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# The Floquet-Lyapunov transformation for fractional discrete-time linear systems with periodic parameters

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The Floquet-Lyapunov transformation is extended to fractional discrete-time linear systems with periodic parameters. A procedure for computation of the transformation is proposed and illustrated by a numerical example.

**Key words:** Floquet-Lyapunov transformation, fractional, discrete-time, linear, periodic parameters

## 1. Introduction

The fractional linear systems have been considered in many papers and books and many well-known results for standard linear systems have been extended to fractional linear systems [6–11]. The Floquet-Lyapunov transformation has been analyzed in many books and papers [1–4]. The transformation for singular 2D linear systems has been extended in [5]. The Floquet approach has been extended to discrete-time periodic systems in [12] and to hybrid periodic linear systems in [2].

In this paper the transformation will be extended to the fractional discrete-time linear systems with periodic varying parameters.

The paper is organized as follows. In Section 2 the Floquet-Lyapunov transformation is recalled for standard linear discrete-time systems with periodic varying parameters. The main result of the paper is presented in Section 3, where the Floquet-Lyapunov transformation has been extended to fractional discrete-time linear systems with periodic varying parameters. In Section 4 procedure and illustrating example are given. Concluding remarks are given in Section 5.



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## 2. The Floquet-Lyapunow transformation of discrete-time linear systems

Consider the discrete-time linear system

$$x_{i+1} = A_i x_i\,, \quad i = 0, 1, \ldots \tag{1}$$

with periodic variable parameters

$$x_{i+K} = A_i x_i\,, \quad i = 0, 1, \ldots, \tag{2}$$

where $x_i \in \Re^n$ is the state vector $A_i \in \Re^{n\times n}$ and $K > 0$ is the period.

**Theorem 1.** *For the system (1) there exists a nonsingular periodic matrix*

$$P_{i+K} = P_i\,, \quad P_0 = I_n\,, \quad i = 0, 1, \ldots \tag{3}$$

*such that the change of variables*

$$x_i = P_i z_i\,, \quad z_i \in \Re^n, \quad i = 0, 1, \ldots \tag{4}$$

*transforms the system (1) into the system*

$$z_{i+1} = B z_i\,, \quad i = 0, 1, \ldots, \tag{5}$$

*where $B \in \Re^{n\times n}$ is the constant matrix.*

Proof is given in [2].

In the paper [12] two algorithms for computation of the transformation matrix $P_i$ have been proposed.

## 3. The Floquet–Lyapunov transformation of fractional discrete-time linear systems

Consider the fractional periodic discrete-time linear system

$$\Delta^\alpha x_{i+1} = A_i x_i\,, \quad A_{i+K} = A_i \in \Re^{n\times n}, \quad 0 < \alpha < 2, \quad i = 0, 1, \ldots, \tag{6}$$

where

$$\Delta^\alpha x_i = \sum_{j=0}^{i} c_j(\alpha) x_{i-j}\,, \tag{7}$$

$$c_j(\alpha) = (-1)^j \binom{\alpha}{j}, \quad \binom{\alpha}{j} = \begin{cases} 0 & \text{for } j < 0, \\ 1 & \text{for } j = 0, \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \ldots \end{cases} \tag{8}$$

$x_i \in \Re^n$ is the state vector and $K > 0$ is the period.

Substituting (7) into (6) we obtain

$$x_{i+1} = \overline{A}_{\alpha i} x_i + \sum_{j=2}^{i+1} \bar{c}_j(\alpha) x_{i-j+1}\,, \tag{9}$$

where

$$\overline{A}_{\alpha i} = A_i + I_n \alpha, \quad \bar{c}_j(\alpha) = -c_j(\alpha), \quad j = 1, 2, \ldots \tag{10}$$

Note that in this case the system (9) is periodic with the period $K$. Using (9) for $i = 0, 1, \ldots, K-1$ we obtain

$$\bar{x}_{i+1} = \overline{A}_i \bar{x}_i\,, \quad i = 0, 1, \ldots, \tag{11}$$

where

$$\bar{x}_i = \begin{bmatrix} x_{i0} \\ x_{i1} \\ \vdots \\ x_{i,K-1} \end{bmatrix}, \quad \overline{A}_i = \begin{bmatrix} \overline{A}_{\alpha 0}(i) & 0 & 0 & \ldots & 0 & 0 \\ c_2(\alpha) & \overline{A}_{\alpha 1}(i) & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_K(\alpha) & c_{K-1}(\alpha) & c_{K-2}(\alpha) & \ldots & c_2(\alpha) & \overline{A}_{\alpha,K-1}(i) \end{bmatrix}. \tag{12}$$

Note that $\det \overline{A}_i \neq 0$ if and only if $\det \overline{A}_{\alpha j}(i) \neq 0$ for $i,\ j = 0, 1, \ldots, K-1$.

To the periodic system (11) we may apply the approach presented in Section 2. The problem under the considerations for the system (11) can be stated as follows.

Given the nonsingular periodic matrix $\overline{A}_i$ and its period $K$. Find a periodic nonsingular matrices $T_k$, $k = 0, 1, \ldots, K-1$ with the period $K$ of the transformation

$$\bar{x}_k = T_k \hat{x}_k\,, \quad T_{k+K} = T_k\,, \quad k = 0, 1, \ldots, K-1 \tag{13}$$

which transforms the system (11) into a linear system with constant matrix $A \in \Re^{n \times n}$

$$\hat{x}_{k+1} = A \hat{x}_k\,, \quad k = 0, 1, \ldots, K-1. \tag{14}$$

The solution of the problem is based on the following.

**Lemma 1.** *The matrices $A$, $\overline{A}_k$ and $T_k$ for $k = 0, 1, \ldots, K-1$ are related by*

$$A = T_{k+1}^{-1} \overline{A}_k T_k\,, \quad k = 0, 1, \ldots, K-1 \tag{15}$$

*and*

$$T_0^{-1} \overline{A} T_0 = A^K, \tag{16}$$

*where*

$$\overline{A} = \overline{A}_{K-1} \overline{A}_{K-2} \ldots \overline{A}_0\,. \tag{17}$$

**Proof.** Using (13) and (14) we obtain

$$\hat{x}_{k+1} = T_{k+1}^{-1}\bar{x}_{k+1} = T_{k+1}^{-1}\overline{A}_k\bar{x}_k = T_{k+1}^{-1}\overline{A}_k T_k\hat{x}_k = A\hat{x}_k \tag{18}$$

and the relation (15).

Note that

$$\begin{aligned} T_K^{-1}\overline{A}_{K-1}T_K T_K^{-1}\overline{A}_{K-2}T_{K-1}\dots T_1^{-1}\overline{A}_0 T_0 &= T_0^{-1}\left(\overline{A}_{K-1}\overline{A}_{K-2}\dots\overline{A}_0 T_0\right) \\ &= T_0^{-1}\overline{A}T_0 \end{aligned} \tag{19}$$

since $T_K = T_0$. □

**Lemma 2.** *Let the scalar function $f(\lambda)$ be well-defined on the spectrum of the matrix $A \in \Re^{n\times n}$, i.e. the values*

$$\begin{gathered} f^{(i)}(\lambda_k) = \left.\frac{d^i f(\lambda)}{d\lambda^i}\right|_{\lambda=\lambda_k}, \\ i = 0, 1, \dots, m_k - 1, \quad k = 1, \dots, r \quad \left(\sum_{k=1}^{r} m_k = n\right) \end{gathered} \tag{20}$$

*are finite. Then the function $f(A)$ of the matrix $A$ is given by [4]*

$$f(A) = \sum_{k=1}^{r}\left[Z_{k1}f(\lambda_k) + Z_{k2}f^{(1)}(\lambda_k) + \dots + Z_{km_k}f^{(m_k-1)}(\lambda_k)\right], \tag{21}$$

*where*

$$Z_{kj} = \sum_{i=j-1}^{m_k-1}\frac{\Psi_k(A)(A - I_n\lambda_k)^i}{(i-j+1)!(j-1)!}\frac{d^{(i-j+1)}}{d\lambda^{(i-j+1)}}\left[\frac{1}{\Psi_k(\lambda)}\right]_{\lambda=\lambda_k} \tag{22}$$

*and the minimal polynomial of the matrix $A$ has the form*

$$\begin{gathered} \Psi(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\dots(\lambda-\lambda_r)^{m_r}, \\ \Psi_k(\lambda) = \frac{\Psi(\lambda)}{(\lambda-\lambda_k)^{m_k}}, \quad k = 1, \dots, r. \end{gathered} \tag{23}$$

In particular case when $m_1 = m_2 = \dots = m_r = 1$ $(r = n)$ we have

$$\Psi(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_n) \tag{24}$$

and

$$f(A) = \sum_{k=1}^{n} Z_k f(\lambda_k), \tag{25}$$

where

$$Z_k = \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{(A - I_n \lambda_i)}{\lambda_k - \lambda_i}, \quad k = 1, \ldots, n. \tag{26}$$

## 4. Procedure and example

To solve the problem under considerations using the method presented in Section 3 the following procedure can be used.

**Procedure:**

given: the matrix $\overline{A}_i$ defined by (12) and the positive number $K$,

find: the matrix $A$ defined by (15) and the matrices $T_k$ for $k = 0, 1, \ldots, K-1$.

**Step 1.** Assume $T_0 = I_n$.

**Step 2.** For given $\overline{A}$ and $K$ using the Lagrange-Sylvester formula (Lemma 2) compute the matrix

$$A = \sum_{k=1}^{\bar{n}} Z_{kj} \sqrt[K]{\lambda_k}, \tag{27}$$

where $Z_{kj}$ has the form (22) in general case or (26) if the matrix $\overline{A}$ has distinct nonzero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{\bar{n}}$.

**Step 3.** Using the formula (which follows from (15))

$$T_{k+1} = A_k T_k A^{-1} \tag{28}$$

compute the matrices $T_k$ for $k = 0, 1, \ldots, K-1$.

The procedure will be illustrated by the following example.

**Example 1.** *Consider the fractional periodic discrete-time linear system (6) with*

$$A_i = \left[2 - \sin \frac{\pi}{4} i\right], \quad \alpha = 0.6 \tag{29}$$

*and the period $K = 4$.*

*The matrix (10) has the form*

$$\overline{A}_{\alpha i} = A_i + I_n \alpha = \left[2.6 - \sin \frac{\pi}{4} i\right], \quad i = 0, 1, \ldots, 3 \tag{30}$$

*and the matrix (12)*

$$\overline{A} = \begin{bmatrix} \overline{A}_{\alpha 0} & 0 & 0 & 0 \\ c_2(\alpha) & \overline{A}_{\alpha 1} & 0 & 0 \\ c_3(\alpha) & c_2(\alpha) & \overline{A}_{\alpha 2} & 0 \\ c_4(\alpha) & c_3(\alpha) & c_2(\alpha) & \overline{A}_{\alpha 3} \end{bmatrix} = \begin{bmatrix} 2.6 & 0 & 0 & 0 \\ 0.12 & 1.893 & 0 & 0 \\ 0.056 & 0.12 & 1.6 & 0 \\ 0.034 & 0.056 & 0.12 & 1.893 \end{bmatrix}. \tag{31}$$

Using Procedure we obtain.

**Step 1.** Assume $T_0 = I_n$.

**Step 2.** The characteristic (minimal) polynomial of the matrix (31) has the form

$$\Psi(\lambda) = \det\left[I_n\lambda - \overline{A}\right] = (\lambda - 1.893)^2(\lambda - 1.6)(\lambda - 2.6). \tag{32}$$

Taking into account (22) and the eigenvalues $\lambda_1 = 1.893$, $\lambda_2 = 1.6$, $\lambda_3 = 2.6$ of the matrix $\overline{A}$ we obtain

$$Z_{11} = \begin{bmatrix} -57.62 & -34.28 & -38.66 & -36.49 \\ -32.12 & -56.11 & -43.19 & -41.02 \\ -32.55 & -42.67 & -57.38 & -42.85 \\ -29.62 & -39.21 & -44.52 & -57.63 \end{bmatrix},$$

$$Z_{12} = \begin{bmatrix} -48.24 & -6.679 & 6.948 & 0.0039 \\ -33.24 & -7.474 & 7.733 & 0.0044 \\ -32.79 & -9.281 & 11.67 & 0.0046 \\ -30.08 & -8.566 & 6.13 & 0.0061 \end{bmatrix},$$

$$Z_{21} = \begin{bmatrix} 510.2 & 736.8 & 759 & 747.5 \\ 697.1 & 670.9 & 776.6 & 762.9 \\ 713.8 & 746.1 & 739.7 & 772.8 \\ 708 & 737.1 & 758.2 & 678.6 \end{bmatrix},$$

$$Z_{31} = \begin{bmatrix} -32.21 & -66.88 & -67.9 & -67.35 \\ -64.31 & -55.56 & -70.92 & -69.98 \\ -67.18 & -68.48 & -64.6 & -71.7 \\ -66.19 & -66.92 & -67.76 & -55.51 \end{bmatrix}.$$

Using (27) we obtain

$$A = Z_{11}\sqrt[4]{\lambda_1} + Z_{12}\sqrt[4]{\lambda_1} + Z_{21}\sqrt[4]{\lambda_2} + Z_{31}\sqrt[4]{\lambda_3} = \begin{bmatrix} 408.8 & 695.7 & 730.2 & 712.4 \\ 625.7 & 609.4 & 741.7 & 721 \\ 640.8 & 691.2 & 696.3 & 727.9 \\ 642.2 & 687.9 & 721.6 & 625.2 \end{bmatrix}.$$

**Step 3.** Compute $T_1, T_2, T_3$ using $T_{k+1} = A_k T_k A^{-1}$

$$T_1 = A_0 T_0 A^{-1} = \begin{bmatrix} -0.0081 & 0.0013 & 0.0044 & 0.0027 \\ 0.0034 & -0.018 & 0.0091 & 0.0062 \\ 0.0035 & 0.013 & -0.024 & 0.0095 \\ 0.00063 & 0.0037 & 0.013 & -0.017 \end{bmatrix},$$

$$T_2 = A_1 T_1 A^{-1} = \begin{bmatrix} 0.000056 & 0.000021 & -0.000061 & -0.000012 \\ -0.000034 & 0.0003 & -0.00018 & -0.00008 \\ -0.00004 & -0.00032 & 0.00054 & -0.0002 \\ 0.000027 & 0.000025 & -0.00033 & 0.00029 \end{bmatrix},$$

$$T_3 = A_2 T_2 A^{-1} = \begin{bmatrix} -0.0000003 & -0.00000056 & 0.00000088 & -0.00000048 \\ 0.0000003 & -0.000004 & 0.000003 & 0.00000071 \\ 0.0000005 & 0.0000059 & -0.0000094 & 0.0000032 \\ -0.00000055 & -0.0000018 & 0.0000061 & -0.000004 \end{bmatrix}.$$

## 5. Concluding remarks

The fractional discrete-time linear systems with periodic varying parameters have been analyzed. The Floquet-Lyapunov transformation has been extended to fractional linear discrete-time systems with periodic varying parameters. The procedure and the example of fractional linear discrete-time system with periodic varying parameters illustrating the method has been presented. An open problem is an extension of these considerations to fractional different orders linear systems with periodic coefficients.

## References

[1] P.J. Antsaklis and A.N. Michel: *Linear Systems*, Birkhauser, Boston, 1997.

[2] J.J. DaCunha and J.M. Davis: A unified Floquet theory for discrete, continuous and hybrid periodic linear systems. *Journal of Differential Equations*, **251**(11), (2011), 2987–3027. DOI: 10.1016/j.jde.2011.07.023

[3] G. Floquet: Sur les équations différentielles linéaires à coefficients périodiques. *Annales scientifiques de l'École Normale Supérieure*, Ser. 2, **12** (1883), 47–88.

[4] F.G. Gantmacher: *The Theory of Matrices*. Chelsea, New York, 1959.

[5] T. Kaczorek: Floquet-Lyapunov transformation for regular 2-D linear systems. *Bulletin of the Polish Academy of Sciences. Technical Sciences*, **39** (1991), 673–679.

[6] T. Kaczorek: *Selected Problems of Fractional Systems Theory*. Springer, Berlin, 2011.

[7] T. Kaczorek and K. Rogowski: *Fractional Linear Systems and Electrical Circuits*. Springer, Cham, 2015.

[8] W. Mitkowski *Outline of Control Theory*. AGH Publisher, Kraków, 2019, ( in Polish).

[9] P. Ostalczyk: *Discrete Fractional Calculus*. World Scientific, River Edgle NJ, 2016.

[10] I. Podlubny: *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[11] T.J. Osler: Leibniz rule for fractional derivatives generalized and application to infinite. *SIAM Journal on Applied Mathematics*, **18**(3), (1970), 658–674. DOI: 10.1137/0118059

[12] J. Sreedhar and P. Van Dooren: Discrete-time Periodic Systems: A Floquet Approach. In *Proceedings of the Conference on Information Sciences and Systems*, Baltimore, (1993), 194–195.