicators relating to the level of implementation of intelligent systems, so-called Smart City, which allow for real-time monitoring and management of the extensive infrastructure of cities. This is important because the intensive increase in the number of vehicles causes an increase in traffic volume, for which the urban street network is not prepared, hence the need for optimization. This phenomenon consequently causes difficulties for individual and public transportation, reduces travel time and decreases the regularity of vehicles, and the cost of their operation increases. Advanced traffic control and management algorithms using elements of artificial intelligence (AI) can be successfully applied in Smart City surveillance systems. Through the use of AI, we can attempt to forecast the traffic situation based on historical data, as well as control signaling to improve the passage through the city during peak traffic hours. In optimizing traffic control and management, the parameters responsible for traffic flow and capacity are important. The authors attempted to find appropriate patterns and mathematical models that can be used in road traffic prediction. Such a solution could assist in the design and optimization of road infrastructure, especially in urban areas, which would also have a bearing on the analysis of traffic volume changes in individual city sectors as well as in the overall agglomeration view.

II. Objectives

The purpose of this paper is to present the original traffic modeling process in the example of the city of Rzeszow, Poland. Collected historical traffic data became the source of determining traffic characteristics and allowed to create of mathematical models, then improved them to represent traffic volume as accurately as possible. The article analyzed in detail a number of prediction models. In particular, the work included polynomial, trigonometric and polynomial-trigonometric regression models, the Random Forest algorithm and linear regression.

III. Methods

In modeling the selected intersection, which is part of a complex system of streets in the city, the universal concept of graph presentation was used. A graph is defined as a set of points (vertices or nodes), while the connections between these points form edges [1-3]. The analyses assume that a traffic intersection is described by vertices and edges. To account for different types of streets (e.g., one-way streets), a directed graph model should be used. Knowing the value of traffic volume on each street, traffic analyses can be made by creating an appropriate mathematical function. The data collected by the city's traffic management center allowed a detailed description of the traffic grid using the aforementioned graphs.

Each edge of the graph is assigned a set of coefficients necessary to correctly describe the regression model [4, 5]. The difference between the actual value and the value predicted by the regression model is called the residual of the model. In order to obtain the best possible model fit, approximations were made

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by testing a number of models, including a polynomial model, a trigonometric model, a polynomial-trigonometric model, and one based on the Random Forest algorithm. In total, 13 generations of different models were developed and tested.

IV. TRAFFIC FLOW DATASET

The traffic analysis used real data collected within the intersection of Powstańców Warszawy Avenue and Batalionów Chłopskich Avenue. Data was collected at hourly intervals. July 12, 2022. The described road intersection is one of the largest in the city, connecting both streets of different statuses/categories: local, district and provincial streets. The streets have been given appropriate signatures: Dąbrowskiego (Db), Podkarpacka (Pk), Powstańców Warszawy (PW), Batalionów Chłopskich (BCh). Notation was adopted for analysis - the name of each connection is an abbreviation of the street names, e.g. BCh_Pk Street (Batalionów Chłopskich Avenue -> Podkarpacka Street).

In total, this intersection has 10 monitored connections: (Db_Pk), (PW_BCh), (Pk_Db), (BCh_PW), (Db_PW), (Db_BCh), (PW_Db), (Pk_PW), (BCh_Db), (BCh_Pk).
Figure 1 presents the map layer of the Powstańców Warszawy Avenue (PW) and Batalionów Chłopskich (BCh) intersection for which an equivalent city road network model as a graph was created.

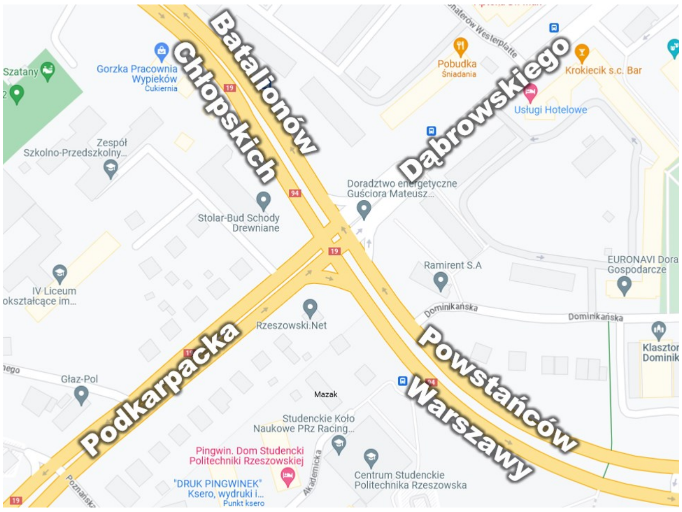

Fig. 1. The map of the Powstańców Warszawy Avenue and Batalionów Chłopskich intersection

V. MODEL DEVELOPMENT

As previously mentioned, the development process spanned across 13 generations of various models. The proposed models were trained and tested at the described road intersection. The tests showed that the fit of the models could vary, depending on the approach to the problem. Some of the models learned from the mistakes of previous generations, some started from scratch, and for a few only minor adjustments were made to their predecessors.

In generation 1, a polynomial-trigonometric model with parameters: t, t^2, $sin(t\pi/12)$, $cos(t\pi/12)$. The first generation is very basic in its nature as its main purpose is to suggest a path for further development (Fig. 2 and 3).

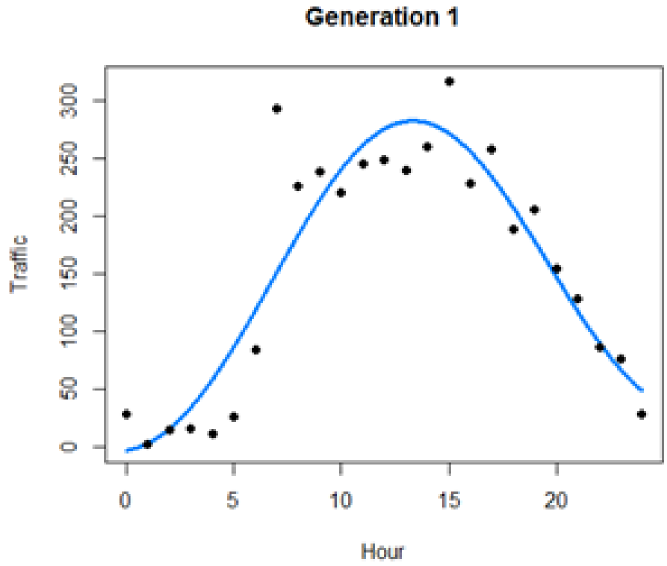

Fig. 2. Generation 1 model fit plot

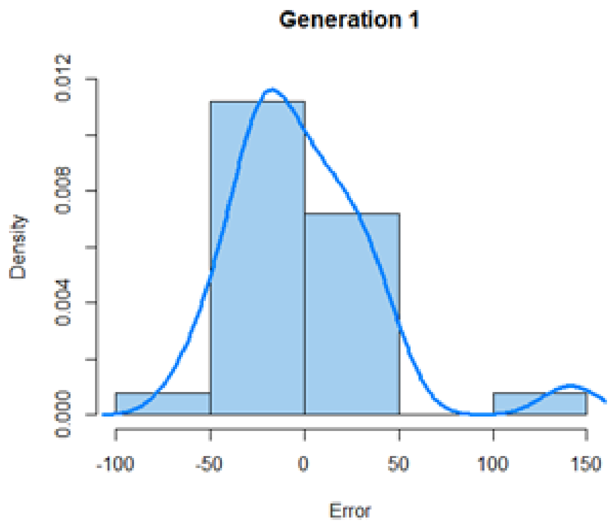

Fig. 3. Generation 1 residual distribution

Both the model fit plot and residual distribution indicate a good overall fit, but the model fails to adhere to the extreme values (which represent rush hours) which is crucial in this case.

In generation 2, a polynomial-trigonometric model with parameters: t^2, $sin(t\pi/12)$, $cos(t\pi/12)$ was implemented. This generation got rid of the t parameter for the result comparison. The model fit changed only a little. The low absolute value of the t^2 parameter coefficient suggests that a purely trigonometric model may be a better solution (Fig. 4 and 5).

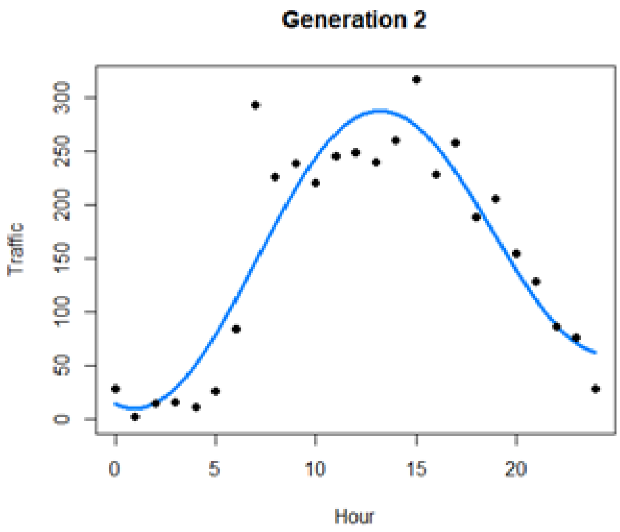

Fig. 4. Generation 2 model fit plot

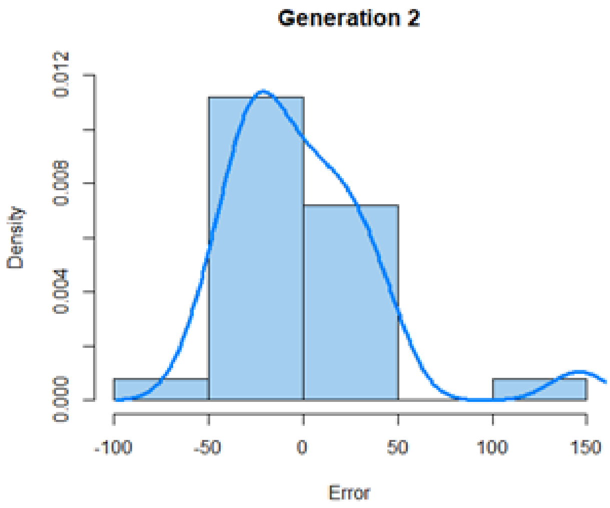

Fig. 5. Generation 2 residual distribution

In generation 3, a trigonometric model with parameters: $sin(t\pi/12)$, $cos(t\pi/12)$, $sin(t\pi/6)$, $cos(t\pi/6)$ was implemented. To see the effectiveness of a purely trigonometric model, the t^2 parameter was replaced with a sine/cosine pair with a period of 12 hours. The change of parameters led to the improvement of model characteristics. This generation also proved the cyclicality of this kind of model which is a major advantage over the other types. However, despite the trigonometric model being the most promising, another try will be given to the polynomial-trigonometric model in the fourth generation which had parameters: t, t^2, $sin(t\pi/12)$, $cos(t\pi/12)$, $sin(t\pi/6)$, $cos(t\pi/6)$. The aim of this generation is the final evaluation of the polynomial-trigonometric model in search of any substantial advantages over a trigonometric model.

After comparing the performances of generations 1-4 it was concluded that the polynomial-trigonometric model has no meaningful advantages over the trigonometric model in the context of this research. Thus its development was discontinued in favor of the trigonometric model.

The generation number 5, a trigonometric model with parameters: $sin(t\pi/24)$, $cos(t\pi/24)$, $sin(t\pi/12)$, $cos(t\pi/12)$, $sin(t\pi/6)$, $cos(t\pi/6)$ was implemented. This generation was based on generation 3. An additional sine/cosine pair with a period of 48 hours was added.

No substantial progress was made in this generation as it barely improved the model's fit and did not solve the issue of omitting the rush hours. Interestingly, due to the parameter selection, this model is cyclical with a two-day period.

For generation 6, a trigonometric model with parameters: $sin(t\pi/48)$, $cos(t\pi/48)$, $sin(t\pi/24)$, $cos(t\pi/24)$, $sin(t\pi/12)$, $cos(t\pi/12)$, $sin(t\pi/6)$, $cos(t\pi/6)$, $sin(t\pi/3)$, $cos(t\pi/3)$ was implemented. In this generation, more sine/cosine pairs were added to enhance the model's precision. The sixth generation was the first to somewhat reflect the rush hours, which is a major leap forward for the model development. This is clearly the best model so far, but there is still room for improvement.

Generation 7, a model based on the Random Forest algorithm with 24 trees was implemented. The aim of this generation was a comparison of the effectiveness of the Random Rorest algorithm with the linear regression one (Fig. 6).

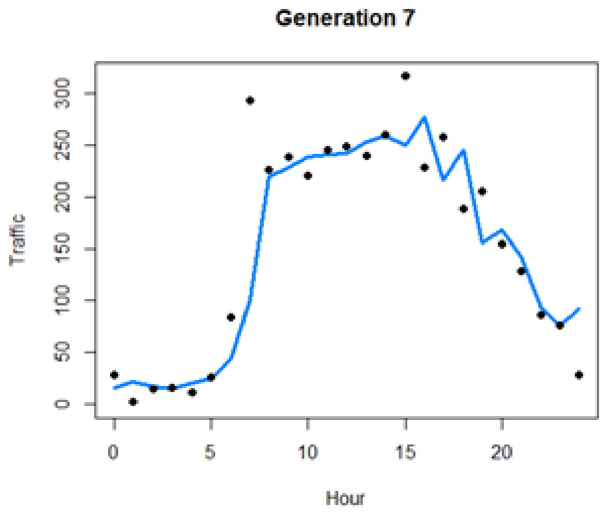

Fig. 6. Generation 7 model fit plot

The tests revealed that the Random Forest algorithm does not surpass the linear regression in the context of this research. As no perspectives on any possible improvements were apparent, the development of this type of model was abandoned.

Generation 8, a trigonometric model with parameters: $sin(t\pi/12)$, $sin((t\text{-}1)\pi/12)$, $sin((t\text{-}2)\pi/12)$, $sin((t\text{-}3)\pi/12)$, $sin((t\text{-}4)\pi/12)$, $sin((t\text{-}5)\pi/12)$, $sin((t\text{-}6)\pi/12)$, $sin((t\text{-}7)\pi/12)$, $sin((t\text{-}8)\pi/12)$, $sin((t\text{-}9)\pi/12)$, $sin((t\text{-}10)\pi/12)$, $sin((t\text{-}11)\pi/12)$ was implemented. Instead of different frequencies, the parameters of this model differ by their phase. In the end, shifting the phase turned out to be of no consequence for the model's quality as most of the parameters were assigned 0 as the coefficient. This led to the conclusion that only changing the frequency matters.

In generation 9, a polynomial model with parameters t, t^2, t^3, t^4. In an attempt to achieve a better fit within the proximity of the extreme points, an entirely different approach was tried. In this model, only 5 selected points (marked red) were used for training. The result was essentially a Lagrange interpolating polynomial. Despite the disappointing results, a polynomial model was tried again in the next generation.

Generation 10, a polynomial model with parameters t, t^2, t^3, t^4, t^5, t^6 was implemented. In this generation, an attempt was made to improve the model's characteristics by increasing the polynomial's degree to 6. Due to even poorer results than the previous generation, the polynomial model was completely abandoned.

In generation number 11, a trigonometric model with parameters: *sin(tπ/12)*, *cos(tπ/12)*, *sin(tπ/6)*, *cos(tπ/6)*, *sin(tπ/3)*, *cos(tπ/3)*, *sin(2tπ/3)*, *cos(2tπ/3)*, *sin(4tπ/3)*, *cos(4tπ/3)*, *sin(8tπ/3)*, *cos(8tπ/3)*, *sin(16tπ/3)*, *cos(16tπ/3)* was used. In this generation, the measurement points were connected with lines (marked grey) to artificially create a larger training dataset for the model (Fig. 7).

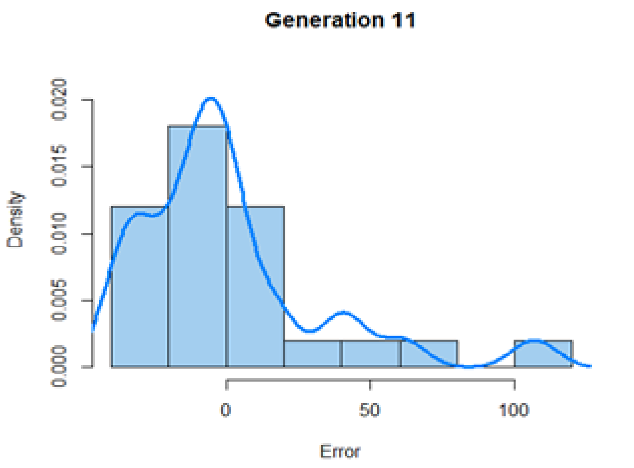

Fig. 7. Generation 11 residual distribution

Despite still not being able to improve the fit in the rush hour vicinity, this model produced some valuable insight. It became clear that increasing the frequency by the consecutive powers of 2 cannot increase the model's precision any further.

Generation 12, a trigonometric model with parameters: *sin(tπ/12)*, *cos(tπ/12)*, *sin(tπ/6)*, *cos(tπ/6)*, *sin(tπ/4)*, *cos(tπ/4)*, *sin(tπ/3)*, *cos(tπ/3)*, *sin(5tπ/12)*, *cos(5tπ/12)*, *sin(tπ/2)*, *cos(tπ/2)* was used. An important change was made in this generation. It used the sine/cosine pairs with the frequency increasing by 1 in each consecutive pair instead of following the powers of 2. The dataset remained the same as in the prior generation (Fig. 8).

Generation 12 turned out to be a game changer, achieving an almost perfect fit, including the troublesome rush hours. However, there was still one change to make, and that was to return to the original training dataset because the artificial one used in generations 11 and 12 took too long to prepare and it was unreasonable to use this method on a larger scale.

For generation 13, a trigonometric model with parameters: *sin(tπ/12)*, *cos(tπ/12)*, *sin(tπ/6)*, *cos(tπ/6)*, *sin(tπ/4)*, *cos(tπ/4)*, *sin(tπ/3)*, *cos(tπ/3)*, *sin(5tπ/12)*, *cos(5tπ/12)*, *sin(tπ/2)*, *cos(tπ/2)* was used. This was the final iteration of the model. The time-consuming method of creating an artificial dataset was abandoned and this model was trained on the original dataset. The parameters remained the same as in the generation 12 (Fig. 9 and 10).

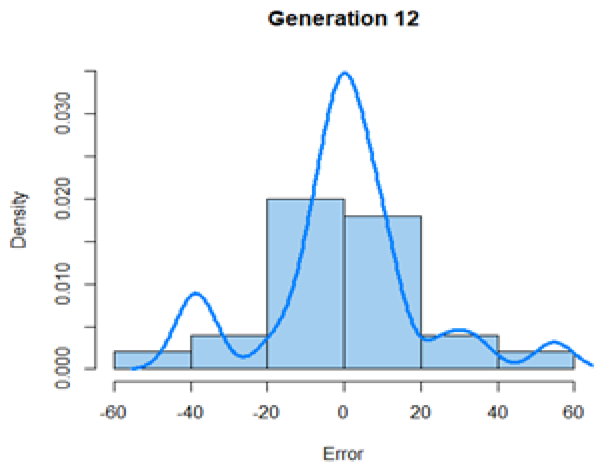

Fig. 8. Generation 12 residual distribution

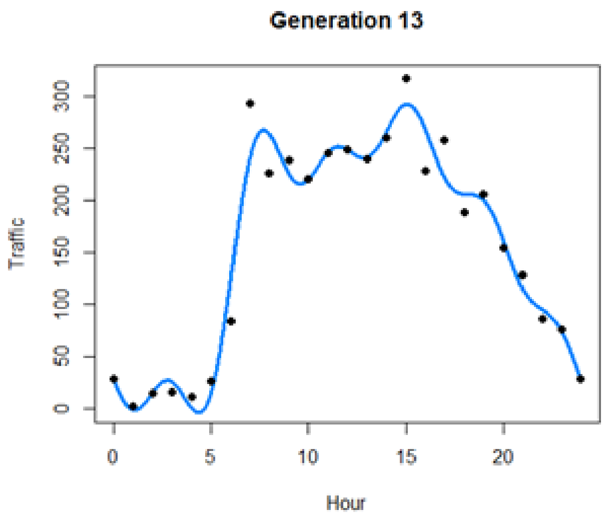

Fig. 9. Generation 13 model fit plot

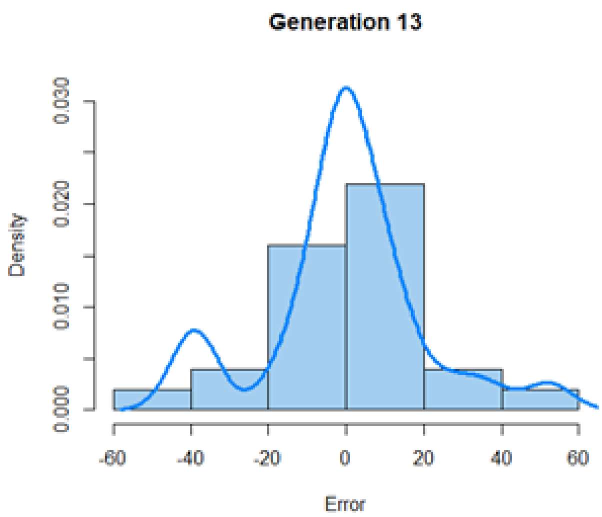

Fig. 10. Generation 13 residual distribution

Swapping the dataset only marginally affected the results, but in return substantially reduced the time needed to prepare the model. In conclusion, generation 13 emerged as the final form of the model, having eliminated all issues of the earlier generations.

Table I presents the summary of the model statistics comparison.

TABLE I
MODEL STATISTICS COMPARISON

Generation	Multiple R	R^2	Adjusted R^2	Standard Error
1 (TP)	0.922	0.850	0.820	44.408
2 (TP)	0.920	0.846	0.824	43.928
3 (T)	0.940	0.884	0.860	39.054
4 (TP)	0.944	0.891	0.854	39.935
5 (T)	0.944	0.891	0.855	39.901
6 (T)	0.965	0.932	0.883	35.861
7 (RF)	0.899	0.809	-	-
8 (T)	0.915	0.837	0.322	44.205
9 (P)	0.385	0.148	-0.022	187.821
10 (P)	0.353	0.124	-0.168	1430.158
11 (T)	0.949	0.900	0.760	51.344
12 (T)	0.980	0.961	0.921	29.378
13 (T)	0.980	0.961	0.922	29.320

VI. RESULTS

With the model ready, the next step was to calculate the sets of coefficients for each connection in the graph. The obtained coefficients were stored in a single data frame. The trained model is suitable for predicting traffic volumes on any of the streets described and monitored by the sensors [8-10]. Fig. 11-20 presents a fit model for various intersection connections.

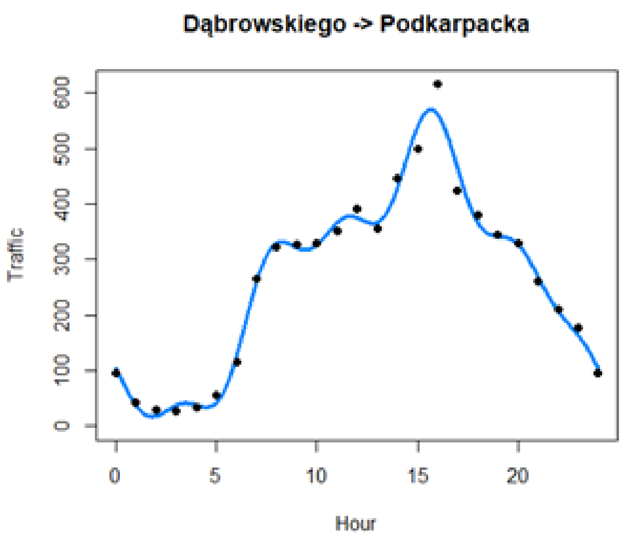

Fig. 11. Model fit on the Db_Pk connection

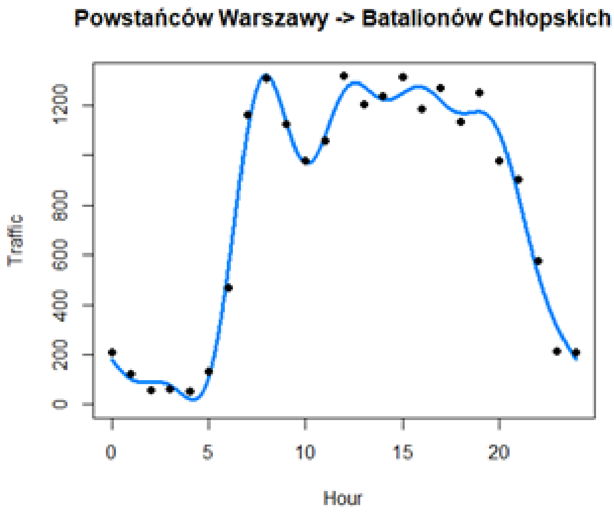

Fig. 12. Model fit on the PW BCh connection

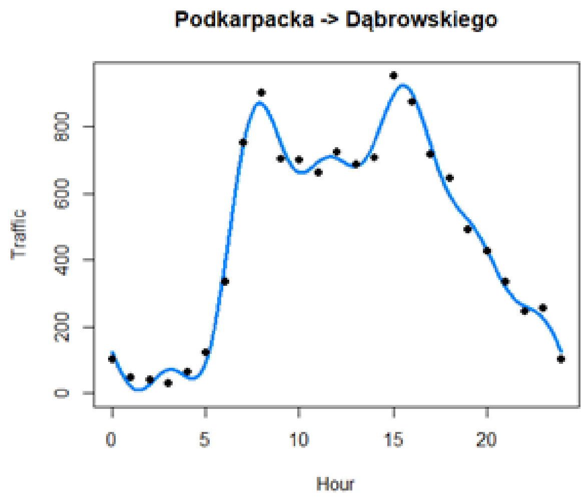

Fig. 13. Model fit on the Pk_Db connection

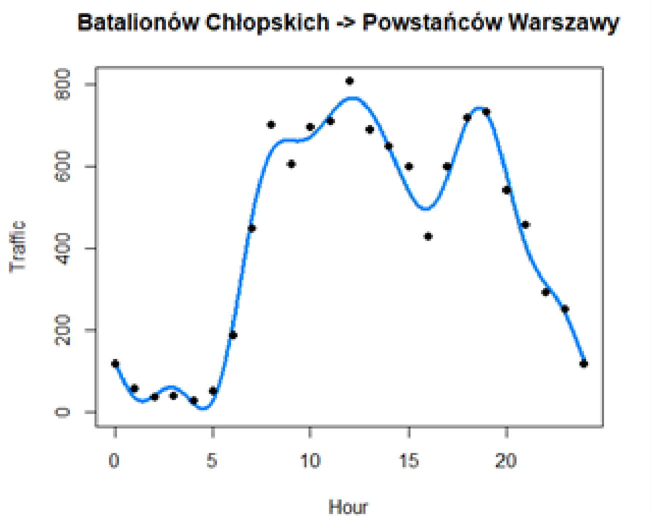

Fig. 14. Model fit on the BCh_PW connection

Fig. 15. Model fit on the Db_PW connection

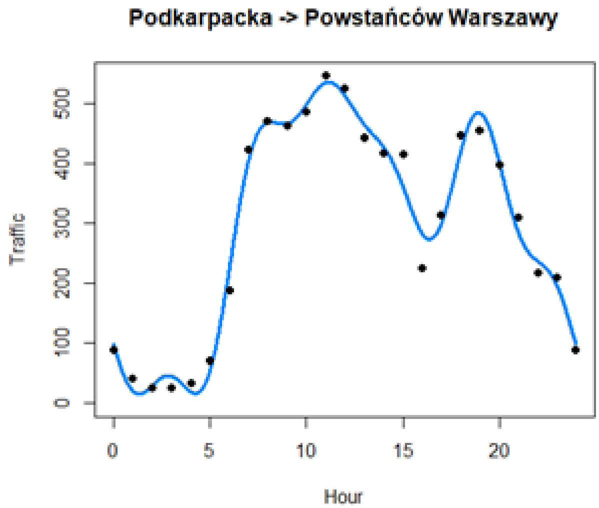

Fig. 16. Model fit on the Pk_PW connection

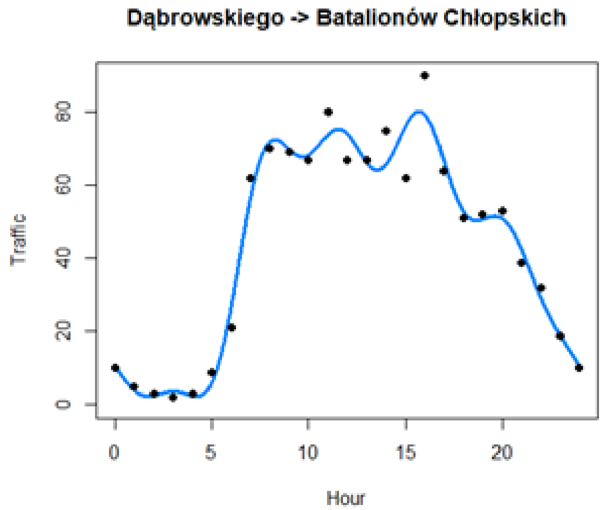

Fig. 17. Model fit on the Db_BCh connection

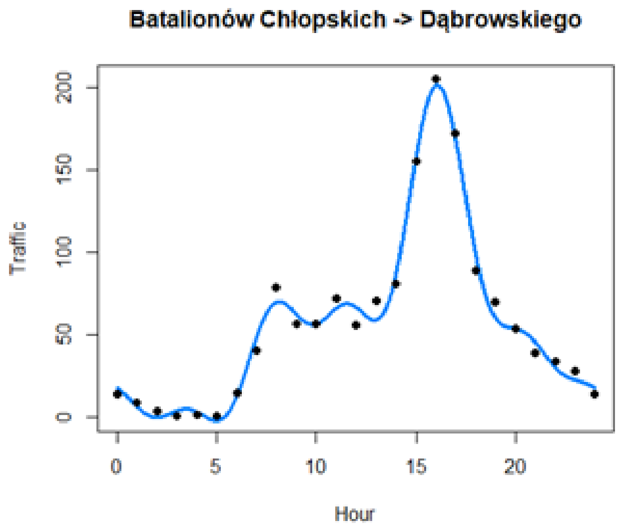

Fig. 18. Model fit on the BCh_Db connection

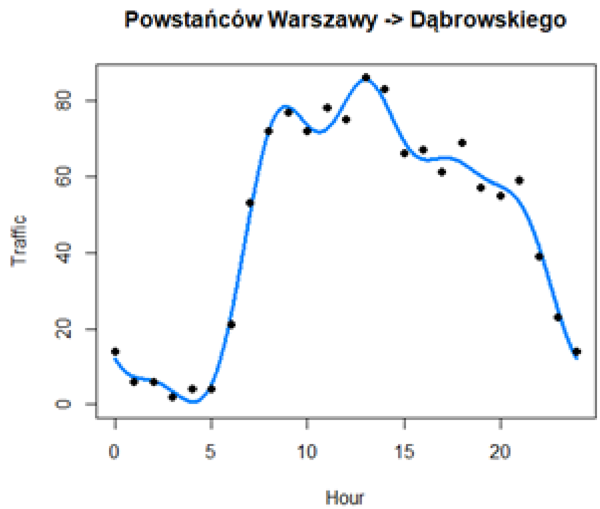

Fig. 19. Model fit on the PW_Db connection

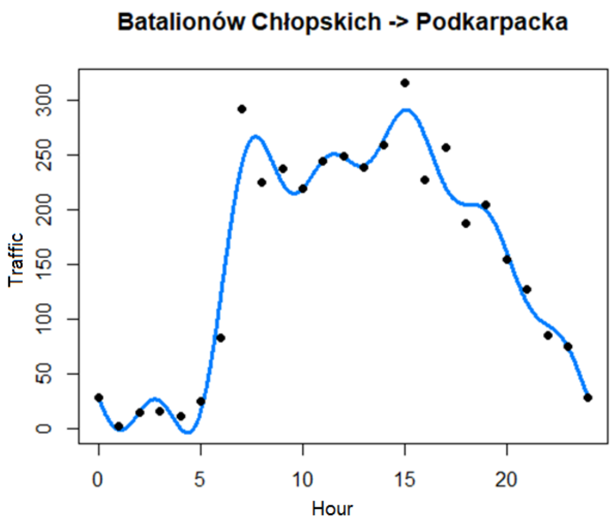

Fig. 20. Model fit on the BCh_Pk connection

As seen in Table II, the model has achieved an excellent fit on all of the measured connections. The calculated coefficients are stored in the table (Table III) which allows for easy access in the programming environment. Along with the formula of the model, this table is what makes the core of the model. Using those two elements is the easiest way to model multiple connections effectively.

TABLE II
MODEL STATISTICS BY CONNECTION

Connection	Multiple R	R^2	Adjusted R^2	Standard Error
Db_Pk	0.994	0.987	0.974	26.367
PW_BCh	0.994	0.988	0.975	78.338
Pk_Db	0.995	0.989	0.979	45.389
PCh_PW	0.992	0.984	0.969	48.469
Db_PW	0.991	0.983	0.966	6.138
Db_BCh	0.985	0.970	0.939	7.110
PW_Db	0.995	0.989	0.978	4.374
Pk_PW	0.991	0.983	0.966	33.524
BCh_Db	0.993	0.987	0.974	8.650
BCh_Pk	0.980	0.961	0.922	29.320
Mean	0.991	0.982	0.964	28.768

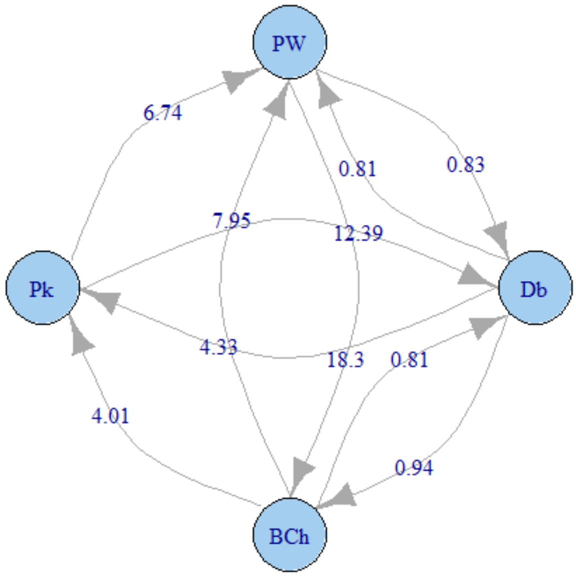

Fig. 1. Graph representing the selected intersection at 7:30 AM. The current traffic volume is displayed on each of the connections

TABLE III
TABLE OF MODEL COEFFICIENTS

Connection	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}
Db_Pk	267.6	-139.7	-165.2	-20.9	-25.3	-8.5	42.5	-4.5	-24.4	10.9	-12.6	-17.0	21.9
PW_BCh	806.0	-310.1	-528.9	-187.5	-119.9	-11.6	66.6	134.0	-18.8	-14.8	-76.2	-25.4	52.2
Pk_Db	481.3	-123.9	-367.2	-71.4	-69.8	-17.7	118.2	29.5	-28.8	-3.8	-41.5	-44.3	33.3
PCh_PW	436.8	-165.0	-290.0	-144.8	-9.6	41.4	-18.7	58.5	17.1	-43.4	-11.3	-27.1	2.9
Db_PW	53.4	-24.1	-28.7	-22.2	-2.7	3.4	-3.7	7.7	-2.7	-6.9	1.2	-0.1	-3.4
Db_BCh	44.6	-14.5	-33.5	-10.2	-2.4	0.2	5.7	1.7	-3.4	1.7	-4.0	-2.0	3.6
PW_Db	48.0	-17.3	-33.0	-13.5	1.5	-0.3	1.7	5.1	-3.2	-3.1	-2.7	1.2	-0.1
Pk_PW	302.2	-72.8	-192.0	-122.3	-3.3	9.5	-14.2	41.6	11.8	-15.3	-1.5	-36.4	-4.0
BCh_Db	58.4	-45.8	-42.3	11.7	-17.5	4.4	27.4	-11.7	-9.0	6.7	-9.7	-1.5	10.5
BCh_Pk	158.1	-47.9	-121.3	-27.5	-16.9	-7.3	24.7	17.8	-6.4	-0.1	-13.2	-21.7	3.8

In order for the model to be fully useful, it must be normalized. normalize the model. A detailed analysis showed that since in the original dataset the measurement points are always placed at the beginning of the measurement interval, which is 1 hour, the curve should be shifted forward by 30 minutes. With this step (forward shift), the model behaves as if the measurement points were in the middle of the measurement interval. The next normalization step was to divide the model results by 60 to get a prediction of the current traffic volume every minute. The resulting data allowed us to create graphs.

Now one can finally construct the graph. Fig. 21 shows details of the traffic load at the analyzed road intersection at 7:30 a.m. The weights of the current traffic volume are entered at each edge. Analogous data is visualized in Fig. 22 for 9:37 p.m.

The vertices of the graph are the streets meeting at the monitored intersection. These are al. Powstańców Warszawy (PW), ul. Dąbrowskiego (Db), al. Batalionów Chłopskich (BCh) and ul. Podkarpacka (Pk). The edges of the graph correspond to the monitored connections. On each of them, closer to the arrowhead, the current traffic intensity on a given connection is written in vehicles per minute. E.g. at 7:30 traffic intensity at the crossing from al. Powstańców Warszawy on al. Batalionów Chłopskich is 12.39 vehicles/min and on the left turn from ul. Dąbrowskiego on al. Powstańców Warszawy it is 0.81 units/min. However, at 9:37 PM these values are 13.33 units/min and 1.13 vehicles/min, respectively.

CONCLUSION

Using statistical data analysis methods and graph theory, the article shows the process of building a fully functional mathematical model of one of the road intersections in the city of Rzeszów. Research and model testing have shown both the advantages and disadvantages of different types of regression models. Detailed analysis allowed for the selection of the best model variant and set of input parameters.

Research has shown that linear regression allows you to create a very accurate model of a small road network but with limited functionality due to the data set used. The study data included only measurements taken over a 1-day period, so subsequent studies will be extended to a broader time frame in the future, which will allow for the creation of a more statistically significant model applicable to real-world implementations.

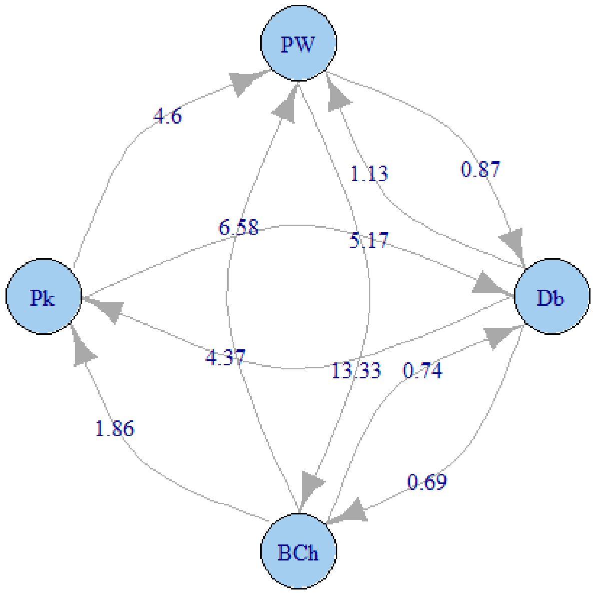

Fig. 2. Graph representing the selected intersection at 9:37 PM. The current traffic volume is displayed on each of the connections

The simulations showed that the polynomial model is the simplest model. Despite the simplicity of its preparation, it has very significant drawbacks, which became apparent in model generations 9 and 11. It turned out that the main drawback of this model is the high susceptibility to over-fitting. Huge discrepancies in results were obtained. The trigonometric model eliminates the main disadvantage of the previous model, i.e. susceptibility to overtraining, which allows for more accurate adjustment of training points without significant discrepancies. An advantage of this model can be considered its cyclicity with the right choice of explanatory variables, which allows the model to be naturally "extended" for further periods, such as days, or weeks, without losing continuity.

It also turned out that the fit of the polynomial-trigonometric model does not outperform the purely trigonometric model, thus losing some of its desirable properties. The use of linear regression made it possible to build a very accurate road network model. Due to the limited dataset used in the calculations, the model has limited functionality. A larger baseline data set would allow for better model fitting and implementation in a real road environment.

Summarizing the results obtained, it is likely that the solution based on the Fourier series was the best fit since traffic is periodic (daily/weekly/seasonal congestion). In future research, it would be worthwhile to investigate models based on Chebyshev polynomials (which are polynomials of the sine and cosine functions). Moreover, the parameters of such trigonometric/multinomial models can be easily updated online [14]. In addition, there is an approach, called aggregate regression, in which all models (not just the best for a given data set) contribute to the result [15], where, strictly speaking, the resulting model is simply a weighted (convex) combination of the input models, and the weights are usually decreasing functions of the models' errors [15].

In addition, the models presented allowed for a detailed analysis of traffic flowing in and out of individual nodes. The balance of this traffic provides key information on traffic characteristics in the study area. It should be noted that not all connections at this intersection are monitored, therefore al. Powstańców Warszawy is deprived of one outgoing road connection, al. Batalionów Chłopskich one incoming road connection, and ul. Podkarpacka incoming road connection and one outgoing road connection.

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