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A New Generalized Family of Weibull-Exponentiated Half Logistic-G Distribution with Applications

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Abstract

We propose a new extension of the Weibull-exponentiated half logistic-G (WEHL-G) distribution, called the Marshall-Olkin-Weibull-exponentiated half logistic-G (MO-WEHL-G) family of distributions. The properties of this family of distributions including quantile function, distribution of the order statistics, hazard rate function, Rényi entropy and moments of residual life are presented. To estimate the parameters of the MO-WEHL-G family of distributions, six different estimation approaches are used, namely, maximum likelihood, Anderson-Darling, Ordinary Least Squares, Weighted Least Squares, Cramér-von Mises and Maximum Product of Spacing. The consistency properties of the six estimation methods were assessed using Monte Carlo simulations for a special case of the MO-WEHL-G family of distributions. The flexibility and importance of the proposed model was assessed using numerous goodness-of-fit statistics on three different data sets.

Keywords: Marshall-Olkin distribution, Weibull-exponentiated half logistic-G distribution, consistency, properties, Monte Carlo simulations

JEL Classification: 62E99, 60E05

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1 Introduction

Several standard distributions such as Weibull, Rayleigh, log-logistic and many others have been used in the past decades to characterize available data sets. However, these traditional distributions are often not adequate in characterizing modern data sets (Jamal et al. 2019). Thus, the need for generalizing traditional distributions by means of compounding or using various generating techniques is apparent. Cordeiro and Lemonte (2011) stated that one of the well-established methods for getting a more flexible distribution is by adding parameter(s) to a baseline distribution. These methods and generators have been detailed in literature, including methods such as Lomax generator by Cordeiro et al. (2014), Weibull-G by Bourguignon et al. (2014), transformed-transformer (T-X) by Alzaghal et al. (2013), Kumaraswamy-G family of distributions by Cordeiro and Lemonte (2011), McDonald-G family of distributions by Alexander et al. (2012), among others. Some recent distributions generated using these methods include: Exponentiated Weibull-exponential distribution by Elgarhy et al. (2017), Harris-Topp-Leone-G family of distributions by Oluyede and Moakofi (2022), generalized Gamma-Weibull distribution by Dauda et al. (2023), half logistic log-logistic Weibull distribution by Moakofi et al. (2022), half-Cauchy generalized exponential distribution by Chaudhary et al. (2022), generalized exponential extended exponentiated family of distributions by Hussain et al. (2022), gamma odd power generalized Weibull-G family of distributions by Gabanakgosi et al. (2022), gamma power half logistic distribution by Arshad et al. (2022) and Topp-Leone Weibull-Lomax distribution by Jamal et al. (2019).

Marshall and Olkin (1997) proposed a way to add a shape parameter to a baseline distribution in order to improve its flexibility. According to Santos-Nero et al. (2014), the Marshall-Olkin transformation offers a variety of behaviors subject to the choice of baseline distribution. This transformation provides a wide range of advantages as the addition of a shape parameter is a well established technique of coming up with more flexible distributions. Well-known generalized distributions via Marshall-Olkin technique include Marshall-Olkin log-logistic extended Weibull distribution by Lepetu et al. (2017), Marshall-Olkin exponential Weibull distribution by Pogány et al. (2017), Marshall-Olkin additive Weibull distribution by Afify et al. (2018), generalized Marshall-Olkin Kumaraswamy-G distribution by Chakraborty and Handique (2017), generalized Marshall-Olkin exponentiated exponential distribution by Ozkan et al. (2023), Marshall-Olkin Fréchet distribution by Krishna et al. (2013) and unit generalized Marshall-Olkin Weibull distribution by Karakaya (2022). Barreto et al. (2013) provided general results for the Marshall-Olkin-G distribution. The Marshall-Olkin extended Weibull family of distributions by Santos-Nero et al. (2014), Korkmaz et al. (2019) focuses on the regression modelling and application of censored data for the Weibull Marshall-Olkin family of distributions. Other papers worth noting are Kumaraswamy Marshall-Olkin-G family of distributions by Alizadeh et al. (2015), Marshall-Olkin generalized gamma distribution by Barriga et al. (2018), and Marshall-Olkin generalized exponential distribution by Ristić and Kundu (2015). The cumulative distribution function (cdf) and probability density function (pdf) of the Marshall-Olkin-G (MO-G) family of distributions by Marshall and Olkin (1997) are given by

$$F_{MO-G}(x;\delta,\xi) = 1 - \frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}, \tag{1}$$

and

$$f_{MO-G}(x;\delta,\xi) = \frac{\delta g(x;\xi)}{\left(1-\bar{\delta}\bar{G}(x;\xi)\right)^2}, \tag{2}$$

respectively, where $\delta > 0$ is the tilt paramater, $\bar{\delta} = 1-\delta$. $\bar{G}(x;\xi) = 1 - G(x;\xi)$, $G(x;\xi)$ and $g(x;\xi)$ are the survival function, cdf and pdf of the baseline distribution, respectively, with the parameter vector ξ.
The Weibull-exponentiated half logistic-G (WEHL-G) family of distributions was introduced by Peter et al. (2022). The cdf and pdf are given by

$$F_{WEHL-G}(x;\alpha,\beta,\xi) = 1 - \exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right), \tag{3}$$

and

$$\begin{aligned} f_{WEHL-G}(x;\alpha,\beta,\xi) = 2\alpha\beta \exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right) \times \\ \times \left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta-1} \times \\ \times \left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1} \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha-1} \times \\ \times \frac{g(x;\xi)}{(1+\bar{G}(x;\xi))^2}, \end{aligned} \tag{4}$$

respectively, where $\alpha, \beta > 0$ are shape parameters and ξ is a vector of parameters.
The proposed distribution offer intriguing qualities and produce improved fits to many types of real-world data than the baseline and several extended distributions in the literature. Also, the additional shape parameter will improve the flexibility of skewness, kurtosis and also modulate the weight of the tails of any baseline distribution, thus allowing our model to capture characteristics of various types of data including heavy-tailed data.
The main objective of this work is to introduce a new flexible family of distributions that can characterize several available or emerging data sets. The distribution is named MO-WEHL-G family of distributions.
This paper is organized as follows. In Sections 2, we present the Marshall-Olkin

Weibull-exponentiated half logistic-G family of distributions, hazard rate function, and quantile function. Section 3 focuses on the special cases of three different kinds of baseline distributions namely log-logistic, Weibull and standard half logistic distributions. In Section 4, properties such as order statistics, Rényi entropy, moments and generating functions are discussed. Section 5 is dedicated to obtaining the maximum likelihood estimates (MLEs), as well as using different estimation techniques. Sections 6 and 7 are dedicated to Monte Carlo simulations study and some applications, followed by concluding remarks in Section 8.

2 Marshall-Olkin-Weibull-Exponentiated Half Logistic-G distribution

In this section, we present the cdf and pdf of the new MO-WEHL-G family of distributions. We insert the WEHL-G distribution into equations (1) and (2) to obtain the new family of distributions. The cdf and pdf of the MO-WEHL-G family of distributions are given by

$$F_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi) = 1 - \frac{\delta \exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)}{1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)}, \tag{5}$$

and

$$\begin{aligned} & f_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi) = \\ & = 2\alpha\beta\delta \exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right) \times \\ & \times \left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta-1} \times \\ & \times \left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1} \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha-1} \times \\ & \times \left(1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{-2} \times \\ & \times \frac{g(x;\xi)}{(1+\bar{G}(x;\xi))^2}, \end{aligned}$$

respectively, for $\delta, \alpha, \beta > 0$, being shape parameters, $\bar{\delta} = 1 - \delta$ and baseline vector of parameters ξ. The hazard rate function (hrf) is given by

$$h(x; \delta, \alpha, \beta, \xi) = 2\alpha\beta \left[-\log\left(1 - \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta-1} \times$$
$$\times \left(1 - \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1} \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha-1} \times$$
$$\times \left(1 - \bar{\delta}\exp\left(-\left[-\log\left(1 - \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{-1} \times$$
$$\times \frac{g(x;\xi)}{(1+\bar{G}(x;\xi))^2},$$

for $\delta, \alpha, \beta > 0$, $\bar{\delta} = 1 - \delta$ and ξ as a baseline vector of parameters.

2.1 Quantile function

The quantile function plays an important role in statistics when it comes to generating random numbers from a probability distribution. Suppose the random variable U follows the uniform distribution. Then the quantile function of the MO-WEHL-G family of distributions is obtained by solving the non-linear equation:

$$1 - \frac{\delta \exp\left(-\left[-\log\left(1 - \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)}{1 - \bar{\delta}\exp\left(-\left[-\log\left(1 - \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)} = u,$$

for $0 \le u \le 1$, that is,

$$G(x;\xi) = 2\left(\left(\left[1 - \exp\left(-\left[-\log\left(\frac{1-u}{\delta + (1-u)\bar{\delta}}\right)\right]^{1/\beta}\right)\right]^{-1/\alpha}\right) + 1\right)^{-1}. \tag{6}$$

Consequently, the quantile function of the MO-WEHL-G family of distributions is given by

$$Q_X(u) = G^{-1}\left[2\left(\left(\left[1 - \exp\left(-\left[-\log\left(\frac{1-u}{\delta + (1-u)\bar{\delta}}\right)\right]^{1/\beta}\right)\right]^{-1/\alpha}\right) + 1\right)^{-1}\right]. \tag{7}$$

Thus, variates from the MO-WEHL-G family of distributions can be obtained using equation (7), for specified cdf G.

2.2 Expansion of density function

In this sub-section, we use the general results for Marshall and Olkin's family of distributions to express the pdf of the MO-WEHL-G family of distributions as an infinite linear combination of the pdf of exponentiated-G (Exp-G) distribution. Note that the pdf of the MO-WEHL-G family of distributions can be written as

$$f_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi)=\frac{\delta f_{WEHL-G}(x;\alpha,\beta,\xi)}{\left(1-\bar{\delta}\bar{F}_{WEHL-G}(x;\alpha,\beta,\xi)\right)^{2}}, \tag{8}$$

where $F_{WEHL-G}(x;\alpha,\beta,\xi)$ and $f_{WEHL-G}(x;\alpha,\beta,\xi)$ are given in equations (3) and (4), respectively. We apply the series expansion

$$(1-z)^{-k}=\sum_{t=0}^{\infty}\binom{k+t-1}{t}z^{t}, \tag{9}$$

which is valid for $|z|<1$ and $k>0$ (Gradshteyn et al. 2014). If $\delta\in(0,1)$, we can obtain

$$f_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi)= \\ = f_{WEHL-G}(x;\alpha,\beta,\xi)\sum_{t=0}^{\infty}\sum_{k=0}^{t}\phi_{t,k}\left(F_{WEHL-G}(x;\alpha,\beta,\xi)\right)^{t-k} \tag{10}$$

where $\phi_{t,k}=\phi_{t,k}(\delta)=\delta(t+1)(1-\delta)^{t}(-1)^{t-k}\binom{t}{k}$. For $\delta>1$, we have

$$f_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi)=f_{WEHL-G}(x;\alpha,\beta,\xi)\sum_{t=0}^{\infty}\vartheta_{t}F^{t}_{WEHL-G}(x;\alpha,\beta,\xi), \tag{11}$$

where $\vartheta_t=\vartheta_t(\delta)=\frac{(t+1)(1-1/\delta)^{t}}{\delta}$ (Barreto et al. 2013). (See details in the appendix). Consequently, for $\delta\in(0,1)$, equation (6) becomes

$$f_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi) = \sum_{r=0}^{\infty}\varphi_{r+1}g_{r+1}(x;\xi), \tag{12}$$

where

$$\begin{aligned}\varphi_{r+1} &= 2\alpha\beta\sum_{t,q,p,m,s,l,w,j=0}^{\infty}\sum_{k=0}^{t}\phi_{t,k}b_{s,m}\binom{\beta(p+1)-1}{m}\binom{t-k}{q}(-1)^{q+p+j+r}\times\\ &\times\frac{(q+1)^{p}}{p!}\frac{\Gamma(l+1)}{\Gamma(1)l!(r+1)}\binom{\alpha\left(m+s+\beta(p+1)+l\right)+w}{w}\times\\ &\times\binom{\alpha\left(m+s+\beta(p+1)+l\right)-1}{j}\binom{w+j}{r},\end{aligned} \tag{13}$$

and $g_{r+1}(x;\xi) = (r+1)G^r(x;\xi)g(x;\xi)$ is the Exp-G distribution with the power parameter $(r+1)$. Also, for $\delta > 1$, equation (6) can be written as

$$f_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi) = \sum_{r=0}^{\infty} \varrho_{r+1} g_{r+1}(x;\xi), \tag{14}$$

where

$$\begin{aligned}\varrho_{r+1} &= 2\alpha\beta \sum_{t,q,p,m,s,l,w,j=0}^{\infty} \vartheta_t b_{s,m} \binom{\beta(p+1)-1}{m}\binom{t}{q}(-1)^{q+p+j+r} \times \\ &\times \frac{(q+1)^p}{p!}\frac{\Gamma(l+1)(r+1)}{\Gamma(1)l!}\binom{\alpha\left(m+s+\beta(p+1)+l\right)+w}{w} \times \\ &\times \binom{\alpha\left(m+s+\beta(p+1)+l\right)-1}{j}\binom{w+j}{r},\end{aligned} \tag{15}$$

and $g_{r+1}(x;\xi) = (r+1)G^r(x;\xi)g(x;\xi)$ is the Exp-G distribution with the power parameter $(r+1)$. Therefore, for both cases, the pdf of MO-WEHL-G family of distributions can be expressed as a linear combination of the Exp-G distribution with power parameter $(r+1)$.

3 Special cases

This section provided some special cases of the MO-WEHL-G family of distributions when the baseline distribution is specified. We consider the cases when the baseline distributions are log-logistic, Weibull and standard half logistic distributions.

3.1 Marshall-Olkin Weibull-Exponentiated Half Logistic-Log-Logistic (MO-WEHL-LLoG) distribution

Consider the log-logistic distribution as the baseline distribution with parameter $c > 0$ having cdf and pdf $G(x;c) = 1-(1+x^c)^{-1}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-2}$, respectively. Then, the cdf and pdf of MO-WEHL-LLoG distribution are given by

$$F(x;\delta,\alpha,\beta,c) = 1 - \frac{\delta \exp\left(-\left[-\log\left(1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right)\right]^{\beta}\right)}{1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right)\right]^{\beta}\right)}, \tag{16}$$

and

$$
\begin{aligned}
f(x;\delta,\alpha,\beta,c) &= 2\alpha\beta\delta\exp\left(-\left[-\log\left(1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right)\right]^{\beta}\right)\times\\
&\times \left[-\log\left(1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right)\right]^{\beta-1}\times\\
&\times \left(1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right)^{-1}\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha-1}\times\\
&\times \left(1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{-2}\times\\
&\times \frac{cx^{c-1}(1+x^c)^{-2}}{(1+(1+x^c)^{-1})^2},
\end{aligned}
$$

respectively for $\delta,\alpha,\beta,c>0$, being shape parameters, $x>0$, and $\bar{\delta}=1-\delta$. The hrf is given by

$$
\begin{aligned}
h(x;\delta,\alpha,\beta,c) &= 2\alpha\beta\left[-\log\left(1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right)\right]^{\beta-1}\times\\
&\times \left(1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{-1}\times\\
&\times \left(1-\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha}\right)^{-1}\left(\frac{1-(1+x^c)^{-1}}{1+(1+x^c)^{-1}}\right)^{\alpha-1}\times\\
&\times \frac{cx^{c-1}(1+x^c)^{-2}}{(1+(1+x^c)^{-1})^2},
\end{aligned}
$$

for $\delta,\alpha,\beta,c>0$, $x>0$, and $\bar{\delta}=1-\delta$.
Figure 1 illustrates the flexibility of the MO-WEHL-LLoG distribution for selected parameter values. The pdf of the MO-WEHL-LLoG distribution can take various shapes that include J, reverse-J, uni-modal, left-skewed or right-skewed shapes. The hrf of the MO-WEHL-LLoG distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.
As shown by Figure 2, for $\delta=1.9$ and $\beta=1.6$, the MO-WEHL-LLoG exhibits higher levels of skewness for larger values of the parameter c across varying levels of α. We can also see high values of skewness also for large values of α and c, when $\delta=4.0$ and $\beta=1.1$.
MO-WEHL-LLoG has the highest values of kurtosis for larger values of the parameter c and smaller values α for $\delta=1.9$ and $\beta=1.6$. We can see high values of kurtosis for lower values of both α and c, also for higher values of c and smaller values of α, when we set $\delta=4.0$ and $\beta=1.1$.

Figure 1: Plots of the pdf and hrf for MO-WEHL-LLoG distribution

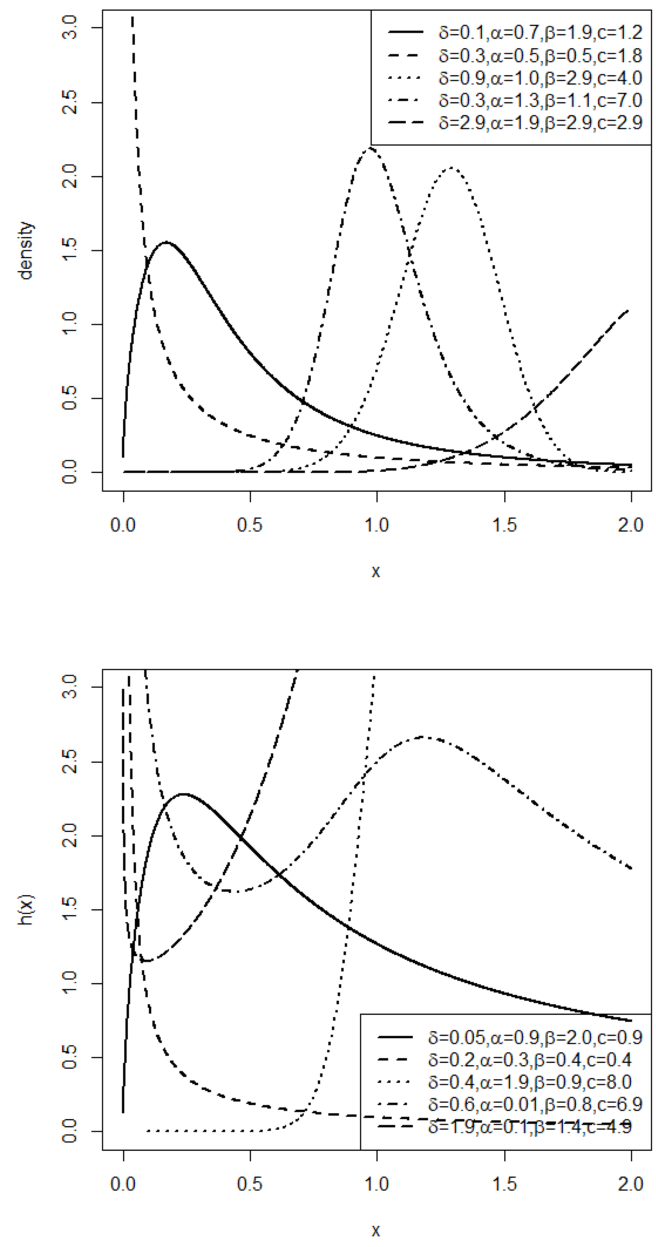
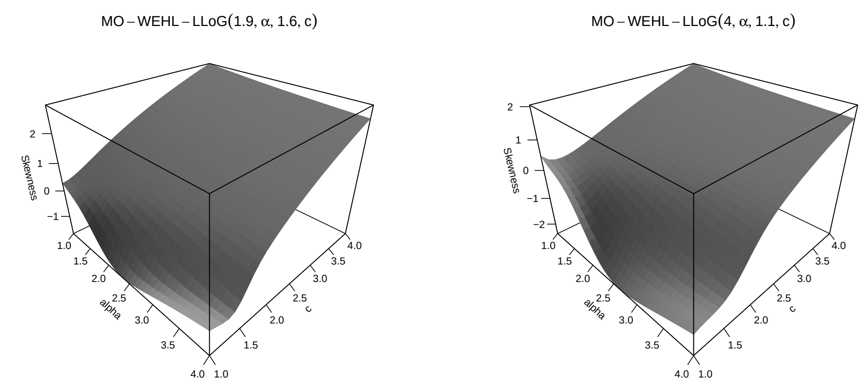


Figure 2: 3D-Plots of the skewness for MO-WEHL-LLoG distribution



3.2 Marshall-Olkin-Weibull-Exponentiated Half Logistic-Weibull (MO-WEHL-W) distribution

Let the one parameter Weibull distribution be the baseline distribution with pdf and cdf given by $g(x;\lambda) = \lambda x^{\lambda-1}\exp(-x^{\lambda})$ and $G(x;\lambda) = 1-\exp(-x^{\lambda})$, for $\lambda > 0$, respectively. Then, the cdf and pdf of MO-WEHL-W distribution are given by

$$F(x;\delta,\alpha,\beta,\lambda) = 1 - \frac{\delta \exp\left(-\left[-\log\left(1-\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha}\right)\right]^{\beta}\right)}{1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha}\right)\right]^{\beta}\right)}, \tag{17}$$

and

$$\begin{aligned} f(x;\delta,\alpha,\beta,\lambda) &= \\ &= 2\alpha\beta\delta\exp\left(-\left[-\log\left(1-\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha}\right)\right]^{\beta}\right)\times \\ &\times\left[-\log\left(1-\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha}\right)\right]^{\beta-1}\times \\ &\times\left(1-\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha}\right)^{-1}\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha-1}\frac{\lambda x^{\lambda-1}\exp(-x^{\lambda})}{(1+\exp(-x^{\lambda}))^{2}}\times \end{aligned}$$

$$\times\left(1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{-2},$$

respectively for $\delta,\alpha,\beta,\lambda>0$, being shape parameters, $x>0$ and $\bar{\delta}=1-\delta$. The hrf is given by

$$\begin{aligned} h(x;\delta,\alpha,\beta,\lambda) &= 2\alpha\beta\left[-\log\left(1-\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha}\right)\right]^{\beta-1}\times \\ &\times \left(1-\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha}\right)^{-1}\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha-1}\times \\ &\times \frac{\lambda x^{\lambda-1}\exp(-x^{\lambda})}{(1+\exp(-x^{\lambda}))^{2}}\times \\ &\times \left(1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{1-\exp(-x^{\lambda})}{1+\exp(-x^{\lambda})}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{-1}, \end{aligned}$$

for $\delta,\alpha,\beta,\lambda>0$, $x>0$ and $\bar{\delta}=1-\delta$.

Figure 3: 3D-Plots of the kurtosis for MO-WEHL-LLoG distribution

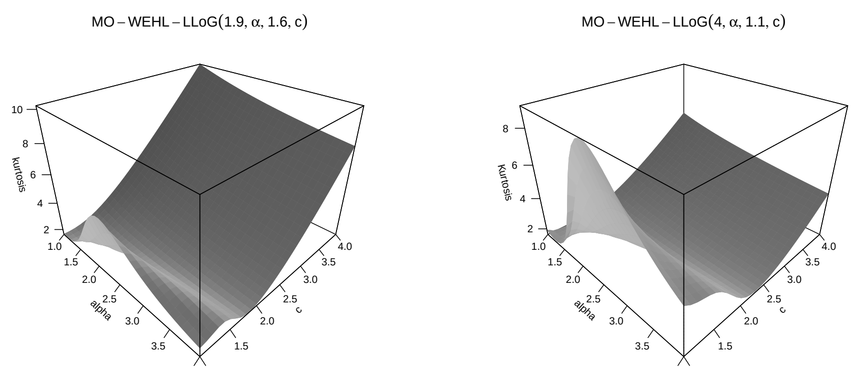
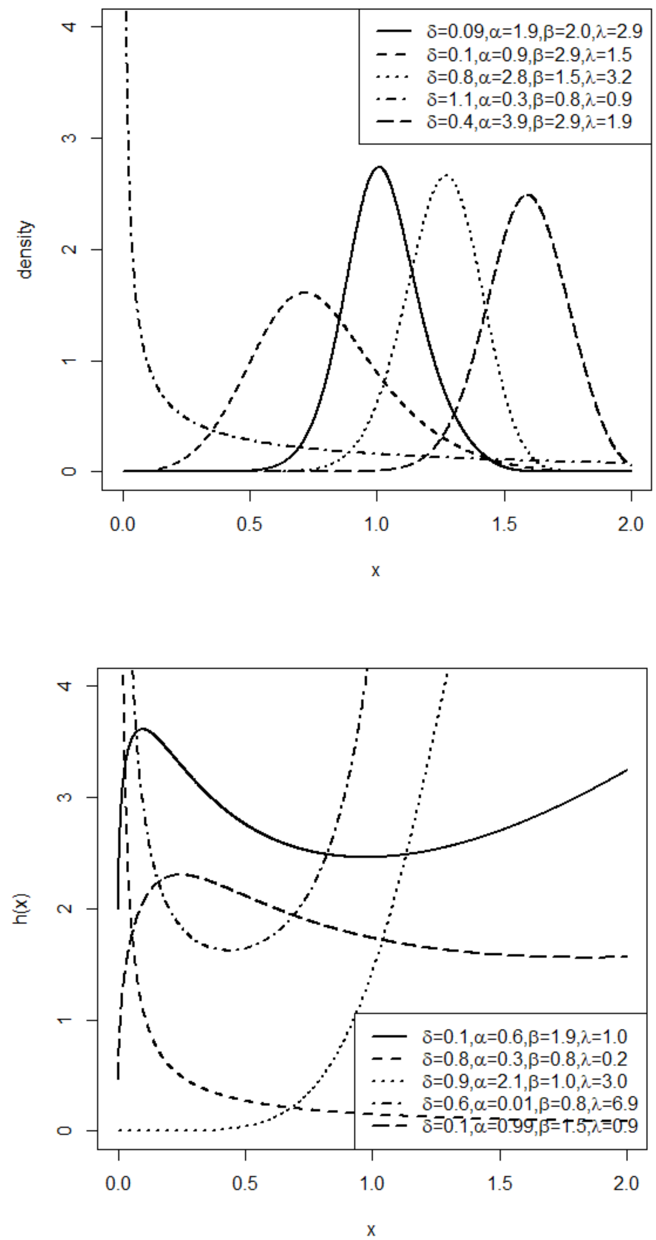


Figure 4 demonstrates the flexible nature of the MO-WEHL-W distribution for selected parameter values. The pdf of the MO-WEHL-W distribution exhibits various shapes that include reverse-J, uni-modal, left-skewed or right-skewed shapes. Also, the hrf of the MO-WEHL-W distribution gives increasing, decreasing, bathtub, upside down bathtub and upside down bathtub followed by bathtub shapes.

Figure 4: Plots of the pdf and hrf for MO-WEHL-W distribution



3.3 Marshall-Olkin-Weibull-Exponentiated Half Logistic-Standard Half Logistic (MO-WEHL-SHL) distribution

Let the baseline distribution be standard half logistic distribution with pdf and cdf given by $g(x) = \frac{2\exp(-x)}{(1+\exp(-x))^2}$ and $G(x) = \frac{1-\exp(-x)}{1+\exp(-x)}$, for $x > 0$, respectively. Then, the cdf and pdf of MO-WEHL-SHL distribution are given by

$$F(x;\delta,\alpha,\beta) = 1 - \frac{\delta\exp\left(-\left[-\log\left(1-\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha}\right)\right]^{\beta}\right)}{1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha}\right)\right]^{\beta}\right)}, \tag{18}$$

and

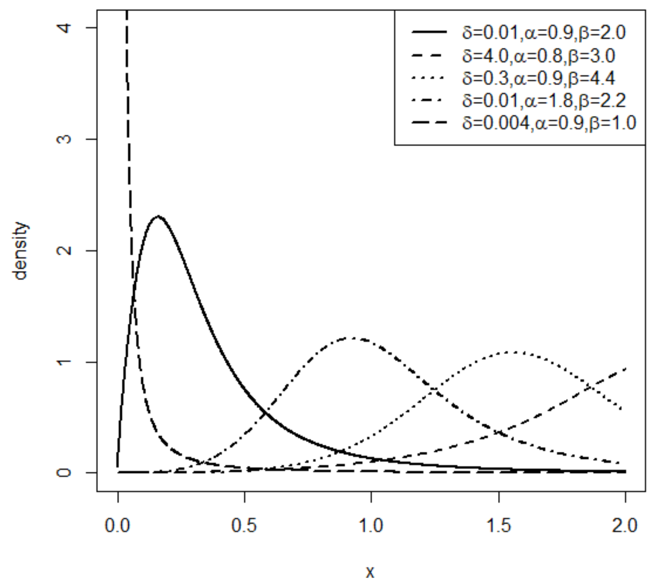
$$\begin{aligned}
&f(x;\delta,\alpha,\beta) = \\
&= 4\alpha\beta\delta\exp\left(-\left[-\log\left(1-\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha}\right)\right]^{\beta}\right)\times \\
&\times\left[-\log\left(1-\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha}\right)\right]^{\beta-1}\times \\
&\times\left(1-\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha}\right)^{-1}\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha-1}\frac{\exp(-x)}{(1+3\exp(-x))^2}\times \\
&\times\left(1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{-2},
\end{aligned}$$

respectively for $\delta,\alpha,\beta > 0$, being shape parameters, $x > 0$ and and $\bar{\delta} = 1-\delta$. The hrf is given by

$$\begin{aligned}
h(x;\delta,\alpha,\beta) &= 4\alpha\beta\left[-\log\left(1-\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha}\right)\right]^{\beta-1}\times \\
&\times\left(1-\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha}\right)^{-1}\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha-1}\times \\
&\times\frac{\exp(-x)}{(1+3\exp(-x))^2}\times \\
&\times\left(1-\bar{\delta}\exp\left(-\left[-\log\left(1-\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{-1},
\end{aligned}$$

for $\delta,\alpha,\beta > 0$, $x > 0$ and $\bar{\delta} = 1-\delta$.

Figure 5: Plots of the pdf and hrf for MO-WEHL-SHL distribution



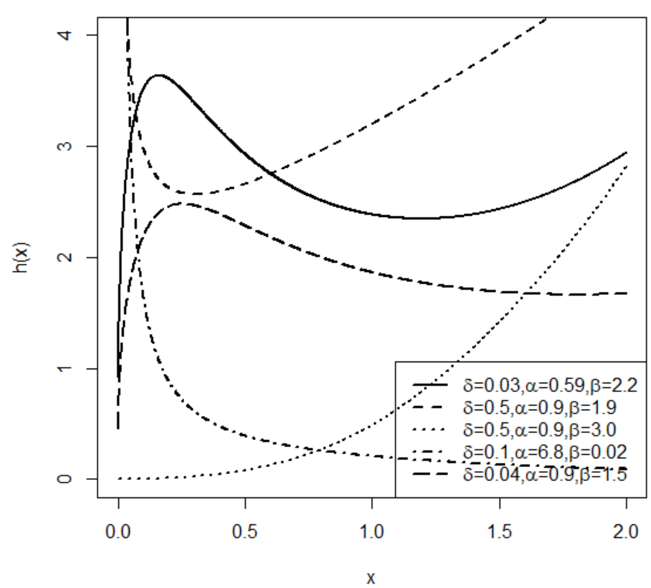


Figure 5 demonstrates the flexible nature of the MO-WEHL-SHL distribution for selected parameter values. The pdf of the MO-WEHL-SHL distribution exhibits various shapes that include J, reverse-J, uni-modal, left-skewed or right-skewed shapes. Also, the hrf of the MO-WEHL-SHL distribution gives increasing, decreasing, bathtub, upside down bathtub and upside down bathtub followed by bathtub shapes.

4 Statistical properties

Some of the useful statistical properties for the MO-WEHL-G family of distributions including the distribution of order statistics, Rényi entropy, moments and generating function, incomplete moments and moments of residual life are presented in this section.

4.1 Distribution of order statistics

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d) random variables distributed according to equation (6). Then, the pdf of the ρ^{th} order statistic is given by

$$\begin{aligned} f_{\rho:n}(x;\delta,\alpha,\beta,\xi) &= \delta n! f_{_{WEHL-G}}(x;\alpha,\beta,\xi) \times \\ &\times \sum_{z=0}^{n-\rho} \frac{(-1)^z}{(\rho-1)!(n-\rho)!} \frac{F^{z+\rho-1}_{_{WEHL-G}}(x;\alpha,\beta,\xi)}{[1-\bar{\delta}\bar{F}_{_{WEHL-G}}(x;\alpha,\beta,\xi)]^{z+\rho-1}}. \end{aligned} \quad (19)$$

If $\delta \in (0,1)$, we have

$$f_{\rho:n}(x;\delta,\alpha,\beta,\xi) = f_{_{WEHL-G}}(x;\alpha,\beta,\xi) \sum_{t=0}^{\infty}\sum_{z=0}^{n-\rho}\sum_{k=0}^{t} U_{t,z,k} F^{t+z-k+\rho-1}_{_{WEHL-G}}(x;\alpha,\beta,\xi), \quad (20)$$

where

$$U_{t,z,k} = U_{t,z,k}(\delta) = \frac{\delta n!(-1)^z(1-\delta)^t(-1)^{t-k}}{(\rho-1)!(n-\rho)!}\binom{t}{k}\binom{z+\rho+t}{t}. \quad (21)$$

For $\delta > 1$, we write

$$\left(1-\bar{\delta}\bar{F}_{_{WEHL-G}}(x;\alpha,\beta,\xi)\right) = \delta\left\{1-(\delta-1)F_{_{WEHL-G}}(x;\alpha,\beta,\xi)/\delta\right\},$$

such that

$$f_{\rho:n}(x;\delta,\alpha,\beta,\xi) = f_{_{WEHL-G}}(x;\alpha,\beta,\xi) \sum_{t=0}^{\infty}\sum_{z=0}^{n-\rho} c_{t,z} F^{t+z+\rho-1}_{_{WEHL-G}}(x;\alpha,\beta,\xi), \quad (22)$$

where

$$c_{t,z} = c_{t,z}(\delta) = \frac{(-1)^l(\delta-1)^t n!}{\delta^{z+t+\rho}(\rho-1)!(n-\rho)!}\binom{z+\rho+t}{t}. \tag{23}$$

(See details in the appendix).
For $\delta \in (0,1)$, we have the pdf of the ρ^{th} order statistic expressed as

$$f_{\rho:n}(x;\delta,\alpha,\beta,\xi) = \sum_{t,r=0}^{\infty}\sum_{z=0}^{n-\rho}\sum_{k=0}^{t} U_{t,z,k}\, a_{r+1}\, g_{r+1}(x;\xi), \tag{24}$$

where

$$\begin{aligned} a_{r+1} &= 2\alpha\beta\sum_{q,p=0}^{\infty}\sum_{m,s,l,w,j=0}^{\infty} b_{s,m}\binom{\beta(p+1)-1}{m}\binom{t+z-k+\rho-1}{q}\times \\ &\times (-1)^{q+p+j+r}\frac{(q+1)^p}{p!}\frac{\Gamma(l+1)}{\Gamma(1)l!(r+1)}\binom{\alpha\,(m+s+\beta(p+1)+l)+w}{w}\times \\ &\times \binom{\alpha\,(m+s+\beta(p+1)+l)-1}{j}\binom{w+j}{r}, \end{aligned}$$

and $g_{r+1}(x;\xi) = (r+1)G^r(x;\xi)g(x;\xi)$ is the Exp-G distribution with the power parameter $(r+1)$.
Similarly, for $\delta > 1$, then equation (22) can be written as

$$f_{\rho:n}(x;\delta,\alpha,\beta,\xi) = \sum_{t=0}^{\infty}\sum_{z=0}^{n-\rho} c_{t,z}\, a^{*}_{r+1}\, g_{r+1}(x;\xi), \tag{25}$$

where

$$\begin{aligned} a^{*}_{r+1} &= 2\alpha\beta\sum_{q,p=0}^{\infty}\sum_{m,s,l,w,j=0}^{\infty} b_{s,m}\binom{\beta(p+1)-1}{m}\binom{t+z+\rho-1}{q}(-1)^{q+p+j+r}\times \\ &\times \frac{(q+1)^p}{p!}\frac{\Gamma(l+1)}{\Gamma(1)l!(r+1)}\binom{\alpha\,(m+s+\beta(p+1)+l)+w}{w}\times \\ &\times \binom{\alpha\,(m+s+\beta(p+1)+l)-1}{j}\binom{w+j}{r}, \end{aligned}$$

and $g_{r+1}(x;\xi) = (r+1)G^r(x;\xi)g(x;\xi)$ is the Exp-G distribution with the power parameter $(r+1)$.

4.2 Rényi entropy

Rényi entropy is defined to be the measure of variation or uncertainty for a random variable X with pdf $f(x)$. Rényi entropy is defined as

$$I_R(v) = (1-v)^{-1} log\left[\int_{-\infty}^{\infty} f^v(x)dx\right],$$

where $v > 0$ and $v \neq 1$. Using the expansion from Barreto et al. (2013), for $\delta \in (0,1)$

$$f^{\nu}_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi) = \frac{\delta^{\nu} f^{\nu}_{WEHL-G}(x;\alpha,\beta,\xi)}{\Gamma(2\nu)} \sum_{i,t=0}^{\infty} \binom{i}{t} (-1)^t (1-\delta)^i \times \times \Gamma(2\nu+i) \frac{[F_{WEHL-G}(x;\alpha,\beta,\xi)]^t}{i!}$$

and for $\delta > 1$,

$$f^{\nu}_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi) = \frac{f^{\nu}_{WEHL-G}(x;\alpha,\beta,\xi)}{\delta^{\nu+t}\Gamma(2\nu)} \times \times \sum_{t=0}^{\infty} (\delta-1)^t \Gamma(2\nu+t) \frac{F^t_{WEHL-G}(x;\alpha,\beta,\xi)}{t!}.$$

Thus, Rényi entropy for $\delta \in (0,1)$ and $\delta > 1$ are given by

$$I_R(\nu) = (1-\nu)^{-1} \log \left(\sum_{i=0}^{\infty} e_i \int_0^{\infty} f^{\nu}_{WEHL-G}(x;\alpha,\beta,\xi)(F_{WEHL-G}(x;\alpha,\beta,\xi))^t dx \right) \quad (26)$$

and

$$I_R(\nu) = (1-\nu)^{-1} \log \left(\sum_{t=0}^{\infty} h_t \int_0^{\infty} f^{\nu}_{WEHL-G}(x;\alpha,\beta,\xi) F^t_{WEHL-G}(x;\alpha,\beta,\xi) dx \right), \quad (27)$$

where

$$e_i = e_i(\delta) = \frac{\sum_{t=0}^{\infty} \delta^{\nu}(1-\delta)^i \Gamma(2\nu+i)\binom{i}{t}(-1)^t}{\Gamma(2\nu) i!}$$

and

$$h_t = h_t(\delta) = \frac{(\delta-1)^t \Gamma(2\nu+t)}{\delta^{\nu+t}\Gamma(2\nu) t!}.$$

(See details in the appendix).
Now, for $\delta \in (0,1)$ and from equation (26), we have

$$I_R(\nu) = (1-\nu)^{-1} \log \left[\sum_{r=0}^{\infty} \tau_r \exp\left((1-\nu) I_{REG}\right) \right], \quad (28)$$

where

$$\begin{aligned}\tau_r &= (2\alpha\beta)^{\nu}\sum_{i,q,p=0}^{\infty}\sum_{m,s,l,w,j=0}^{\infty} e_i b_{s,m}\binom{\beta(p+\nu)-\nu}{m}\binom{t}{q}(-1)^{q+p+j+r}\times\\ &\times \frac{(q+\nu)^p}{p!}\frac{\Gamma(l+\nu)}{\Gamma(\nu)l!}\binom{\alpha\,(m+s+\beta(p+\nu)+l)+\nu+w-1}{w}\times\\ &\times \binom{\alpha\,(m+s+\beta(p+\nu)+l)-\nu}{j}\binom{w+j}{r}\frac{1}{\left(\frac{r}{\nu}+1\right)^{\nu}},\end{aligned} \quad (29)$$

and $I_{REG}=(1-\nu)^{-1}\log\left[\int_0^{\infty}\left(\left(\frac{r}{\nu}+1\right)g(x;\xi)[G(x;\xi)]^{\frac{r}{\nu}}\right)^{\nu}dx\right]$ is the Rényi entropy of the Exp-G distribution with power parameter $(\frac{r}{\nu}+1)$.
Similarly, for $\delta>1$, we have

$$I_R(\nu) = (1-\nu)^{-1}\log\left[\sum_{r=0}^{\infty}\tau_r^{*}\exp\left((1-\nu)I_{REG}\right)\right], \quad (30)$$

where

$$\begin{aligned}\tau_r^{*} &= (2\alpha\beta)^{\nu}\sum_{t,q,p=0}^{\infty}\sum_{m,s,l,w,j=0}^{\infty} h_t b_{s,m}\binom{\beta(p+\nu)-\nu}{m}\binom{t}{q}(-1)^{q+p+j+r}\times\\ &\times \frac{(q+\nu)^p}{p!}\frac{\Gamma(l+\nu)}{\Gamma(\nu)l!}\binom{\alpha\,(m+s+\beta(p+\nu)+l)+\nu+w-1}{w}\times\\ &\times \binom{\alpha\,(m+s+\beta(p+\nu)+l)-\nu}{j}\binom{w+j}{r}\frac{1}{\left(\frac{r}{\nu}+1\right)^{\nu}},\end{aligned} \quad (31)$$

and $I_{REG}=(1-\nu)^{-1}\log\left[\int_0^{\infty}\left(\left(\frac{r}{\nu}+1\right)g(x;\xi)[G(x;\xi)]^{\frac{r}{\nu}}\right)^{\nu}dx\right]$ is the Rényi entropy of the Exp-G distribution with power parameter $(\frac{r}{\nu}+1)$.

4.3 Moments and generating function

Let X be from the MO-WEHL-G family of distributions and Y_{r+1} denote the Exp-G random variable with power parameter $(r+1)$. Then the n^{th} raw moment of MO-WEHL-G family of distributions, say μ_n' can be obtained from (12) and (14). For $\delta\in(0,1)$,

$$\mu_n'=E(X^n) = \sum_{r=0}^{\infty}\varphi_{r+1}E(Y_{r+1}^n), \quad (32)$$

where φ_{r+1} is as given in equation (13) and $E(Y_{r+1}^n)$ is the n^{th} raw moment of Y_{r+1}. For $\delta > 1$,

$$\mu_n' = E(X^n) \quad = \quad \sum_{r=0}^{\infty} \varrho_{r+1} E(Y_{r+1}^n), \tag{33}$$

where ϱ_{r+1} is as given in (15) and $E(Y_{r+1}^n)$ is the n^{th} raw moment of Y_{r+1}.
The moment generating function (mgf) of X, denoted $M_X(t) = E(e^{tX})$ can be derived as follows. For $\delta \in (0, 1)$,

$$M_X(t) = \sum_{r=0}^{\infty} \varphi_{r+1} M_r(t).$$

For $\delta > 1$, the mgf of X is

$$M_X(t) = \sum_{r=0}^{\infty} \varrho_{r+1} M_r(t),$$

where $M_r(t)$ is the mgf of Y_{r+1}, φ_{r+1} and ϱ_{r+1} are obtained from equations (13) and (15). Hence, $M_X(t)$ can be determined from the Exp-G generating function. Furthermore, the characteristic function can be obtained by $\phi(t) = E(e^{itX})$, where $i = \sqrt{-1}$. For $\delta \in (0, 1)$, we have

$$\phi(t) = \sum_{r=0}^{\infty} \varphi_{r+1} \Phi_{r+1}(t),$$

and for $\delta > 1$,

$$\phi(t) = \sum_{r=0}^{\infty} \varrho_{r+1} \Phi_{r+1}(t),$$

where $\Phi_{r+1}(t)$ is the characteristic function of Exp-G distribution with power parameter $(r + 1)$.

4.4 Incomplete moments

Incomplete moments are useful when it comes to obtaining inequality measures (Bonferroni and Lorenz curves). The n^{th} incomplete moment of the MO-WEHL-G family of distributions is obtained as follows:
For $\delta \in (0, 1)$,

$$m_n(t) \quad = \quad \int_{-\infty}^{t} x^n f(x; \delta, \alpha, \beta, \xi) dx = \sum_{r=0}^{\infty} \varphi_{r+1} \int_{-\infty}^{t} x^n g_{r+1}(x; \xi) dx. \tag{34}$$

For $\delta > 1$,

$$m_n(t) = \int_{-\infty}^{t} x^n f(x;\delta,\alpha,\beta,\xi)dx = \sum_{r=0}^{\infty} \varrho_{r+1} \int_{-\infty}^{t} x^n g_{r+1}(x;\xi)dx. \quad (35)$$

Note that $m_1(t)$ and $m_2(t)$ can be used in the construction of Bonferroni and Lorenz curves. These curves are of great importance in insurance, demography, reliability, medicine and economics. Clearly, for $\delta \in (0,1)$

$$m_1(t) = \sum_{r=0}^{\infty} \varphi_{r+1} J_{r+1}(t), \quad (36)$$

and

$$m_2(t) = \sum_{r=0}^{\infty} \varrho_{r+1} J_{r+1}(t),$$

for $\delta > 1$, where $J_{r+1}(t) = \int_{-\infty}^{t} x g_{r+1}(x;\xi)dx$ is the first incomplete moment of the Exp-G distribution.

4.5 Moment of residual and reversed residual life

To calculate the mean, variance and coefficient of variation of residual life in reliability and survival analysis, one needs the moment of residual and reversed residual life. The n^{th} moment of the residual life, say $b_n(u)$ of a random variable X is given by

$$b_n(u) = E\left[(X-u)^n \mid X > u\right] = \frac{1}{\overline{F}(u)} \int_{u}^{\infty} (x-u)^n f(x;\delta,\alpha,\beta,\xi)dx.$$

Consequently, $b_n(u)$ for the MO-WEHL-G family of distributions is given as follows: For $\delta \in (0,1)$,

$$b_n(u) = \frac{1}{\overline{F}(u)} \sum_{r,i=0}^{\infty} \binom{n}{i} (-u)^{n-i} \varphi_{r+1} \int_{u}^{\infty} x^i g_{r+1}(x;\xi)dx, \quad (37)$$

where φ_{r+1} is as defined in equation (13) and $g_{r+1}(x;\xi)$ denotes the Exp-G distribution with power parameter $(r+1)$. Also, For $\delta > 1$,

$$b_n(u) = \frac{1}{\overline{F}(u)} \sum_{r,i=0}^{\infty} \varrho_{r+1} \binom{n}{i} (-u)^{n-i} \int_{u}^{\infty} x^i g_{r+1}(x;\xi)dx, \quad (38)$$

where ϱ_{r+1} is as defined in equation (15) and $g_{r+1}(x;\xi)$ denotes the Exp-G distribution with power parameter $(r+1) > 0$.
The n^{th} moment of the reversed residual life, say $e_n(u)$ of a random variable X is

$$e_n(u) = E\left[(u-X)^n \mid X \le u\right] = \frac{1}{F(u)} \int_0^u (u-x)^n f(x;\delta,\alpha,\beta,\xi)dx.$$

Subsequently, $e_n(u)$ for the MO-WEHL-G family of distributions is given as follows:
For $\delta \in (0,1)$,

$$e_n(u) = \frac{1}{F(u)} \sum_{r,i=0}^{\infty} \binom{n}{i} (-u)^{n-i} \varphi_{r+1} \int_0^u x^i g_{r+1}(x;\xi)dx,$$

where φ_{r+1} is as defined in equation (13) and $g_{r+1}(x;\xi)$ denotes the Exp-G distribution with power parameter $(r+1)$. Also, For $\delta > 1$,

$$e_n(u) = \frac{1}{F(u)} \sum_{r,i=0}^{\infty} \binom{n}{i} (-u)^{n-i} \varrho_{r+1} \int_0^u x^i g_{r+1}(x;\xi)dx,$$

where ϱ_{r+1} is as defined in equation (15) and $g_{r+1}(x;\xi)$ denotes the Exp-G distribution with power parameter $(r+1)$.

5 Estimation methods

In this section, we use different estimation methods to estimate the unknown parameters of the MO-WEHL-G family of distributions. The estimation methods include Maximum Likelihood (ML), Anderson-Darling (AD), Ordinary Least Squares (OLS), Weighted Least Squares (WLS), Cramér-von Mises (CVM) and Maximum Product of Spacing (MPS). For these estimation methods, a simple random sample is employed.
We maximize the objective functions using the nlm function in R (R Development Core Team (2011)), which optimizes the objective function from each estimation method, finds local estimates that are convergent. These functions were applied and executed across a wide range of initial values, incorporating both maximizing and minimizing techniques depending on the estimation method employed. This process often results in multiple maxima or minima. In such cases, we select the estimates corresponding to the largest observed maximum value or the smallest observed minimum value of the objective function, which are considered our best estimates. Occasionally, no maximum or minimum is identified for the chosen initial values. In these instances, new initial values are tried until a maximum or minimum is obtained.
The issues of existence and uniqueness of the estimates are of theoretical interest and have been explored by various authors for different distributions, including Seregin

(2010), Santos Silva and Tenreyro (2010) and Zhou (2009). At this point we are not able to address the theoretical aspects (existence, uniqueness) of the estimates of the parameters of the MO-WEHL-G family of distributions.

5.1 Maximum likelihood estimation

Let $X \sim MO-WEHL-G(\delta,\alpha,\beta,\xi)$ and $\Delta=(\delta,\alpha,\beta,\xi)^T$ be the vector of model parameters, then the log-likelihood function $\ell_n=\ell_n(\Delta)$ based on a random sample of size n from the MO-WEHL-G family of distributions is given by

$$\begin{aligned}
\ell(\Delta) &= (n)\ln(2\delta\alpha\beta)-\beta\sum_{i=1}^{n}\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\overline{G}(x_i;\xi)}\right)^{\alpha}\right)\right]+\\
&+ (\beta-1)\sum_{i=1}^{n}\ln\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\overline{G}(x_i;\xi)}\right)^{\alpha}\right)\right]-\\
&- \sum_{i=1}^{n}\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\overline{G}(x_i;\xi)}\right)^{\alpha}\right)+(\alpha-1)\sum_{i=1}^{n}\ln\left(\frac{G(x_i;\xi)}{1+\overline{G}(x_i;\xi)}\right)+\\
&+ \sum_{i=1}^{n}\ln\left(g(x_i;\xi)\right)-2\sum_{i=1}^{n}\ln\left(1+\overline{G}(x_i;\xi)\right)-\\
&- 2\sum_{i=1}^{n}\ln\left(1-\bar{\delta}\exp\left(-\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right).
\end{aligned}$$

In order to obtain the estimates of the unknown parameters from the MO-WEHL-G family of distributions, we solve $U=(\frac{\partial\ell_n}{\partial\delta},\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\beta},\frac{\partial\ell_n}{\partial\xi_k})^T=\mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The elements of the score vector U are given in the appendix.

5.2 Anderson-Darling estimation

Suppose $x_{(1)},x_{(2)},...,x_{(n)}$ are the order statistics of a random sample of size n from the MO-WEHL-G family of distributions. Then, the Anderson-Darling estimates (ADEs) of the MO-WEHL-G family of distributions are obtained by minimizing the following function

$$A(\delta,\alpha,\beta,\xi)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(F(x_{(i)};\delta,\alpha,\beta,\xi)\right)+\log\left(S(x_{(i)};\delta,\alpha,\beta,\xi)\right)\right],$$

where $F(x_{(i)};\delta,\alpha,\beta,\xi)$ and $S(x_{(i)};\delta,\alpha,\beta,\xi)$ be the cdf and survival function of the i^{th} order statistic from the MO-WEHL-G family of distributions.

The ADEs can also be derived by solving the non-linear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\vartheta_z\left(x_{(i)};\delta,\alpha,\beta,\xi\right)}{F(x_{(i)};\delta,\alpha,\beta,\xi)}-\frac{\vartheta_z\left(x_{(n+1-i)};\delta,\alpha,\beta,\xi\right)}{S\left(x_{(n+1-i)};\delta,\alpha,\beta,\xi\right)}\right]=0, z=1,2,3,4, \quad (39)$$

where

$$\begin{aligned}\vartheta_1\left(x_{(i)};\delta,\alpha,\beta,\xi\right) &= \frac{\partial F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)}{\partial\delta},\\ \vartheta_2\left(x_{(i)};\delta,\alpha,\beta,\xi\right) &= \frac{\partial F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)}{\partial\alpha},\\ \vartheta_3\left(x_{(i)};\delta,\alpha,\beta,\xi\right) &= \frac{\partial F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)}{\partial\beta},\end{aligned}$$

and

$$\vartheta_4\left(x_{(i)};\delta,\alpha,\beta,\xi\right) = \frac{\partial F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)}{\partial\xi_k}. \quad (40)$$

5.3 Ordinary Least Squares estimation

The Ordinary Least Squares estimates (OLSEs) of the parameters of the MO-WEHL-G family of distributions are obtained by minimizing the function

$$V\left(\delta,\alpha,\beta,\xi\right)=\sum_{i=1}^{n}\left[F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)-\frac{i}{n+1}\right]^2.$$

The OLSEs can be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\left[F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)-\frac{i}{n+1}\right]\vartheta_z\left(x_{(i)};\delta,\alpha,\beta,\xi\right)=0, z=1,2,3,4,$$

where $\vartheta_z\left(x_{(i)};\delta,\alpha,\beta,\xi\right)$ are defined in equation (40).

5.4 Weighted Least Squares estimation

The Weighted Least Squares estimates (WLSEs) of the parameters of the MO-WEHL-G family of distributions are obtained by minimizing the function

$$W\left(\delta,\alpha,\beta,\xi\right)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-1+1)}\left[F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)-\frac{i}{n+1}\right]^2,$$

with respect to δ,α,β and parameter vector ξ. The WLSEs can be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-1+1)}\left[F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)-\frac{i}{n+1}\right]\vartheta_z\left(x_{(i)};\delta,\alpha,\beta,\xi\right)=0, z=1,2,3,4,$$

where $\vartheta_z\left(x_{(i)};\delta,\alpha,\beta,\xi\right)$ are defined in equation (40).

5.5 Cramér-von Mises estimation

The Cramér-von Mises estimates (CVMEs) of the parameters of the MO-WEHL-G family of distributions parameters are obtained through the minimization of the function

$$C\left(\delta,\alpha,\beta,\xi\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)-\frac{2i-1}{2n}\right]^2,$$

with respect to δ,α,β and parameter vector ξ. The CVMEs can also be obtained by solving the non-linear equations

$$\sum_{i=1}^{n}\left[F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)-\frac{2i-1}{2n}\right]\vartheta_z\left(x_{(i)};\delta,\alpha,\beta,\xi\right)=0, z=1,2,3,4,$$

where $\vartheta_z\left(x_{(i)};\delta,\alpha,\beta,\xi\right)$ are defined in equation (40).

5.6 Maximum Product of Spacing estimation

The Maximum Product of Spacing method is used to estimate the parameters of a distribution as an alternative to the maximum likelihood method. Let $D_i\left(x_{(i)};\delta,\alpha,\beta,\xi\right)=F\left(x_{(i)};\delta,\alpha,\beta,\xi\right)-F\left(x_{(i-1)};\delta,\alpha,\beta,\xi\right)$, for $i=1,2,..,n+1$, be the uniform spacing of a random sample from the MO-WEHL-G family of distributions, where $F\left(x_(0);\delta,\alpha,\beta,\xi\right)=0$, $F\left(x_{(n+1)};\delta,\alpha,\beta,\xi\right)=1$ and $\sum_{i=1}^{n+1}D_i\left(x_{(i)};\delta,\alpha,\beta,\xi\right)=1$. The Maximum Product of Spacing estimates (MPSEs) for δ,α,β and parameter vector ξ can be obtained by maximizing

$$H\left(\delta,\alpha,\beta,\xi\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left(D_i\left(x_{(i)};\delta,\alpha,\beta,\xi\right)\right).$$

The MPSEs of the MO-WEHL-G family of distributions can be obtained by solving the non-linear equations:

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i\left(x_{(i)};\delta,\alpha,\beta,\xi\right)}\left[\vartheta_z\left(x_{(i)};\delta,\alpha,\beta,\xi\right)-\vartheta_z\left(x_{(i-1)};\delta,\alpha,\beta,\xi\right)\right]=0,$$

$z=1,2,3,4$, where $\vartheta_z\left(x_{(i)};\delta,\alpha,\beta,\xi\right)$ are defined in equation (40).

6 Monte Carlo simulation study

With the six estimation methods discussed in Section 5, the performance of the MO-WEHL-LLoG distribution is examined by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800) via the R package. We simulate $N = 1000$ samples for the true parameters values of $(\alpha, \delta, \beta, c)$ given in Table 1 and Table 2. The tables list the average bias (ABIAS) and root mean squared errors (RMSEs) for the six estimation methods: ML, LS, WLS, MPS, CVM, and AD, with different sample sizes. The ABIAS and RMSE for the estimated parameter, say, $\hat{\theta}$, say, are given by:

$$ABIAS(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. In Tables 1 and 2, the row indicating $\sum$ Ranks corresponds to the partial sum of the ranks. Among all the estimators for a given metric, the superscript indicates their rank. Table 2 presents, for example, the ABIAS of $\hat{\alpha}$ obtained via MLE method as $0.0091^{\{1\}}$ for $n = 25$. This indicates that the ABIAS of $\hat{\alpha}$ obtained using the MLE method ranks first among all other estimators. So, when $n = 25$, in comparison with all other estimators, MLE provides the best ABIAS of $\hat{\alpha}$.
Table 3 shows the partial and overall ranks of all the estimation methods of MO-WEHL-LLoG distribution by various model parameter values. Based on the results in Tables 1 and 2, the MO-WEHL-LLoG distribution is stable, as the ABIAS and RMSE values for its four parameters are modest. It can be observed that the bias occasionally decreases with increasing sample size, while RMSE decreases as sample size increases for all estimations. Figures 6 and 7 demonstrate how the RMSEs of parameters decreased with increasing sample size for each estimation method. It appears that, for large sample sizes, all estimation methods provide accurate bias and mean squared error estimates. Table 3 shows that MLE method allows us to obtain better estimates of MO-WEHL-LLoG parameters, followed by AD and then MPS methods. According to the rankings, the LS method performs the least well.

7 Applications

This section applies the MO-WEHL-LLoG distribution to three data sets in order to highlight the significance and applicability of the new distribution. The fit of the MO-WEHL-LLoG distribution is compared to those of the Marshall-Olkin log-logistic Weibull (MOLLW) distribution (Lepetu et al. 2017), Marshall-Olkin exponential-Gompertz (MOEGo) distribution by Khaleel et al. (2020), Marshall-Olkin generalized-log-logistic (MOG-LL) distribution by Yousof et al. (2018), Marshall-Olkin modified Weibull (MOMW) distribution by Santos-Nero et al. (2014), Weibull exponentiated half logistic log-logistic distribution (Peter et al. 2022), odd log-logistic exponentiated Weibull (OLLEW) by Afify et al. (2018), Weibull Lomax

Table 1: Simulation results for different estimation methods for $\alpha = 0.2, \delta = 0.2, \beta = 0.9, c = 2.3$

		MLE		LS		WLS		MPS		CVM		AD	
n	Parameter	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
25	α	$0.01808^{\{4\}}$	$0.23415^{\{5\}}$	$0.01096^{\{2\}}$	$0.21435^{\{3\}}$	$0.0141^{\{3\}}$	$0.20285^{\{2\}}$	$-0.02773^{\{5\}}$	$0.13493^{\{1\}}$	$0.00505^{\{1\}}$	$0.22056^{\{4\}}$	$0.05934^{\{6\}}$	$0.35014^{\{6\}}$
	δ	$0.05753^{\{1\}}$	$0.26177^{\{1\}}$	$1.3845^{\{6\}}$	$4.77267^{\{6\}}$	$1.0082^{\{4\}}$	$3.76933^{\{4\}}$	$0.26329^{\{2\}}$	$1.91841^{\{2\}}$	$1.20854^{\{5\}}$	$3.8836^{\{5\}}$	$0.37483^{\{3\}}$	$2.54928^{\{3\}}$
	β	$0.62092^{\{5\}}$	$3.19519^{\{6\}}$	$0.25681^{\{2\}}$	$0.60152^{\{2\}}$	$0.28287^{\{3\}}$	$1.28482^{\{4\}}$	$0.81786^{\{6\}}$	$3.10555^{\{5\}}$	$0.30875^{\{4\}}$	$0.65224^{\{3\}}$	$0.15653^{\{1\}}$	$0.5291^{\{1\}}$
	c	$-0.12006^{\{4\}}$	$0.53335^{\{2\}}$	$-0.2376^{\{6\}}$	$0.70798^{\{6\}}$	$-0.09497^{\{2\}}$	$0.58718^{\{4\}}$	$-0.0418^{\{1\}}$	$0.51509^{\{1\}}$	$-0.16661^{\{5\}}$	$0.66933^{\{5\}}$	$-0.10458^{\{3\}}$	$0.5646^{\{3\}}$
$\sum$ ranks		28		33		26		23		32		26	
50	α	$0.0092^{\{4\}}$	$0.07557^{\{2\}}$	$0.00859^{\{3\}}$	$0.18472^{\{5\}}$	$-0.00674^{\{1\}}$	$0.09396^{\{3\}}$	$-0.02047^{\{6\}}$	$0.07401^{\{1\}}$	$0.0077^{\{2\}}$	$0.20072^{\{6\}}$	$0.01223^{\{5\}}$	$0.13456^{\{4\}}$
	δ	$0.00209^{\{1\}}$	$0.04207^{\{1\}}$	$0.64263^{\{6\}}$	$2.24799^{\{6\}}$	$0.17597^{\{4\}}$	$0.41554^{\{3\}}$	$0.06077^{\{2\}}$	$0.18198^{\{2\}}$	$0.43945^{\{5\}}$	$1.77283^{\{5\}}$	$0.06796^{\{3\}}$	$0.75608^{\{4\}}$
	β	$0.0121^{\{1\}}$	$0.09059^{\{1\}}$	$0.22579^{\{4\}}$	$0.58243^{\{4\}}$	$0.14565^{\{3\}}$	$0.47817^{\{3\}}$	$0.33056^{\{6\}}$	$1.78713^{\{6\}}$	$0.27347^{\{5\}}$	$0.59345^{\{5\}}$	$0.04814^{\{2\}}$	$0.27763^{\{2\}}$
	c	$0.02822^{\{2\}}$	$0.20522^{\{1\}}$	$-0.17804^{\{5\}}$	$0.69154^{\{6\}}$	$-0.03854^{\{4\}}$	$0.41628^{\{4\}}$	$-0.00473^{\{1\}}$	$0.26611^{\{2\}}$	$-0.17868^{\{6\}}$	$0.65642^{\{5\}}$	$-0.0289^{\{3\}}$	$0.29759^{\{3\}}$
$\sum$ ranks		13		39		25		26		39		26	
100	α	$0.00738^{\{5\}}$	$0.0689^{\{4\}}$	$0.00407^{\{2\}}$	$0.10266^{\{6\}}$	$-0.00605^{\{4\}}$	$0.06172^{\{3\}}$	$-0.00789^{\{6\}}$	$0.03723^{\{2\}}$	$-0.00475^{\{3\}}$	$0.08847^{\{5\}}$	$0.00304^{\{1\}}$	$0.0348^{\{1\}}$
	δ	$0.00256^{\{1\}}$	$0.03698^{\{1\}}$	$0.1999^{\{6\}}$	$0.67853^{\{6\}}$	$0.06683^{\{4\}}$	$0.17676^{\{4\}}$	$0.01281^{\{3\}}$	$0.06647^{\{2\}}$	$0.12304^{\{5\}}$	$0.3317^{\{5\}}$	$0.00873^{\{2\}}$	$0.11153^{\{3\}}$
	β	$0.00681^{\{1\}}$	$0.06091^{\{1\}}$	$0.15774^{\{5\}}$	$0.48402^{\{6\}}$	$0.03406^{\{3\}}$	$0.22746^{\{4\}}$	$0.06298^{\{4\}}$	$0.1674^{\{3\}}$	$0.20272^{\{6\}}$	$0.45905^{\{5\}}$	$0.01264^{\{2\}}$	$0.13549^{\{2\}}$
	c	$0.0237^{\{4\}}$	$0.16454^{\{4\}}$	$-0.1236^{\{6\}}$	$0.54296^{\{6\}}$	$0.0114^{\{2\}}$	$0.10008^{\{2\}}$	$0.00728^{\{1\}}$	$0.05232^{\{1\}}$	$-0.11221^{\{5\}}$	$0.51344^{\{5\}}$	$-0.01275^{\{3\}}$	$0.14273^{\{3\}}$
$\sum$ ranks		21		43		26		22		39		17	
200	α	$0.00658^{\{4\}}$	$0.06808^{\{4\}}$	$0.00809^{\{5\}}$	$0.0757^{\{6\}}$	$0.00287^{\{3\}}$	$0.0501^{\{3\}}$	$-0.00049^{\{1\}}$	$0.01249^{\{1\}}$	$0.00972^{\{6\}}$	$0.06988^{\{5\}}$	$0.00129^{\{2\}}$	$0.0155^{\{2\}}$
	δ	$0.00173^{\{3\}}$	$0.03016^{\{3\}}$	$0.0566^{\{5\}}$	$0.15499^{\{5\}}$	$0.02361^{\{4\}}$	$0.07551^{\{4\}}$	$0.00053^{\{1\}}$	$0.02095^{\{2\}}$	$0.06005^{\{6\}}$	$0.17764^{\{6\}}$	$0.0006^{\{2\}}$	$0.01754^{\{1\}}$
	β	$0.00319^{\{2\}}$	$0.04296^{\{1\}}$	$0.08661^{\{6\}}$	$0.3662^{\{6\}}$	$0.01667^{\{4\}}$	$0.1831^{\{4\}}$	$0.00764^{\{3\}}$	$0.05022^{\{3\}}$	$0.08323^{\{5\}}$	$0.35285^{\{5\}}$	$0.00049^{\{1\}}$	$0.04952^{\{2\}}$
	c	$0.01405^{\{4\}}$	$0.12848^{\{4\}}$	$-0.08907^{\{5\}}$	$0.44158^{\{6\}}$	$-0.0081^{\{3\}}$	$0.05265^{\{3\}}$	$0.00053^{\{1\}}$	$0.02021^{\{1\}}$	$-0.09515^{\{6\}}$	$0.43882^{\{5\}}$	$-0.00361^{\{2\}}$	$0.05192^{\{2\}}$
$\sum$ ranks		25		44		28		13		44		14	
400	α	$0.00015^{\{3\}}$	$0.00596^{\{3\}}$	$0.00741^{\{6\}}$	$0.05443^{\{6\}}$	$0.00312^{\{4\}}$	$0.03474^{\{4\}}$	$0.00004^{\{1\}}$	$0.00471^{\{2\}}$	$0.00578^{\{5\}}$	$0.04881^{\{5\}}$	$0.00015^{\{2\}}$	$0.0037^{\{1\}}$
	δ	$-0.00032^{\{3\}}$	$0.00484^{\{2\}}$	$0.03312^{\{6\}}$	$0.09707^{\{5\}}$	$0.00937^{\{4\}}$	$0.04068^{\{4\}}$	$-0.00021^{\{2\}}$	$0.00578^{\{3\}}$	$0.0315^{\{5\}}$	$0.12053^{\{6\}}$	$-0.00002^{\{1\}}$	$0.0041^{\{1\}}$
	β	$0.00038^{\{2\}}$	$0.01246^{\{2\}}$	$0.04573^{\{6\}}$	$0.2988^{\{6\}}$	$-0.00674^{\{4\}}$	$0.11495^{\{4\}}$	$0.00081^{\{3\}}$	$0.01518^{\{3\}}$	$0.03957^{\{5\}}$	$0.2689^{\{5\}}$	$-0.00022^{\{1\}}$	$0.00967^{\{1\}}$
	c	$0.00046^{\{3\}}$	$0.03217^{\{3\}}$	$-0.0894^{\{6\}}$	$0.40286^{\{6\}}$	$0.00215^{\{4\}}$	$0.04724^{\{4\}}$	$0.00034^{\{1\}}$	$0.00717^{\{1\}}$	$-0.04209^{\{5\}}$	$0.31839^{\{5\}}$	$-0.00044^{\{2\}}$	$0.00899^{\{2\}}$
$\sum$ ranks		21		47		32		16		41		11	
800	α	$0.00013^{\{3\}}$	$0.00333^{\{3\}}$	$0.00425^{\{6\}}$	$0.04038^{\{5\}}$	$0.00154^{\{5\}}$	$0.02592^{\{4\}}$	$-0.00002^{\{1\}}$	$0.0005^{\{1\}}$	$-0.00015^{\{4\}}$	$0.04561^{\{6\}}$	$0.00002^{\{2\}}$	$0.00168^{\{2\}}$
	δ	$-0.00002^{\{2\}}$	$0.00067^{\{2\}}$	$0.02308^{\{6\}}$	$0.07147^{\{6\}}$	$0.00438^{\{4\}}$	$0.03284^{\{4\}}$	$-0.00001^{\{1\}}$	$0.00051^{\{1\}}$	$0.01974^{\{5\}}$	$0.06333^{\{5\}}$	$-0.00007^{\{3\}}$	$0.00154^{\{3\}}$
	β	$-0.00003^{\{1\}}$	$0.00241^{\{1\}}$	$0.0633^{\{6\}}$	$0.28337^{\{6\}}$	$0.00305^{\{4\}}$	$0.08212^{\{4\}}$	$0.0002^{\{3\}}$	$0.00498^{\{2\}}$	$0.04882^{\{5\}}$	$0.26376^{\{5\}}$	$0.00013^{\{2\}}$	$0.00562^{\{3\}}$
	c	$0.00099^{\{4\}}$	$0.03145^{\{4\}}$	$-0.0653^{\{6\}}$	$0.39505^{\{6\}}$	$0.00097^{\{3\}}$	$0.02705^{\{3\}}$	$-0.00001^{\{1\}}$	$0.00016^{\{1\}}$	$-0.02284^{\{5\}}$	$0.25148^{\{5\}}$	$-0.0002^{\{2\}}$	$0.00611^{\{2\}}$
$\sum$ ranks		20		47		31		11		40		19	

Figure 6: Plots of RMSEs of parameters in Table 1

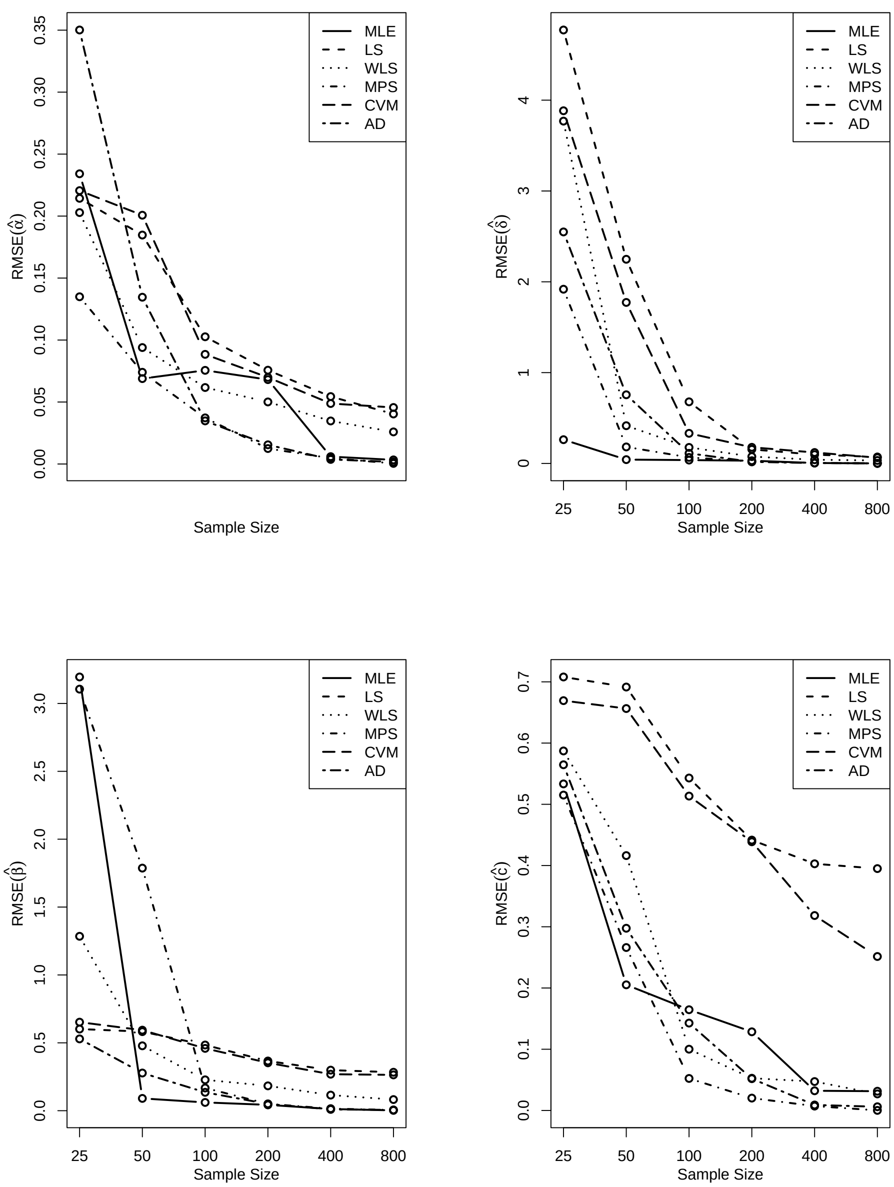
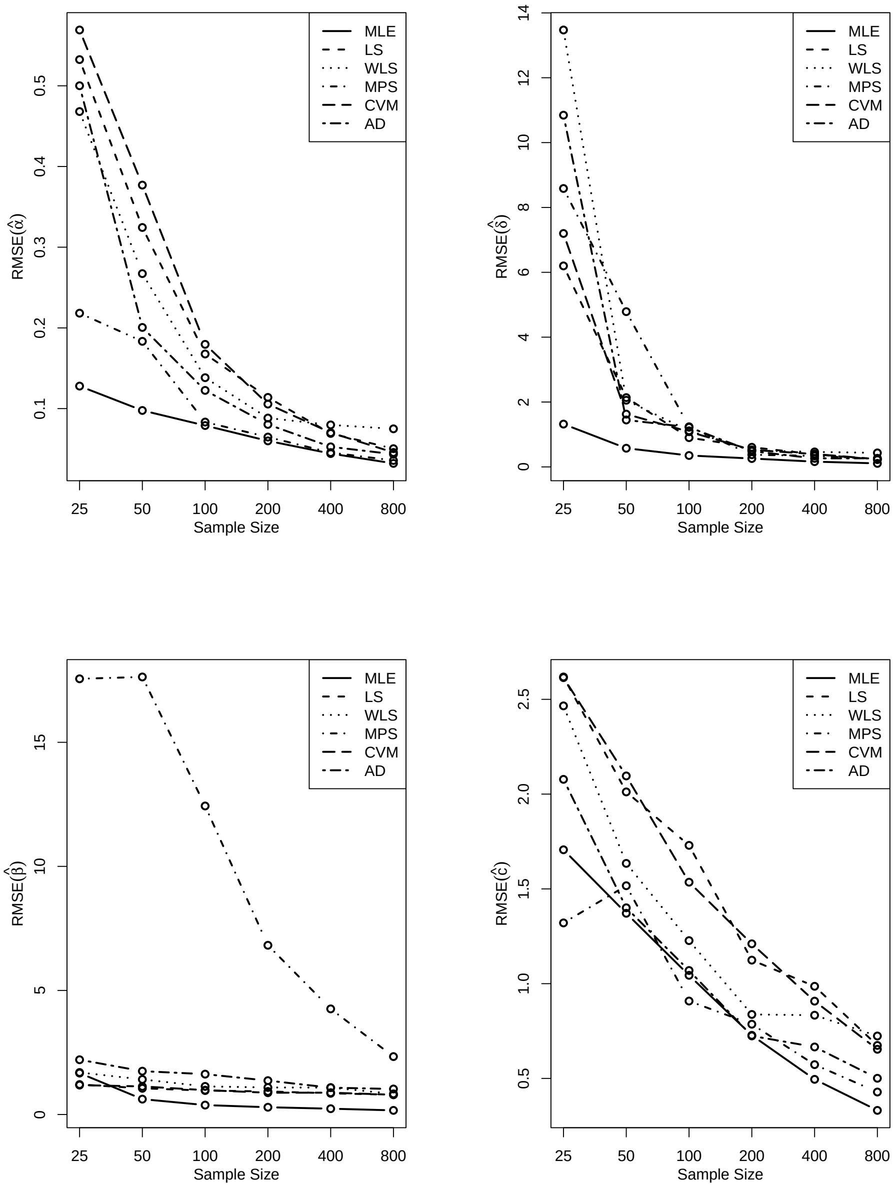


Table 2: Simulation results for different estimation methods for $\alpha = 0.4, \delta = 0.9, \beta = 1.7, c = 1.2$

		MLE		LS		WLS		MPS		CVM		AD	
n	Parameter	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE	ABIAS	RMSE
25	α	$0.0091^{\{1\}}$	$0.1279^{\{1\}}$	$0.1252^{\{5\}}$	$0.53248^{\{5\}}$	$0.08725^{\{3\}}$	$0.46811^{\{3\}}$	$0.02764^{\{2\}}$	$0.21823^{\{2\}}$	$0.14546^{\{6\}}$	$0.56916^{\{6\}}$	$0.11414^{\{4\}}$	$0.50009^{\{4\}}$
	δ	$0.3985^{\{1\}}$	$1.3204^{\{1\}}$	$1.32497^{\{4\}}$	$6.19846^{\{2\}}$	$2.40714^{\{6\}}$	$13.47274^{\{6\}}$	$1.06703^{\{2\}}$	$8.5852^{\{4\}}$	$1.20934^{\{3\}}$	$7.1983^{\{3\}}$	$1.69562^{\{5\}}$	$10.84579^{\{5\}}$
	β	$0.5394^{\{5\}}$	$1.6694^{\{3\}}$	$-0.35223^{\{4\}}$	$1.20907^{\{2\}}$	$-0.11833^{\{1\}}$	$1.69416^{\{4\}}$	$12.26125^{\{6\}}$	$17.56081^{\{6\}}$	$-0.29313^{\{3\}}$	$1.1854^{\{1\}}$	$0.29138^{\{2\}}$	$2.20716^{\{5\}}$
	c	$-0.5325^{\{2\}}$	$1.7064^{\{2\}}$	$1.48018^{\{6\}}$	$2.61809^{\{6\}}$	$1.31031^{\{4\}}$	$2.46563^{\{4\}}$	$-0.33966^{\{1\}}$	$1.3206^{\{1\}}$	$1.45687^{\{5\}}$	$2.61456^{\{5\}}$	$0.96726^{\{3\}}$	$2.07792^{\{3\}}$
	$\sum$ ranks	16		34		31		24		32		31	
50	α	$0.0081^{\{1\}}$	$0.0977^{\{1\}}$	$0.07413^{\{5\}}$	$0.32443^{\{5\}}$	$0.04786^{\{4\}}$	$0.26728^{\{4\}}$	$0.02421^{\{3\}}$	$0.18335^{\{2\}}$	$0.08703^{\{6\}}$	$0.37696^{\{6\}}$	$0.01463^{\{2\}}$	$0.20057^{\{3\}}$
	δ	$0.1256^{\{1\}}$	$0.5755^{\{1\}}$	$0.4877^{\{6\}}$	$2.13699^{\{5\}}$	$0.43288^{\{5\}}$	$2.0558^{\{4\}}$	$0.2906^{\{2\}}$	$4.78771^{\{6\}}$	$0.37273^{\{3\}}$	$1.6273^{\{3\}}$	$0.4185^{\{4\}}$	$1.44712^{\{2\}}$
	β	$0.1153^{\{1\}}$	$0.6192^{\{1\}}$	$-0.2998^{\{5\}}$	$1.05574^{\{2\}}$	$0.14348^{\{2\}}$	$1.4139^{\{4\}}$	$12.10244^{\{6\}}$	$17.63389^{\{6\}}$	$-0.28771^{\{4\}}$	$1.13094^{\{3\}}$	$0.22731^{\{3\}}$	$1.74684^{\{5\}}$
	c	$-0.3945^{\{2\}}$	$1.3713^{\{1\}}$	$1.10773^{\{5\}}$	$2.01176^{\{5\}}$	$0.89044^{\{4\}}$	$1.63462^{\{4\}}$	$-0.30071^{\{1\}}$	$1.51669^{\{3\}}$	$1.21169^{\{6\}}$	$2.09597^{\{6\}}$	$0.66156^{\{3\}}$	$1.40031^{\{2\}}$
	$\sum$ ranks	9		38		31		29		37		24	
100	α	$0.0087^{\{2\}}$	$0.079^{\{1\}}$	$0.02451^{\{5\}}$	$0.16768^{\{5\}}$	$0.01232^{\{4\}}$	$0.13832^{\{4\}}$	$0.01187^{\{3\}}$	$0.08332^{\{2\}}$	$0.02847^{\{6\}}$	$0.17963^{\{6\}}$	$0.00867^{\{1\}}$	$0.12258^{\{3\}}$
	δ	$0.0514^{\{2\}}$	$0.3521^{\{1\}}$	$0.14784^{\{3\}}$	$0.90186^{\{2\}}$	$0.23349^{\{6\}}$	$1.11314^{\{4\}}$	$0.02327^{\{1\}}$	$1.23824^{\{6\}}$	$0.17418^{\{4\}}$	$1.08494^{\{3\}}$	$0.18705^{\{5\}}$	$1.22208^{\{5\}}$
	β	$0.0496^{\{1\}}$	$0.3798^{\{1\}}$	$-0.25098^{\{4\}}$	$0.96493^{\{2\}}$	$-0.11488^{\{2\}}$	$1.13304^{\{4\}}$	$7.62507^{\{6\}}$	$12.43487^{\{6\}}$	$-0.26383^{\{5\}}$	$0.98191^{\{3\}}$	$0.23372^{\{3\}}$	$1.6267^{\{5\}}$
	c	$-0.2579^{\{2\}}$	$1.044^{\{2\}}$	$1.03704^{\{6\}}$	$1.72969^{\{6\}}$	$0.62281^{\{4\}}$	$1.22715^{\{4\}}$	$-0.21872^{\{1\}}$	$0.9083^{\{1\}}$	$0.89476^{\{5\}}$	$1.53533^{\{5\}}$	$0.45464^{\{3\}}$	$1.06949^{\{3\}}$
	$\sum$ ranks	12		33		32		26		37		28	
200	α	$0.0044^{\{1\}}$	$0.06^{\{1\}}$	$0.00765^{\{3\}}$	$0.11388^{\{6\}}$	$0.01498^{\{5\}}$	$0.08831^{\{4\}}$	$0.01661^{\{6\}}$	$0.06475^{\{2\}}$	$0.01474^{\{4\}}$	$0.10562^{\{5\}}$	$0.0066^{\{2\}}$	$0.0804^{\{3\}}$
	δ	$0.0301^{\{3\}}$	$0.2582^{\{1\}}$	$0.09723^{\{6\}}$	$0.6056^{\{6\}}$	$0.0143^{\{1\}}$	$0.50888^{\{4\}}$	$-0.07179^{\{5\}}$	$0.37582^{\{2\}}$	$0.02958^{\{2\}}$	$0.52488^{\{5\}}$	$0.04049^{\{4\}}$	$0.48439^{\{3\}}$
	β	$0.0142^{\{1\}}$	$0.2929^{\{1\}}$	$-0.3447^{\{5\}}$	$0.9293^{\{3\}}$	$-0.09215^{\{2\}}$	$1.08917^{\{4\}}$	$3.25201^{\{6\}}$	$6.8208^{\{6\}}$	$-0.23401^{\{4\}}$	$0.8784^{\{2\}}$	$0.20684^{\{3\}}$	$1.3659^{\{5\}}$
	c	$-0.1502^{\{1\}}$	$0.7275^{\{2\}}$	$0.70226^{\{5\}}$	$1.12413^{\{5\}}$	$0.39802^{\{4\}}$	$0.83764^{\{4\}}$	$-0.16198^{\{2\}}$	$0.786^{\{3\}}$	$0.70993^{\{6\}}$	$1.21035^{\{6\}}$	$0.25372^{\{3\}}$	$0.72436^{\{1\}}$
	$\sum$ ranks	11		39		28		32		34		24	
400	α	$0.0036^{\{3\}}$	$0.0443^{\{1\}}$	$0.00155^{\{1\}}$	$0.06921^{\{4\}}$	$0.00528^{\{5\}}$	$0.07981^{\{6\}}$	$0.01078^{\{6\}}$	$0.04554^{\{2\}}$	$0.00479^{\{4\}}$	$0.07021^{\{5\}}$	$0.00316^{\{2\}}$	$0.05264^{\{3\}}$
	δ	$0.0114^{\{2\}}$	$0.1622^{\{1\}}$	$0.03262^{\{5\}}$	$0.38967^{\{4\}}$	$0.06262^{\{6\}}$	$0.46235^{\{6\}}$	$-0.01436^{\{4\}}$	$0.33842^{\{3\}}$	$0.01306^{\{3\}}$	$0.3966^{\{5\}}$	$-0.0003^{\{1\}}$	$0.26299^{\{2\}}$
	β	$0.0077^{\{1\}}$	$0.2343^{\{1\}}$	$-0.1267^{\{4\}}$	$0.85436^{\{2\}}$	$-0.09278^{\{2\}}$	$1.08504^{\{5\}}$	$1.8709^{\{6\}}$	$4.25741^{\{6\}}$	$-0.11929^{\{3\}}$	$0.87948^{\{3\}}$	$0.23015^{\{5\}}$	$1.08144^{\{4\}}$
	c	$-0.0575^{\{1\}}$	$0.4952^{\{1\}}$	$0.51062^{\{6\}}$	$0.98615^{\{6\}}$	$0.38295^{\{4\}}$	$0.83351^{\{4\}}$	$-0.14818^{\{2\}}$	$0.57282^{\{2\}}$	$0.48752^{\{5\}}$	$0.90778^{\{5\}}$	$0.21635^{\{3\}}$	$0.66607^{\{3\}}$
	$\sum$ ranks	11		32		38		31		37		23	
800	α	$0.0023^{\{1\}}$	$0.0321^{\{1\}}$	$0.00457^{\{2\}}$	$0.05037^{\{5\}}$	$0.00582^{\{4\}}$	$0.07496^{\{6\}}$	$0.01297^{\{6\}}$	$0.03574^{\{2\}}$	$-0.00528^{\{3\}}$	$0.04565^{\{4\}}$	$0.00747^{\{5\}}$	$0.04347^{\{3\}}$
	δ	$0.0077^{\{2\}}$	$0.1081^{\{1\}}$	$-0.01043^{\{3\}}$	$0.22453^{\{2\}}$	$0.03896^{\{6\}}$	$0.43215^{\{6\}}$	$-0.02094^{\{4\}}$	$0.25013^{\{4\}}$	$0.03574^{\{5\}}$	$0.23801^{\{3\}}$	$0.00319^{\{1\}}$	$0.26601^{\{5\}}$
	β	$0.0052^{\{2\}}$	$0.1667^{\{1\}}$	$-0.06685^{\{4\}}$	$0.79763^{\{3\}}$	$0.02435^{\{3\}}$	$0.83297^{\{4\}}$	$0.8957^{\{6\}}$	$2.33458^{\{6\}}$	$0.00196^{\{1\}}$	$0.79614^{\{2\}}$	$0.15576^{\{5\}}$	$1.02836^{\{5\}}$
	c	$-0.0241^{\{1\}}$	$0.3315^{\{1\}}$	$0.32085^{\{6\}}$	$0.67413^{\{5\}}$	$0.26372^{\{4\}}$	$0.72397^{\{6\}}$	$-0.13822^{\{3\}}$	$0.42828^{\{2\}}$	$0.26782^{\{5\}}$	$0.65374^{\{4\}}$	$0.08509^{\{2\}}$	$0.5009^{\{3\}}$
	$\sum$ ranks	10		30		39		33		27		29	

Figure 7: Plots of RMSEs of parameters in Table 2



(WL) distribution by Tahir et al. (2015) and the exponentiated half logistic odd Lindley-Weibull (EHLOL-W) distribution by Sengweni et al. (2021). The pdf's of these distributions are given in the appendix.

Table 3: Partial and overall ranks of all estimation methods of MO-WEHL-LLoG distribution by various model parameter values

Parameters	n	MLE	LS	WLS	MPS	CVM	AD
	25	4	6	2.5	1	5	2.5
	50	1	5.5	2	3.5	5.5	3.5
$\alpha = 0.2, \delta = 0.2, \beta = 0.9, c = 2.3$	100	2	6	4	3	5	1
	200	3	5.5	4	1	5.5	2
	400	3	6	4	2	5	1
	800	3	6	4	1	5	2
	25	1	6	3.5	2	5	3.5
	50	1	6	4	3	5	2
$\alpha = 0.4, \delta = 0.9, \beta = 1.7, c = 1.2$	100	1	5	4	2	6	3
	200	1	6	3	4	5	2
	400	1	4	6	3	5	2
	800	1	4	6	5	2	3
$\sum$ ranks		22	66	47	30.5	59	27.5
Overall rank		1	6	4	3	5	2

The goodness-of-fit is assessed using the following statistics: -2log-likelihood ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Consistent Akaike Information Criterion ($CAIC = AIC + 2\frac{p(p+1)}{n-p-1}$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$), (n is the number of observations, and p is the number of estimated parameters), Cramér-von Mises statistic (W^*), Anderson-Darling statistic (A^*) (Cheng and Balakrishnann 1995), Kolmogorov-Smirnov (K-S) statistic, and its $p - value$. The values of these statistics are given in Tables 4, 5 and 6. The model with the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic is regarded as the best model. The p-value from the Kolmogorov-Smirnov (K-S) test, is used to assess how well the proposed distribution fits the data. Specifically, the p-value represents the probability of obtaining a test statistic that is as extreme or more extreme than the observed value, assuming the null hypothesis is true (Wasserstein and Lazar 2016). In the K-S test, the null hypothesis states that the data follows the proposed distribution.

7.1 Kevlar 49/Epoxy Strands Failure at 90% Stress Level

The first data set relates to the stress-rupture life of Kevlar 49/epoxy strands that were continuously compressed at a 90% stress level until they all failed (Andrews and Herzberg 2012, Barlow et al. 1984). (See the data in the appendix.) Table 4 gives the maximum likelihood estimates (MLEs) of the fitted distributions together with the standard errors (in parenthesis) and all the values of goodness-of-fit statistics.The MO-WEHL-LLoG distribution suited the epoxy strands failure data set better than the competing models taken into consideration, according to the results above (see Table 4). The goodness-of-fit statistics: $AIC, AICC, BIC$, W^*, A^* and KS have small values under the MO-WEHL-LLoG distribution. Furthermore, the p-value for the MO-WEHL-LLoG distribution is the largest compared to other fitted models. Figure 8 of fitted density plots and probability plots show that the MO-WEHL-LLoG model performed better than other models in terms of fitting the data.
The fitted and empirical cumulative distribution (ECDF), observed and fitted Kaplan-Meier (K-M) survival curves, hazard rate function (hrf) plot and total test on time (TTT) scaled plot are displayed in Figure 9. We can observe that the empirical cdf and Kaplan-Meier survival curves are closely followed by the MO-WEHL-LLoG distribution. The hrf is decreasing-increasing-decreasing in the TTT scaled plot. The estimated hrf of the MO-WEHL-LLoG distribution is non-monotonic.

7.2 Annual Maximum Antecedent Rainfall Measurements

The second data set (Chen et al. 2010) corresponds to 52 ordered annual maximum antecedent rainfall measurements in mm from Maple Ridge in British Columbia, Canada. (See the data in the appendix.)
From the results in Table 5 it is quite clear that the MO-WEHL-LLoG distribution provides the best fit among the competitors since it has the lowest value of $-2\ln(L)$, AIC, $CAIC$, BIC, W^*, A^* and K-S statistic. Furthermore, the p-value of the $K-S$ statistic is largest for the MO-WEHL-LLoG distribution, suggesting that the proposed MO-WEHL-LLoG distribution provides the best fit for the annual maximum antecedent rainfall measurement data.
From Figure 11 above, we see that the fitted cdf is closely following the empirical cdf and also the fitted survival function is estimating the survival probabilities quite well since it is very close to the empirical survival plot. It can be inferred that the MO-WEHL-LLoG distribution is suitable for modeling the annual maximum antecedent rainfall measurement data as both the TTT scaled and hrf plots estimate the hrf of the data to be increasing.

Table 4: Estimates of models and goodness-of-fit statistics for Kevlar 49/Epoxy strands failure data

	Estimates				Statistics							
Model	α	δ	β	c	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
MO-WEHL-LLoG	**0.1738**	**2.2372**	**0.7514**	**4.4466**	**200.8900**	**208.8900**	**209.3000**	**219.3500**	**0.0820**	**0.5605**	**0.0722**	**0.7722**
	(0.0681)	**(0.8131)**	**(0.1901)**	**(2.0699)**								
	c	α	β	δ								
MOLLW	0.6465	1.4111	0.7663	4.4286	203.7490	211.749	212.1656	222.2095	0.1215	0.7486	0.0709	0.6901
	(0.2461)	(0.8143)	(0.2540)	(4.1388)								
	α	λ	β	θ								
MOEGo	1.4062	1.3638	3.9655×10^{-09}	0.7984	207.3198	215.3197	215.7364	225.7802	0.2237	1.2372	0.0840	0.4731
	$(4.6075\times10^{+01})$	$(1.2609\times10^{+02})$	(1.9915×10^{-01})	$(1.5756\times10^{+01})$								
	δ	a	α	β								
MOGLL	1.4398	$1.6399\times10^{+05}$	$4.6854\times10^{+05}$	0.9042	205.2296	215.3197	215.7364	225.7802	0.2237	1.2372	0.0840	0.4731
	(6.0038×10^{-01})	(2.1134×10^{-06})	(6.6083×10^{-07})	(1.7060×10^{-02})								
	α	δ	λ	γ								
MOMW	0.9993	1.0624	0.0113	0.9059	205.8765	213.8765	214.2931	224.337	0.1859	1.0527	0.0779	0.5716
	(0.5486)	(0.8951)	(0.0665)	(0.1495)								
	α	β	θ									
WEHL-LLoG	0.3692	0.9800	2.3596	-	205.5700	211.5700	211.8200	219.4200	0.2042	1.151	0.10479	0.2174
	(0.0707)	(0.4839)	(1.5430)	-								
	α	β	γ	θ								
OLLEW	2.8888	1.3029	0.3593	1.5253	205.64	213.6400	214.0600	224.1000	0.1599	0.9381	0.0755	0.6118
	(4.5164)	(0.7359)	(0.5164)	(1.0846)								
	a	b	α	β								
WL	0.2506	0.7860	1.3581	0.3303	205.1900	213.1900	213.6100	223.6600	0.1440	0.8627	0.0787	0.5587
	(0.4173)	(0.1804)	(0.4579)	(0.6282)								
	λ	α	a	b								
EHLOL-W	7.3266×10^{-04}	1.9999×10^{-01}	2.0258×10^{-01}	9.0434×10^{-05}	246.9521	254.9525	255.3691	265.4129	0.3075	2.2674	0.1594	0.0117
	(1.1321×10^{-03})	(3.8227×10^{-02})	(3.8906×10^{-02})	(2.4811×10^{-04})								

Figure 8: Fitted densities and probability plots for Kevlar 49/Epoxy strands failure data

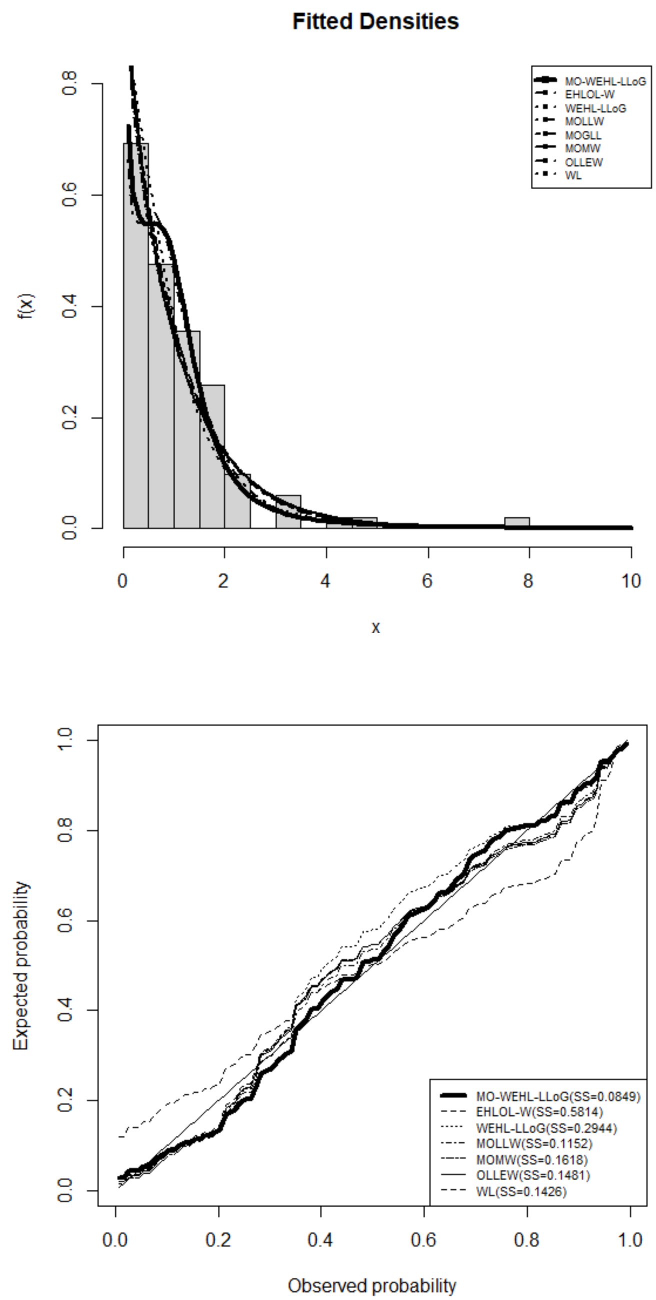
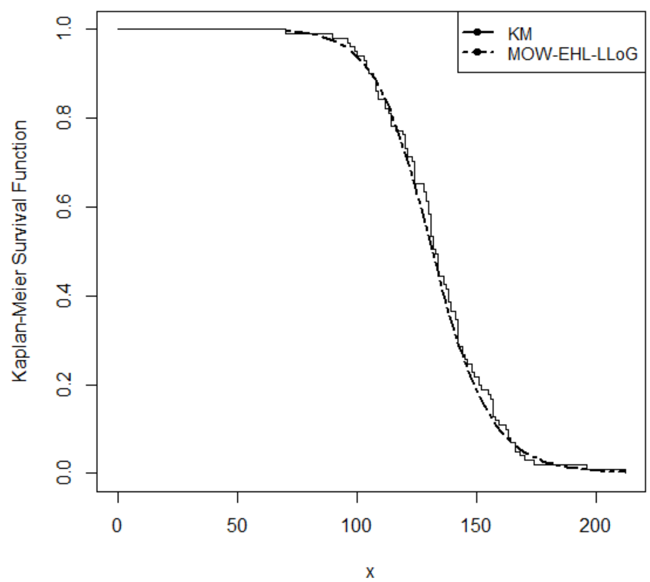


Figure 9: Estimated cdf, Kaplan-Meier survival, fitted hazard rate function and Scaled TTT-Transform plots for the MO-WEHL-LLoG distribution for epoxy strands failure data



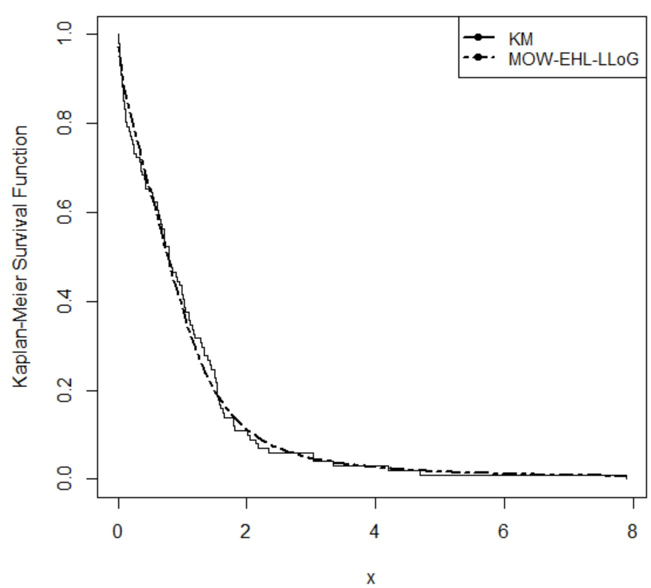
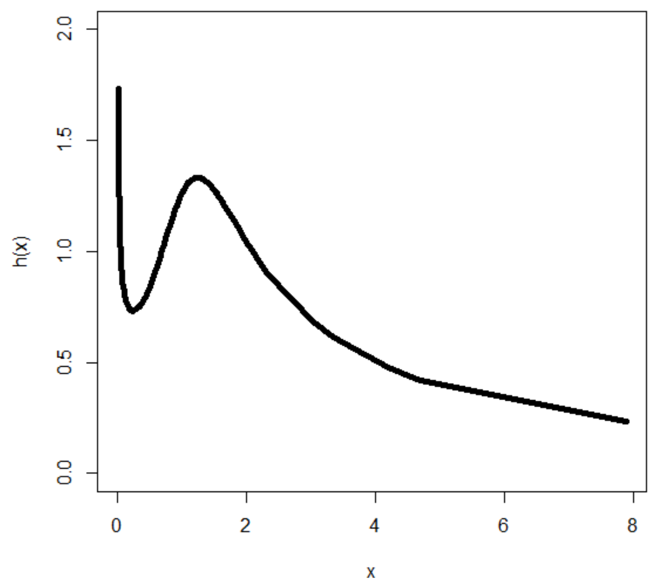


Figure 9: Estimated cdf, Kaplan-Meier survival, fitted hazard rate function and Scaled TTT-Transform plots for the MO-WEHL-LLoG distribution for epoxy strands failure data, cont.



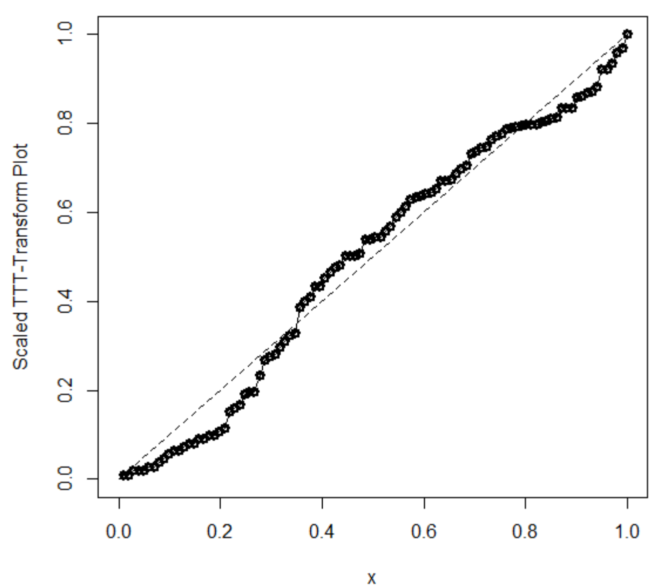
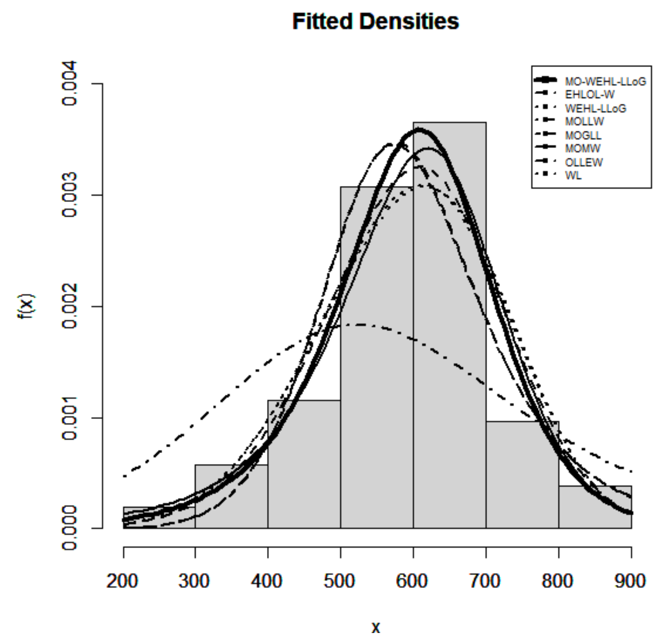


Table 5: Estimates of models and goodness-of-fit statistics for annual maximum antecedent rainfall measurement data

Model	Estimates				Statistics							
	α	δ	β	c	$-2\log,(L)$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
MO-WEHL-LLoG	**62.44429** **(9.1373×10^{-04})**	**22.92158** **(4.1391×10^{-03})**	**4.17916** **(3.4412×10^{-01})**	**0.79537** **(1.2951×10^{-02})**	**649.2837**	**657.2837**	**658.1348**	**665.0887**	**0.0240**	**0.1653**	**0.0597**	**0.9925**
MOLLW	c 1.0799 (1.3873×10^{-01})	α 2.7848×10^{-02} (2.0069×10^{-02})	β 7.7712×10^{-01} (1.0898×10^{-01})	δ 3.9686×10^{+04} (6.9852×10^{-07})	679.6600	687.6600	688.5100	695.4600	0.0716	0.4483	0.2380	0.0055
MOEGo	α 1.0886×10^{01} (8.3212×10^{-08})	λ 7.6478×10^{-03} (4.5331×10^{-07})	β 3.8557×10^{-03} (5.8060×10^{-04})	θ 1.0479×10^{-03} (3.0175×10^{-04})	655.4234	664.2746	664.2746	671.2285	0.0890	0.5887	0.1043	0.623
MOGLL	δ 1872.6707 (0.6961)	a 0.3412 (0.8462)	α 232.3053 (25.5044)	β 23.6157 (59.1513)	653.8041	661.8041	662.6552	669.6091	0.0904	0.5681	0.0840	0.8561
MOMW	α 1.0889×10^{-04} (2.4322×10^{-05})	δ 6.7698×10^{01} (1.9617×10^{-09})	λ 4.8703×10^{-04} (3.2909×10^{-04})	γ 1.6048 (1.0111×10^{-06})	650.3420	658.3410	659.192	666.1459	0.0335	0.2340	0.0721	0.9497
WEHL-LLoG	α 5.3897 (0.7611)	β 14.4006 (0.7451)	θ 0.4817 (0.0222)	-	650.4400	656.4400	656.9400	662.2900	0.0533	0.3368	0.0858	0.8406
OLLEW	α 1.662 (3.0590)	β 0.1332 (2.025×10^{-02})	γ 5.824 (1.9760)	θ 18.9500 (2.935×10^{-01})	653.6700	661.6700	662.5200	669.4700	0.0837	0.5278	0.0833	0.8637
WL	a 2.5725×10^{-04} (5.5192 ×10^{-04})	b 9.7048 (2.0563×10^{-03})	α 3.9928×10^{-01} (5.0578×10^{-02})	β 3.2925×10^{01} (6.9840×10^{-05})	650.5400	658.5200	659.3700	666.3200	0.0541	0.3430	0.08816	0.8137
EHLOL-W	λ 0.0015 (0.0008)	α 1.6835 (0.0003)	a 0.3230 (0.0112)	b 1.1273 (0.0003)	649.8300	657.8300	658.6800	665.63	0.0406	0.2618	0.0781	0.9089

Figure 10: Fitted densities and PP plots for annual maximum antecedent rainfall measurement data



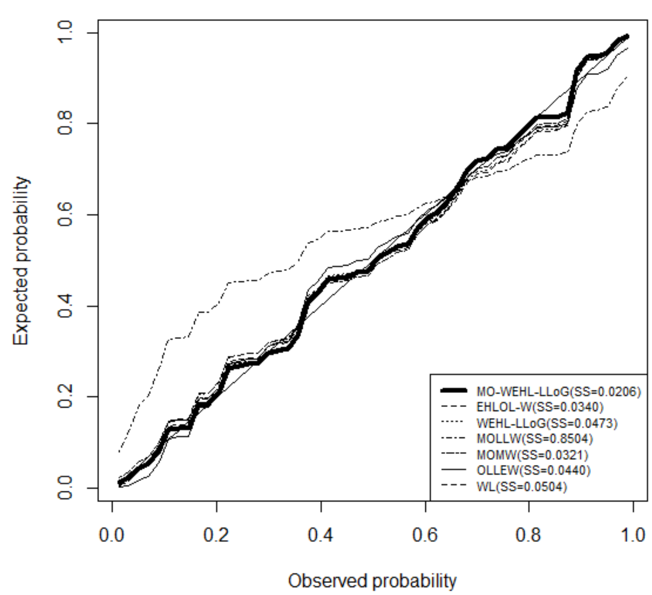
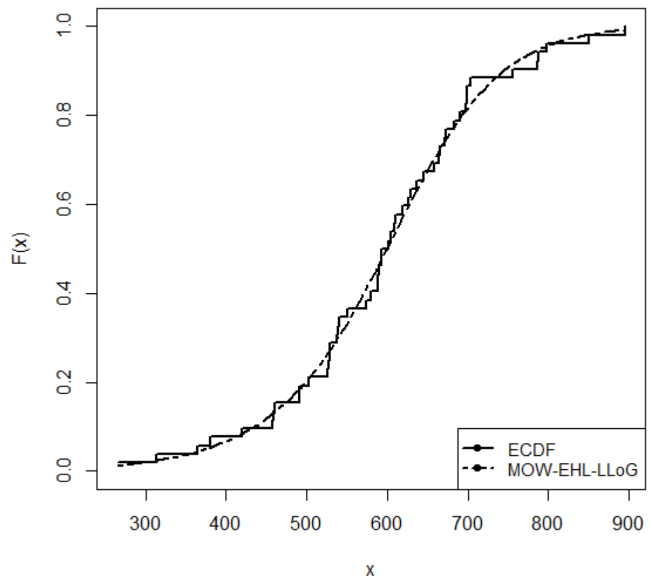


Figure 11: Estimated cdf, Kaplan-Meier survival, fitted hazard rate function and Scaled TTT-Transform plots for the MO-WEHL-LLoG distribution for annual maximum antecedent rainfall measurement data



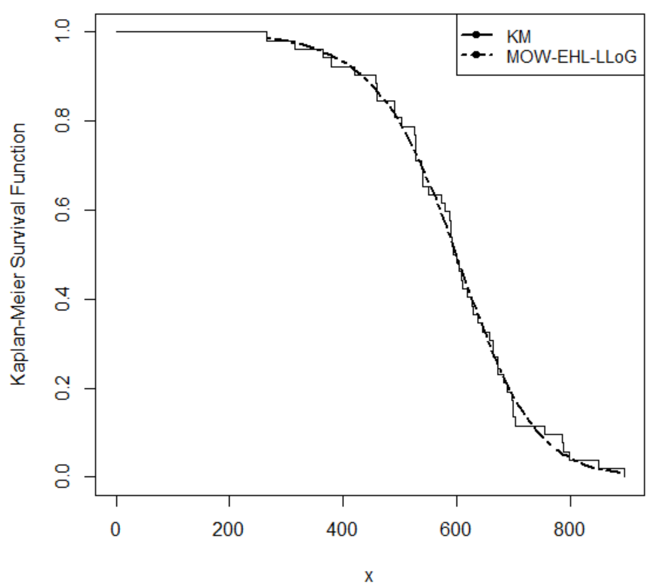
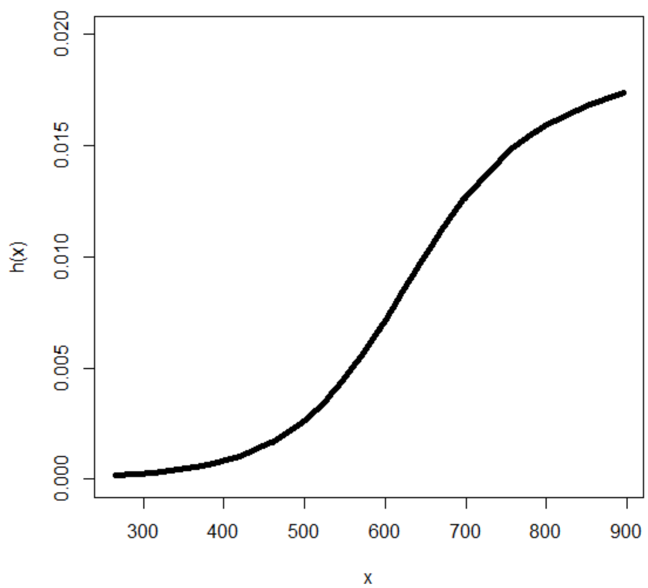
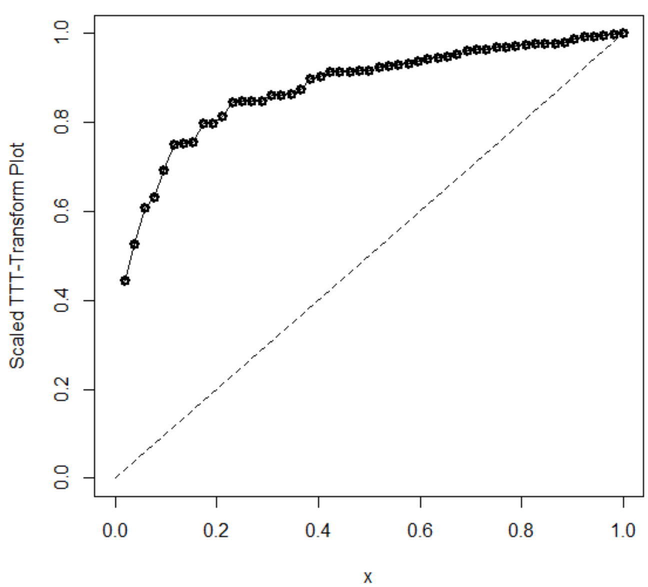


Figure 11: Estimated cdf, Kaplan-Meier survival, fitted hazard rate function and Scaled TTT-Transform plots for the MO-WEHL-LLoG distribution for annual maximum antecedent rainfall measurement data, cont.





7.3 Fatigue time of 101 6061-T6 aluminium coupons

This data set represent the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (Birnbaum et al. 1969). (See the data in the appendix).

From the results in Table 6, we see that the values of the goodness-of-fit statistics: $-2\ln(L)$, $AIC, AICC, BIC$, W^*, A^* and K-S are the smallest for the MO-WEHL-LLoG distribution. This indicates that the MO-WEHL-LLoG distribution is a better fit for the aluminium coupons data as compared to other fitted distributions. The p-value of the K-S statistic is also the largest for MO-WEHL-LLoG distribution.

Figure 13 shows that the fitted cdf is closely following the empirical cdf and also the fitted survival function is estimating the survival probabilities quite well since it is very close to the Kaplan-Meier survival plot. The hrf is estimated to be an increasing shape on both the TTT and hrf plots.

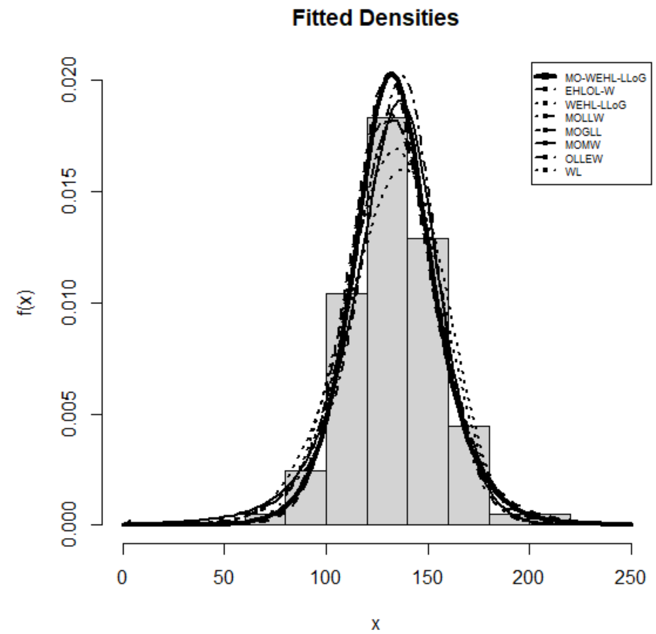
8 Conclusions

We have proposed and studied a new generalized family of distributions called the Marshall-Olkin-Weibull-exponentiated half logistic-G (MO-WEHL-G) distribution. Several mathematical and statistical properties of the new family of distributions were derived. To estimate parameters of the MO-WEHL-LLoG distribution, which is a special case of the MO-WEHL-G family, a variety of estimation techniques are employed. These include maximum likelihood estimation, least-squares estimation, weighted least-squares estimation, maximum product spacing estimation, Cramér-von Mises estimation, and Anderson-Darling estimation. Monte Carlo simulations were used to evaluate the consistency properties of the six estimation methods for the MO-WEHL-LLoG distribution. The results show that MLE method better estimated the MO-WEHL-LLoG parameters as compared to other methods. Finally, to demonstrate the relevance and applicability of the MO-WEHL-G distribution, its special case of MO-WEHL-LLoG distribution was fitted to three data sets.

Table 6: Estimates of models and goodness-of-fit statistics for fatigue time of aluminium coupons data

Model	Estimates				Statistics							
	α	δ	β	c	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
MO-WEHL-LLoG	**164.2700** **(0.0096)**	**2.1400×10^{4}** **(0.1224)**	**1.9500** **(0.1194)**	**1.8500** **(0.5831)**	**910.4800**	**918.4800**	**918.9000**	**928.9500**	**0.03883**	**0.2621**	**0.0526**	**0.9427**
MOLLW	c 1.3541×10^{-01} (9.0504×10^{-11})	α 6.6301×10^{-05} (2.2699×10^{-06})	β 2.2769 (7.4662×10^{-10})	δ $3.1106\times10^{+02}$ (8.8614×10^{-14})	918.1509	926.1501	926.5667	936.6105	0.1014	0.6663	0.0653	0.7810
MOEGo	α 3.0442×10^{-01} (7.2377×10^{-05})	λ 6.9373×10^{-09} (1.0440×10^{-02})	β 4.3407×10^{-02} (3.0054×10^{-03})	θ 3.4750×10^{-05} (1.4124×10^{-05})	942.5371	950.5373	950.9540	960.9978	0.1650	1.1204	0.1311	0.0620
MOGLL	δ 5.6750×10^{03} (2.4288×10^{-09})	a 8.5352×10^{06} (1.4712×10^{-11})	α 2.9508×10^{07} (4.7498×10^{-12})	β 1.1212 (1.5485×10^{-03})	912.8613	920.8613	921.278	931.3218	0.0384	0.2706	0.0657	0.77591
MOMW	α 2.1329×10^{-01} (2.8528×10^{-01})	δ 1.2819×10^{03} (9.0842×10^{-06})	λ 6.5430×10^{-03} (2.6816×10^{-03})	γ 5.3860×10^{-01} (3.4198×10^{-01})	916.8428	924.8427	925.2594	935.3032	0.0844	0.5602	0.0589	0.8743
WEHL-LLoG	α 403.4000 (6.396×10^{-04})	β 5.2260 (0.3699)	θ 1.5090 (5.138×10^{-03})	-	915.0991	921.0991	921.3465	928.9444	0.0675	0.4516	0.07434	0.6321
OLLEW	α 0.4802 (0.5317)	β 0.1582 (0.0189)	γ 7.5475 (1.5178)	θ 19.0869 (1.2688)	911.4400	919.4400	919.8600	929.9000	0.0594	0.3612	0.0613	0.8419
WL	a 0.4454 (2.5089)	b 12.9512 (1.4122)	α 0.2597 (0.0777)	β 9.3330 (0.4927)	922.0500	930.0500	930.4700	940.5100	0.1212	0.7956	0.0944	0.3294
EHLOL-W	λ 0.0032 (0.0096)	α 4.5436 (0.1224)	a 0.2860 (0.1194)	b 0.13405 (0.5831)	911.4700	919.4700	919.8900	929.9400	0.0481	0.3049	0.0623	0.8285

Figure 12: Fitted densities and expected probability plot for aluminium coupons data



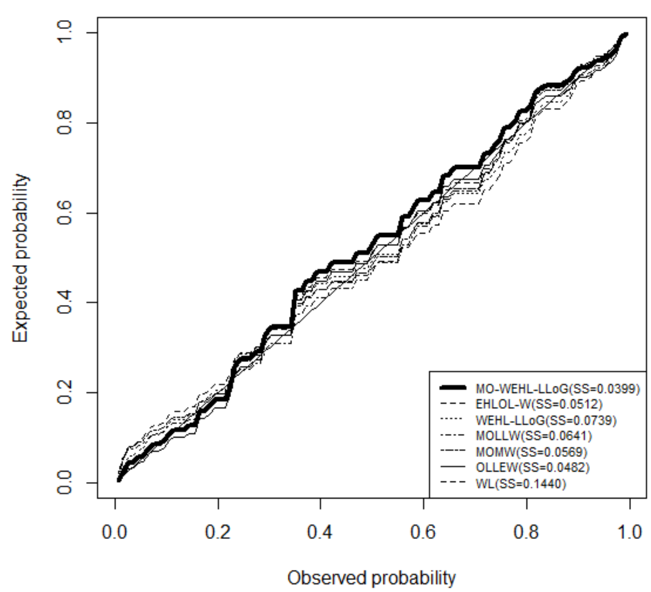
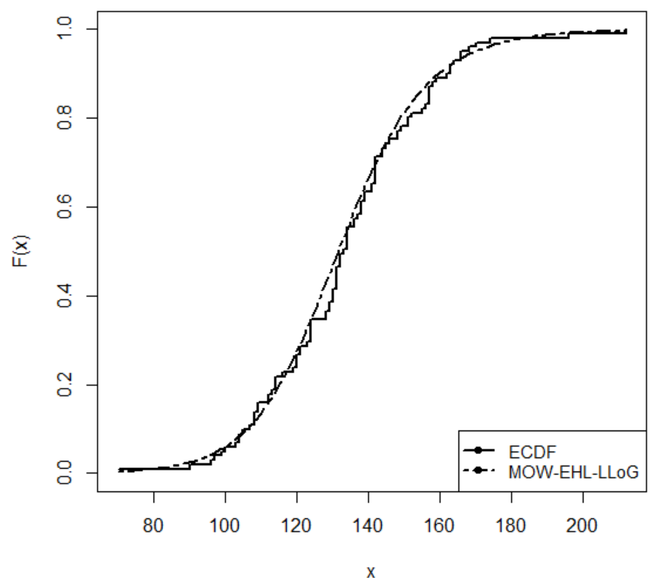


Figure 13: Estimated cdf, Kaplan-Meier survival, fitted hazard rate function and scaled TTT-transform plots for the MO-WEHL-LLoG distribution for aluminium coupons data



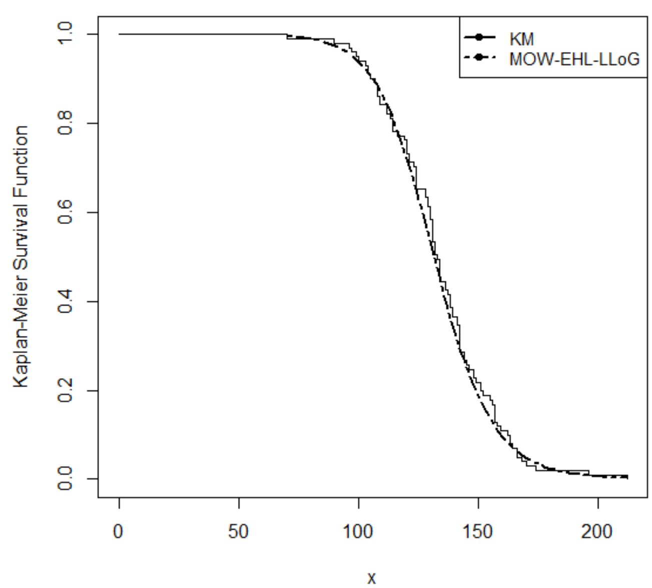
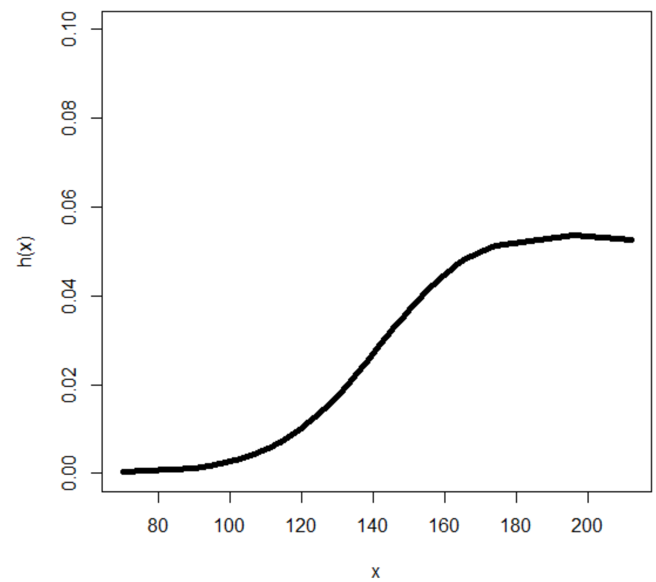
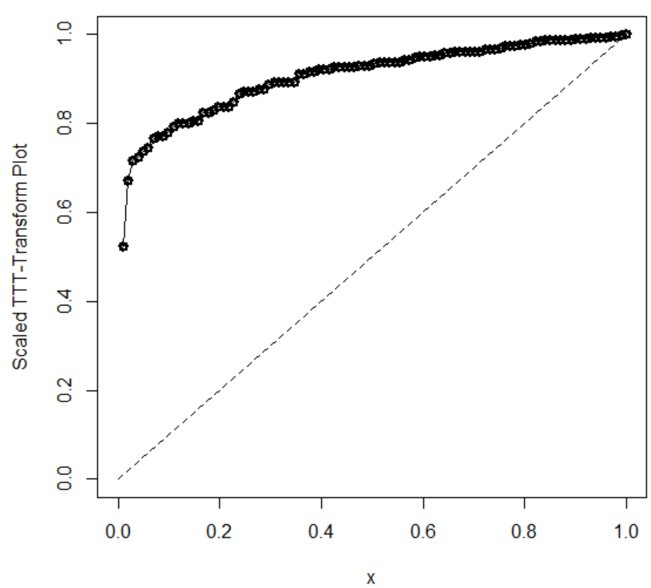


Figure 13: Estimated cdf, Kaplan-Meier survival, fitted hazard rate function and scaled TTT-transform plots for the MO-WEHL-LLoG distribution for aluminium coupons data, cont.





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A Expansion of density function

Note that if we let

$$F_{WEHL-G}(x;\alpha,\beta,\xi) = F(x;\alpha,\beta,\xi)$$

and

$$f_{WEHL-G}(x;\alpha,\beta,\xi) = f(x;\alpha,\beta,\xi),$$

then we have

$$\begin{aligned}
&f(x;\alpha,\beta,\xi)\left(F(x;\alpha,\beta,\xi)\right)^{t-k} = \\
&= 2\alpha\beta\exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\times \\
&\times\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta-1}\times \\
&\times\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1}\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha-1}\times \\
&\times\frac{g(x;\xi)}{(1+\bar{G}(x;\xi))^{2}}\times
\end{aligned}$$

$$\times\left(1-\exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{t-k}=$$
$$=2\alpha\beta\sum_{q=0}^{\infty}\binom{t-k}{q}(-1)^{q}\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1}\times$$
$$\times\exp\left(-(q+1)\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\times$$
$$\times\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta-1}\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha-1}\times$$
$$\times\frac{g(x;\xi)}{(1+\bar{G}(x;\xi))^{2}}.$$

Now, applying the following Taylor series expansion

$$\left(\exp\left(-(q+1)\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)=$$
$$=\sum_{p=0}^{\infty}\frac{(-1)^{p}(q+1)^{p}\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta p}}{p!},$$

and using the results on power series raised to a positive integer, by setting $a_s=\frac{1}{s+2}$, that is $\left(\sum_{s=0}^{\infty}a_s y^s\right)^m=\sum_{s=0}^{\infty}b_{s,m}y^s$, we have

$$\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta(p+1)-1}=$$
$$=y^{\beta(p+1)-1}\left[\sum_{m=0}^{\infty}\binom{\beta(p+1)-1}{m}y^{m}\left(\sum_{s=0}^{\infty}\frac{y^{s}}{s+2}\right)^{m}\right]=$$
$$=y^{\beta(p+1)-1}\left[\sum_{m=0}^{\infty}\binom{\beta(p+1)-1}{m}y^{m}\sum_{s=0}^{\infty}b_{s,m}y^{s}\right]=$$
$$=\left[\sum_{m,s=0}^{\infty}\binom{\beta(p+1)-1}{m}b_{s,m}y^{m+s+\beta(p+1)-1}\right],$$

(Gradshteyn et al. 2014), where $b_{s,m} = (sa_0)^{-1} \sum_{l=1}^{s} [m(l+1)-s] a_l b_{s-l,m}$, $b_{0,m} = a_0^m$, and obtain

$$\begin{aligned} & f(x;\alpha,\beta,\xi)\left(F(x;\alpha,\beta,\xi)\right)^{t-k} = \\ & = 2\alpha\beta \sum_{q,p=0}^{\infty} \sum_{m,s=0}^{\infty} \binom{\beta(p+1)-1}{m} b_{s,m} \binom{t-k}{q} (-1)^{q+p} \frac{(q+1)^p}{p!} \times \\ & \times \left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1} \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha[m+s+\beta(p+1)]-1} \times \\ & \times \frac{g(x;\xi)}{(1+\bar{G}(x;\xi))^2}. \end{aligned} \tag{41}$$

Applying the generalized binomial series expansions

$$\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1} = \sum_{l=0}^{\infty} \frac{\Gamma(l+1)}{\Gamma(1)l!} \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha l},$$

$$(1+\bar{G}(x;\xi))^{-(\alpha[m+s+\beta(p+1)+l]+1)} = \sum_{w=0}^{\infty} \binom{\alpha\,(m+s+\beta(p+1)+l)+w}{w} \bar{G}^{w}(x;\xi),$$

$$G(x;\xi))^{\alpha(m+s+\beta(p+1)+l)-1} = \sum_{j=0}^{\infty} \binom{\alpha\,(m+s+\beta(p+1)+l)-1}{j} (-1)^j \bar{G}^j(x;\xi),$$

and

$$\bar{G}^{w+j}(x;\xi) = \sum_{r=0}^{\infty} \binom{w+j}{r} (-1)^r G^r(x;\xi).$$

Thus, we can reduce equation (41) to

$$\begin{aligned} & f(x;\alpha,\beta,\xi)\left(F(x;\alpha,\beta,\xi)\right)^{t-k} = \\ & = 2\alpha\beta \sum_{q,p=0}^{\infty} \sum_{m,s,l,w,j,r=0}^{\infty} b_{s,m} \binom{\beta(p+1)-1}{m} \binom{t-k}{q} (-1)^{q+p+j+r} \times \\ & \times \frac{(q+1)^p}{p!} \frac{\Gamma(l+1)}{\Gamma(1)l!} \binom{\alpha\,(m+s+\beta(p+1)+l)+w}{w} \times \\ & \times \binom{\alpha\,(m+s+\beta(p+1)+l)-1}{j} \binom{w+j}{r} G^r(x;\xi) g(x;\xi). \end{aligned} \tag{42}$$

Consequently, for $\delta \in (0,1)$, we write

$$f_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi) \quad = \quad \sum_{r=0}^{\infty} \varphi_{r+1} g_{r+1}(x;\xi), \tag{43}$$

where

$$\begin{aligned}\varphi_{r+1} &= 2\alpha\beta \sum_{t,q,p,m,s,l,w,j=0}^{\infty} \sum_{k=0}^{t} \phi_{t,k} b_{s,m} \binom{\beta(p+1)-1}{m} \binom{t-k}{q} (-1)^{q+p+j+r} \times \\ &\times \frac{(q+1)^p}{p!} \frac{\Gamma(l+1)}{\Gamma(1)l!(r+1)} \binom{\alpha\,(m+s+\beta(p+1)+l)+w}{w} \times \\ &\times \binom{\alpha\,(m+s+\beta(p+1)+l)-1}{j} \binom{w+j}{r}, \end{aligned} \quad (44)$$

and $g_{r+1}(x;\xi) = (r+1)G^r(x;\xi)g(x;\xi)$ is the Exp-G distribution with the power parameter $(r+1)$. Also, for $\delta > 1$, we write

$$f_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi) = \sum_{r=0}^{\infty} \varrho_{r+1} g_{r+1}(x;\xi), \quad (45)$$

where

$$\begin{aligned}\varrho_{r+1} &= 2\alpha\beta \sum_{t,q,p,m,s,l,w,j=0}^{\infty} \vartheta_t b_{s,m} \binom{\beta(p+1)-1}{m} \binom{t}{q} (-1)^{q+p+j+r} \times \\ &\times \frac{(q+1)^p}{p!} \frac{\Gamma(l+1)(r+1)}{\Gamma(1)l!} \binom{\alpha\,(m+s+\beta(p+1)+l)+w}{w} \times \\ &\times \binom{\alpha\,(m+s+\beta(p+1)+l)-1}{j} \binom{w+j}{r}, \end{aligned} \quad (46)$$

and $g_{r+1}(x;\xi) = (r+1)G^r(x;\xi)g(x;\xi)$ is the Exp-G distribution with the power parameter $(r+1)$. Therefore, for both cases, the pdf of MO-WEHL-G family of distributions can be expressed as a linear combination of the Exp-G distribution with power parameter $(r+1)$.

B Distribution of order statistics

The pdf of the ρ^{th} order statistic from the MO-WEHL-G family of distributions is given by

$$\begin{aligned}f_{\rho:n}(x;\delta,\alpha,\beta,\xi) &= \delta n! f_{WEHL-G}(x;\alpha,\beta,\xi) \times \\ &\times \sum_{z=0}^{n-\rho} \frac{(-1)^z}{(\rho-1)!(n-\rho)!} \frac{F^{z+\rho-1}_{WEHL-G}(x;\alpha,\beta,\xi)}{[1-\bar{\delta}\bar{F}_{WEHL-G}(x;\alpha,\beta,\xi)]^{z+\rho-1}}. \end{aligned} \quad (47)$$

If $\delta \in (0,1)$, we have

$$\begin{aligned} f_{\rho:n}(x;\delta,\alpha,\beta,\xi) &= \\ &= f_{WEHL-G}(x;\alpha,\beta,\xi)\sum_{t=0}^{\infty}\sum_{z=0}^{n-\rho}\sum_{k=0}^{t}U_{t,z,k}F_{WEHL-G}^{t+z-k+\rho-1}(x;\alpha,\beta,\xi), \end{aligned} \tag{48}$$

where

$$U_{t,z,k}=U_{t,z,k}(\delta)=\frac{\delta n!(-1)^{z}(1-\delta)^{t}(-1)^{t-k}}{(\rho-1)!(n-\rho)!}\binom{t}{k}\binom{z+\rho+t}{t}. \tag{49}$$

For $\delta > 1$, we write

$$\left(1-\bar{\delta}\bar{F}_{WEHL-G}(x;\alpha,\beta,\xi)\right)=\delta\left\{1-(\delta-1)F_{WEHL-G}(x;\alpha,\beta,\xi)/\delta\right\},$$

such that

$$f_{\rho:n}(x;\delta,\alpha,\beta,\xi)=f_{WEHL-G}(x;\alpha,\beta,\xi)\sum_{t=0}^{\infty}\sum_{z=0}^{n-\rho}c_{t,z}F_{WEHL-G}^{t+z+\rho-1}(x;\alpha,\beta,\xi), \tag{50}$$

where

$$c_{t,z}=c_{t,z}(\delta)=\frac{(-1)^{l}(\delta-1)^{t}n!}{\delta^{z+t+\rho}(\rho-1)!(n-\rho)!}\binom{z+\rho+t}{t}. \tag{51}$$

Note that

$$\begin{aligned} &f(x;\alpha,\beta,\xi)\left(F(x;\alpha,\beta,\xi)\right)^{t+z-k+\rho-1}= \\ &=2\alpha\beta\exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\times \\ &\times\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta-1} \\ &\times\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1}\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha-1}\times \end{aligned}$$

$$\times \frac{g(x;\xi)}{(1+\bar{G}(x;\xi))^2} \times$$
$$\times \left(1-\exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{t+z-k+\rho-1} =$$
$$= 2\alpha\beta \sum_{q=0}^{\infty} \binom{t+z-k+\rho-1}{q} (-1)^q \left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1} \times$$
$$\times \exp\left(-(q+1)\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right) \times$$
$$\times \left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta-1} \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha-1} \times$$
$$\times \frac{g(x;\xi)}{(1+\bar{G}(x;\xi))^2}.$$

Now, applying the following Taylor series expansion

$$\left(\exp\left(-(q+1)\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right) =$$
$$= \sum_{p=0}^{\infty} \frac{(-1)^p (q+1)^p \left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta p}}{p!},$$

and using the results on power series raised to a positive integer, by setting $a_s = \frac{1}{s+2}$, that is $\left(\sum_{s=0}^{\infty} a_s y^s\right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s$, we obtain

$$\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta(p+1)-1} =$$
$$= y^{\beta(p+1)-1}\left[\sum_{m=0}^{\infty}\binom{\beta(p+1)-1}{m} y^m \left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right] =$$
$$= y^{\beta(p+1)-1}\left[\sum_{m=0}^{\infty}\binom{\beta(p+1)-1}{m} y^m \sum_{s=0}^{\infty} b_{s,m} y^s\right] =$$
$$= \left[\sum_{m,s=0}^{\infty}\binom{\beta(p+1)-1}{m} b_{s,m} y^{m+s+\beta(p+1)-1}\right],$$

(Gradshteyn et al. 2014), where $b_{s,m} = (sa_0)^{-1} \sum_{l=1}^{s} [m(l+1)-s] a_l b_{s-l,m}$, $b_{0,m} = a_0^m$, and we can write

$$\begin{aligned} & f(x;\alpha,\beta,\xi)\,(F(x;\alpha,\beta,\xi))^{t+z-k+\rho-1} = \\ & = 2\alpha\beta \sum_{q,p=0}^{\infty} \sum_{m,s=0}^{\infty} \binom{\beta(p+1)-1}{m} b_{s,m} \binom{t+z-k+\rho-1}{q} \times \\ & \times (-1)^{q+p} \frac{(q+1)^p}{p!} \left(1 - \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1} \times \\ & \times \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha[m+s+\beta(p+1)]-1} \frac{g(x;\xi)}{(1+\bar{G}(x;\xi))^2}. \end{aligned} \tag{52}$$

Applying the generalized binomial series expansions

$$\left(1 - \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-1} = \sum_{l=0}^{\infty} \frac{\Gamma(l+1)}{\Gamma(1)l!} \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha l},$$

$$(1+\bar{G}(x;\xi))^{-(\alpha[m+s+\beta(p+1)+l]+1)} = \sum_{w=0}^{\infty} \binom{\alpha\,(m+s+\beta(p+1)+l)+w}{w} \bar{G}^{w}(x;\xi),$$

$$G(x;\xi))^{\alpha(m+s+\beta(p+1)+l)-1} = \sum_{j=0}^{\infty} \binom{\alpha\,(m+s+\beta(p+1)+l)-1}{j} (-1)^j \bar{G}^{j}(x;\xi),$$

and

$$\bar{G}^{w+j}(x;\xi) = \sum_{r=0}^{\infty} \binom{w+j}{r} (-1)^r G^r(x;\xi),$$

equation (52) reduces to

$$\begin{aligned} & f(x;\alpha,\beta,\xi)\,(F(x;\alpha,\beta,\xi))^{t+z-k+\rho-1} = \\ & = 2\alpha\beta \sum_{q,p=0}^{\infty} \sum_{m,s,l,w,j,r=0}^{\infty} b_{s,m} \binom{\beta(p+1)-1}{m} \binom{t+z-k+\rho-1}{q} \times \\ & \times (-1)^{q+p+j+r} \frac{(q+1)^p}{p!} \frac{\Gamma(l+1)}{\Gamma(1)l!} \binom{\alpha\,(m+s+\beta(p+1)+l)+w}{w} \times \\ & \times \binom{\alpha\,(m+s+\beta(p+1)+l)-1}{j} \binom{w+j}{r} G^r(x;\xi) g(x;\xi). \end{aligned} \tag{53}$$

For $\delta \in (0,1)$, If we substitute (53) into (48), we have the pdf of the ρ^{th} order statistic expressed as

$$f_{\rho:n}(x;\delta,\alpha,\beta,\xi) = \sum_{t,r=0}^{\infty} \sum_{z=0}^{n-\rho} \sum_{k=0}^{t} U_{t,z,k} a_{r+1} g_{r+1}(x;\xi), \tag{54}$$

where

$$
\begin{aligned}
a_{r+1} &= 2\alpha\beta \sum_{q,p=0}^{\infty} \sum_{m,s,l,w,j=0}^{\infty} b_{s,m} \binom{\beta(p+1)-1}{m} \binom{t+z-k+\rho-1}{q} (-1)^{q+p+j+r} \times \\
&\times \frac{(q+1)^p}{p!} \frac{\Gamma(l+1)}{\Gamma(1)l!(r+1)} \binom{\alpha(m+s+\beta(p+1)+l)+w}{w} \times \\
&\times \binom{\alpha(m+s+\beta(p+1)+l)-1}{j} \binom{w+j}{r},
\end{aligned}
$$

and $g_{r+1}(x;\xi) = (r+1)G^r(x;\xi)g(x;\xi)$ is the Exp-G distribution with the power parameter $(r+1)$.
Similarly, for $\delta > 1$, then equation (50) can be written as

$$
f_{\rho:n}(x;\delta,\alpha,\beta,\xi) = \sum_{t=0}^{\infty} \sum_{z=0}^{n-\rho} c_{t,z} a^*_{r+1} g_{r+1}(x;\xi), \tag{55}
$$

where

$$
\begin{aligned}
a^*_{r+1} &= 2\alpha\beta \sum_{q,p=0}^{\infty} \sum_{m,s,l,w,j=0}^{\infty} b_{s,m} \binom{\beta(p+1)-1}{m} \binom{t+z+\rho-1}{q} (-1)^{q+p+j+r} \times \\
&\times \frac{(q+1)^p}{p!} \frac{\Gamma(l+1)}{\Gamma(1)l!(r+1)} \binom{\alpha\,(m+s+\beta(p+1)+l)+w}{w} \times \\
&\times \binom{\alpha\,(m+s+\beta(p+1)+l)-1}{j} \binom{w+j}{r},
\end{aligned}
$$

and $g_{r+1}(x;\xi) = (r+1)G^r(x;\xi)g(x;\xi)$ is the Exp-G distribution with the power parameter $(r+1)$.

C Rényi entropy

Rényi entropy is defined to be the measure of variation or uncertainty for a random variable X with pdf f(x). Rényi entropy is defined as

$$
I_R(v) = (1-v)^{-1} log \left[\int_{-\infty}^{\infty} f^v(x) dx \right],
$$

where $v > 0$ and $v \neq 1$. Using the following expansion from Barreto et al. (2013), for $\delta \in (0,1)$

$$
\begin{aligned}
&f^{\nu}_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi) = \\
&= \frac{\delta^{\nu} f^{\nu}_{WEHL-G}(x;\alpha,\beta,\xi)}{\Gamma(2\nu)} \sum_{i,t=0}^{\infty} \binom{i}{t} (-1)^t (1-\delta)^i \Gamma(2\nu+i) \frac{[F_{WEHL-G}(x;\alpha,\beta,\xi)]^t}{i!}
\end{aligned}
$$

and for $\delta > 1$

$$f^{\nu}_{MO-WEHL-G}(x;\delta,\alpha,\beta,\xi) = \frac{f^{\nu}_{WEHL-G}(x;\alpha,\beta,\xi)}{\delta^{\nu+t}\Gamma(2\nu)}\sum_{t=0}^{\infty}(\delta-1)^t\Gamma(2\nu+t)\frac{F^t_{WEHL-G}(x;\alpha,\beta,\xi)}{t!}.$$

Thus, Rényi entropy for $\delta \in (0,1)$ and $\delta > 1$ are given by

$$I_R(\nu) = (1-\nu)^{-1}\log\left(\sum_{i=0}^{\infty} e_i \int_0^{\infty} f^{\nu}_{WEHL-G}(x;\alpha,\beta,\xi)(F_{WEHL-G}(x;\alpha,\beta,\xi))^t dx\right) \tag{56}$$

and

$$I_R(\nu) = (1-\nu)^{-1}\log\left(\sum_{t=0}^{\infty} h_t \int_0^{\infty} f^{\nu}_{WEHL-G}(x;\alpha,\beta,\xi)F^t_{WEHL-G}(x;\alpha,\beta,\xi)dx\right), \tag{57}$$

where

$$e_i = e_i(\delta) = \frac{\sum_{t=0}^{\infty}\delta^{\nu}(1-\delta)^i\Gamma(2\nu+i)\binom{i}{t}(-1)^t}{\Gamma(2\nu)i!}$$

and

$$h_t = h_t(\delta) = \frac{(\delta-1)^t\Gamma(2\nu+t)}{\delta^{\nu+t}\Gamma(2\nu)t!}.$$

Note that

$$\begin{aligned}
&f(x;\alpha,\beta,\xi)^{\nu}\left(F(x;\alpha,\beta,\xi)\right)^t = \\
&= (2\alpha\beta)^{\nu}\exp\left(-\nu\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\times \\
&\times\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\nu\beta-\nu}\times \\
&\times\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-\nu}\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\nu\alpha-\nu}\times \\
&\times\frac{g^{\nu}(x;\xi)}{(1+\bar{G}(x;\xi))^{2\nu}}\times
\end{aligned}$$

$$\times\left(1-\exp\left(-\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)^{t}=$$
$$=(2\alpha\beta)^{\nu}\sum_{q=0}^{\infty}\binom{t}{q}(-1)^{q}\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-\nu}\times$$
$$\times\exp\left(-(q+\nu)\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\times$$
$$\times\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\nu\beta-\nu}\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\nu\alpha-\nu}\times$$
$$\times\frac{g^{\nu}(x;\xi)}{(1+\bar{G}(x;\xi))^{2\nu}}.$$

Now, applying the following Taylor series expansion

$$\left(\exp\left(-(q+\nu)\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)=$$
$$=\sum_{p=0}^{\infty}\frac{(-1)^{p}(q+\nu)^{p}\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta p}}{p!},$$

and using the results on power series raised to a positive integer, by setting $a_s=\frac{1}{s+2}$, that is $\left(\sum_{s=0}^{\infty}a_s y^s\right)^m=\sum_{s=0}^{\infty}b_{s,m}y^s$, we obtain

$$\left[-\log\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)\right]^{\beta(p+\nu)-\nu}=$$
$$=y^{\beta(p+\nu)-\nu}\left[\sum_{m=0}^{\infty}\binom{\beta(p+\nu)-\nu}{m}y^{m}\left(\sum_{s=0}^{\infty}\frac{y^{s}}{s+2}\right)^{m}\right]=$$
$$=y^{\beta(p+\nu)-\nu}\left[\sum_{m=0}^{\infty}\binom{\beta(p+\nu)-\nu}{m}y^{m}\sum_{s=0}^{\infty}b_{s,m}y^{s}\right]=$$
$$=\left[\sum_{m,s=0}^{\infty}\binom{\beta(p+\nu)-\nu}{m}b_{s,m}y^{m+s+\beta(p+\nu)-\nu}\right],$$

(Gradshteyn et al. 2014), where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, $b_{0,m} = a_0^m$, and we have

$$f^{\nu}(x;\alpha,\beta,\xi)\,(F(x;\alpha,\beta,\xi))^t = \\ = (2\alpha\beta)^{\nu}\sum_{q,p=0}^{\infty}\sum_{m,s=0}^{\infty}\binom{\beta(p+\nu)-\nu}{m}b_{s,m}\binom{t}{q}(-1)^{q+p}\frac{(q+\nu)^p}{p!}\times \\ \times\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-\nu}\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha[m+s+\beta(p+\nu)]-\nu}\times \\ \times\frac{g^{\nu}(x;\xi)}{(1+\bar{G}(x;\xi))^{2\nu}}. \tag{58}$$

Applying the generalized binomial series expansions

$$\left(1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha}\right)^{-\nu} = \\ = \sum_{l=0}^{\infty}\frac{\Gamma(l+\nu)}{\Gamma(\nu)l!}\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\alpha l},$$

$$(1+\bar{G}(x;\xi))^{-(\alpha(m+s+\beta(p+\nu)+l)+\nu)} = \\ = \sum_{w=0}^{\infty}\binom{\alpha\,(m+s+\beta(p+\nu)+l)+\nu+w-1}{w}\bar{G}^{w}(x;\xi),$$

$$G(x;\xi))^{\alpha(m+s+\beta(p+\nu)+l)-\nu} = \sum_{j=0}^{\infty}\binom{\alpha\,(m+s+\beta(p+\nu)+l)-\nu}{j}(-1)^j\bar{G}^{j}(x;\xi),$$

and

$$\bar{G}^{w+j}(x;\xi) = \sum_{r=0}^{\infty}\binom{w+j}{r}(-1)^r G^r(x;\xi),$$

equation (58) reduces to

$$f(x;\alpha,\beta,\xi)\,(F(x;\alpha,\beta,\xi))^t = \\ = (2\alpha\beta)^{\nu}\sum_{q,p=0}^{\infty}\sum_{m,s,l,w,j,r=0}^{\infty}b_{s,m}\binom{\beta(p+\nu)-\nu}{m}\binom{t}{q}(-1)^{q+p+j+r}\times \\ \times\frac{(q+\nu)^p}{p!}\frac{\Gamma(l+\nu)}{\Gamma(\nu)l!}\binom{\alpha\,(m+s+\beta(p+\nu)+l)+\nu+w-1}{w}\times \\ \times\binom{\alpha\,(m+s+\beta(p+\nu)+l)-\nu}{j}\binom{w+j}{r}G^r(x;\xi)g(x;\xi). \tag{59}$$

Now, for $\delta \in (0,1)$ and from equation (56), we have

$$
\begin{aligned}
&I_R(\nu) = \\
&= (1-\nu)^{-1} \log\Bigg[(2\alpha\beta)^{\nu} \sum_{i,q,p=0}^{\infty} \sum_{m,s,l,w,j,r=0}^{\infty} e_i b_{s,m} \binom{\beta(p+\nu)-\nu}{m} \binom{t}{q} (-1)^{q+p+j+r} \times \\
&\times \frac{(q+\nu)^p}{p!} \frac{\Gamma(l+\nu)}{\Gamma(\nu) l!} \binom{\alpha\,(m+s+\beta(p+\nu)+l)+\nu+w-1}{w} \binom{w+j}{r} \times \\
&\times \binom{\alpha\,(m+s+\beta(p+\nu)+l)-\nu}{j} \frac{1}{\left(\frac{r}{\nu}+1\right)^{\nu}} \int_0^{\infty} \left((\frac{r}{\nu}+1) g(x;\xi) [G(x;\xi)]^{\frac{r}{\nu}} \right)^{\nu} dx \Bigg] = \\
&= (1-\nu)^{-1} \log\left[\sum_{r=0}^{\infty} \tau_r \exp\left((1-\nu) I_{REG}\right)\right], \qquad (60)
\end{aligned}
$$

where

$$
\begin{aligned}
\tau_r \;\; = \;\; & (2\alpha\beta)^{\nu} \sum_{i,q,p=0}^{\infty} \sum_{m,s,l,w,j=0}^{\infty} e_i b_{s,m} \binom{\beta(p+\nu)-\nu}{m} \binom{t}{q} (-1)^{q+p+j+r} \times \\
\times \;\; & \frac{(q+\nu)^p}{p!} \frac{\Gamma(l+\nu)}{\Gamma(\nu) l!} \binom{\alpha\,(m+s+\beta(p+\nu)+l)+\nu+w-1}{w} \times \\
\times \;\; & \binom{\alpha\,(m+s+\beta(p+\nu)+l)-\nu}{j} \binom{w+j}{r} \frac{1}{\left(\frac{r}{\nu}+1\right)^{\nu}}, \qquad (61)
\end{aligned}
$$

and $I_{REG} = (1-\nu)^{-1} \log\left[\int_0^{\infty} \left((\frac{r}{\nu}+1) g(x;\xi) [G(x;\xi)]^{\frac{r}{\nu}} \right)^{\nu} dx\right]$ is the Rényi entropy of the Exp-G distribution with power parameter $(\frac{r}{\nu}+1)$. Similarly, for $\delta > 1$, we have

$$
\begin{aligned}
&I_R(\nu) = \\
&= (1-\nu)^{-1} \log\Bigg[(2\alpha\beta)^{\nu} \sum_{t,q,p=0}^{\infty} \sum_{m,s,l,w,j,r=0}^{\infty} h_t b_{s,m} \binom{\beta(p+\nu)-\nu}{m} \binom{t}{q} (-1)^{q+p+j+r} \times \\
&\times \frac{(q+\nu)^p}{p!} \frac{\Gamma(l+\nu)}{\Gamma(\nu) l!} \binom{\alpha\,(m+s+\beta(p+\nu)+l)+\nu+w-1}{w} \binom{w+j}{r} \times \\
&\times \binom{\alpha\,(m+s+\beta(p+\nu)+l)-\nu}{j} \frac{1}{\left(\frac{r}{\nu}+1\right)^{\nu}} \int_0^{\infty} \left((\frac{r}{\nu}+1) g(x;\xi) [G(x;\xi)]^{\frac{r}{\nu}} \right)^{\nu} dx \Bigg] = \\
&= (1-\nu)^{-1} \log\left[\sum_{r=0}^{\infty} \tau_r^{*} \exp\left((1-\nu) I_{REG}\right)\right], \qquad (62)
\end{aligned}
$$

where

$$\begin{aligned}
\tau_r^* &= (2\alpha\beta)^{\nu} \sum_{t,q,p=0}^{\infty} \sum_{m,s,l,w,j=0}^{\infty} h_t b_{s,m} \binom{\beta(p+\nu)-\nu}{m} \binom{t}{q} (-1)^{q+p+j+r} \times \\
&\times \frac{(q+\nu)^p}{p!} \frac{\Gamma(l+\nu)}{\Gamma(\nu) l!} \binom{\alpha\left(m+s+\beta(p+\nu)+l\right)+\nu+w-1}{w} \times \\
&\times \binom{\alpha\left(m+s+\beta(p+\nu)+l\right)-\nu}{j} \binom{w+j}{r} \frac{1}{\left(\frac{r}{\nu}+1\right)^{\nu}}, \qquad (63)
\end{aligned}$$

and $I_{REG} = (1-\nu)^{-1} \log \left[\int_0^{\infty} \left(\left(\frac{r}{\nu}+1\right) g(x;\xi) [G(x;\xi)]^{\frac{r}{\nu}} \right)^{\nu} dx \right]$ is the Rényi entropy of the Exp-G distribution with power parameter $(\frac{r}{\nu}+1)$.

D Elements of score vector

The partial derivatives of the log-likelihood function with respect to each component of the parameter vector are:

$$\frac{\partial \ell_n}{\partial \delta} = \frac{n}{\delta} - 2\sum_{i=1}^{n} \frac{\exp\left(-\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)}{\left(1-\bar{\delta}\exp\left(-\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)},$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} - \beta \sum_{i=1}^{n} \frac{\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha} \ln\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)}{\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)} = \\
&- (\beta-1) \sum_{i=1}^{n} \frac{\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha} \ln\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)}{\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)} + \\
&+ \sum_{i=1}^{n} \frac{\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha} \ln\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)}{\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)} + \sum_{i=1}^{n} \ln\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right) + \\
&- 2\bar{\delta}\beta \sum_{i=1}^{n} \frac{\exp\left(-\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta-1}}{\left(1-\bar{\delta}\exp\left(-\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)} \times \\
&\times \frac{\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha} \ln\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)}{\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{n}\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right] + \\
&+ \sum_{i=1}^{n}\ln\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right] + \\
&- 2\bar{\delta}\sum_{i=1}^{n}\frac{\exp\left(-\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\ln\left(\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]\right)}{\left(1-\bar{\delta}\exp\left(-\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)} \times \\
&\times \left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \ell}{\partial \xi_k} &= -\alpha\beta\sum_{i=1}^{n}\frac{\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha-1}\left[\frac{\partial G(x_i;\xi)}{\partial \xi_k}\left(1+\bar{G}(x_i;\xi)\right)+\frac{\partial G(x_i;\xi)}{\partial \xi_k}G(x_i;\xi)\right]}{\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\left(1+\bar{G}(x_i;\xi)\right)^{2}} + \\
&+ \alpha(\beta-1)\sum_{i=1}^{n}\frac{\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha-1}\left[\frac{\partial G(x_i;\xi)}{\partial \xi_k}\left(1+\bar{G}(x_i;\xi)\right)+\frac{\partial G(x_i;\xi)}{\partial \xi_k}G(x_i;\xi)\right]}{\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\left(1+\bar{G}(x_i;\xi)\right)^{2}} + \\
&+ \sum_{i=1}^{n}\frac{\alpha\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha-1}\left[\frac{\partial G(x_i;\xi)}{\partial \xi_k}\left(1+\bar{G}(x_i;\xi)\right)+\frac{\partial G(x_i;\xi)}{\partial \xi_k}G(x_i;\xi)\right]}{\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\left(1+\bar{G}(x_i;\xi)\right)^{2}} + \\
&+ (\alpha-1)\sum_{i=1}^{n}\frac{\left[\frac{\partial G(x_i;\xi)}{\partial \xi_k}\left(1+\bar{G}(x_i;\xi)\right)+\frac{\partial G(x_i;\xi)}{\partial \xi_k}G(x_i;\xi)\right]}{\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)\left(1+\bar{G}(x_i;\xi)\right)^{2}} + \\
&+ \sum_{i=1}^{n}\frac{\frac{\partial g(x_i;\xi)}{\partial \xi_k}}{g(x_i;\xi)} - 2\sum_{i=1}^{n}\frac{\frac{\partial(1+\bar{G}(x_i;\xi))}{\partial \xi_k}}{\left(1+\bar{G}(x_i;\xi)\right)} - \\
&- 2\bar{\delta}\alpha\beta\sum_{i=1}^{n}\frac{\exp\left(-\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)}{\left(1-\bar{\delta}\exp\left(-\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta}\right)\right)} \times \\
&\times \frac{\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha-1}\left[-\ln\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)\right]^{\beta-1}}{\left(1-\left(\frac{G(x_i;\xi)}{1+\bar{G}(x_i;\xi)}\right)^{\alpha}\right)} \times \\
&\times \frac{\left[\frac{\partial G(x_i;\xi)}{\partial \xi_k}\left(1+\bar{G}(x_i;\xi)\right)+\frac{\partial G(x_i;\xi)}{\partial \xi_k}G(x_i;\xi)\right]}{\left(1+\bar{G}(x_i;\xi)\right)^{2}}.
\end{aligned}$$

E Competing models

The pdf of Marshall-Olkin log-logistic Weibull (MOLLW) distribution is given by

$$f_{MOLLW}(x;c,\alpha,\beta,\delta) = \frac{\delta(1+x^c)^{-1}e^{-\alpha x^\beta}\{\alpha\beta x^{\beta-1}+cx^{c-1}(1+x^c)^{-1}\}}{\left[1-\bar{\delta}(1+x^c)^{-1}e^{-\alpha x^\beta}\right]^2},$$

for $x > 0, c, \alpha, \beta, \delta > 0$. The pdf of Marshall-Olkin exponential-Gompertz (MOEGo) distribution is given by

$$f_{MOEGo}(x;\alpha,\lambda,\beta,\theta) = \frac{\alpha\lambda(1-e^{-\lambda})\theta e^{\beta x}e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}}{\left(\alpha(1-e^{-\lambda})+\bar{\alpha}\left(1-e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}\right)\right)^2},$$

for $x > 0, \alpha, \lambda, \beta, \theta > 0$.

The pdf of Marshall-Olkin generalized-log-logistic (MOG-LL) distribution is given by

$$f_{MOG-LL}(x;\delta,a,\alpha,\beta) = \frac{\delta a\beta\alpha^{-\beta}x^{\beta-1}[1+(\frac{x}{\alpha})^\beta]^{-a-1}}{(1-\bar{\delta}[1+(\frac{x}{\alpha})^\beta]^{-a})^2},$$

for $x > 0, \delta, a, \alpha, \beta > 0$.

The pdf of Marshall-Olkin modified Weibull (MOMW) distribution is given by

$$f_{MOMW}(x;\alpha,\delta,\lambda,\gamma) = \frac{\delta\alpha(\gamma+\lambda x)x^{\gamma-1}e^{\lambda x-\alpha x^\gamma e^{\lambda x}}}{(1-\bar{\delta}e^{-\alpha x^\gamma e^{\lambda x}})^2},$$

for $x > 0, \alpha, \delta, \lambda, \gamma > 0$.

The pdf of Weibull exponentiated half logistic log-logistic (WEHL-LLoG) distribution is given by

$$\begin{aligned} f_{WEHL-LLoG}(x;\alpha,\beta,\theta) &= 2\alpha\beta\exp\left(-\left[-\log\left(1-\left(\frac{1-(1+x^\theta)^{-1}}{1+(1+x^\theta)^{-1}}\right)^\alpha\right)\right]^\beta\right)\times \\ &\times \left[-\log\left(1-\left(\frac{1-(1+x^\theta)^{-1}}{1+(1+x^\theta)^{-1}}\right)^\alpha\right)\right]^{\beta-1}\times \\ &\times \left(1-\left(\frac{1-(1+x^\theta)^{-1}}{1+(1+x^\theta)^{-1}}\right)^\alpha\right)^{-1}\left(\frac{1-(1+x^\theta)^{-1}}{1+(1+x^\theta)^{-1}}\right)^{\alpha-1}\times \\ &\times \frac{\theta x^{\theta-1}(1+x^\theta)^{-2}}{(1+(1+x^\theta)^{-1})^2}, \end{aligned}$$

for $\alpha, \beta, \theta > 0$ and $x > 0$. The pdf of odd log-logistic exponentiated Weibull (OLLEW) distribution is given by

$$f_{OLLEW}(x;\alpha,\beta,\gamma,\theta) = \frac{\theta\beta\gamma x^{\beta-1}e^{-(x/\alpha)^{\beta}}[1-e^{-(x/\alpha)^{\beta}}]^{\gamma\theta-1}(1-[1-e^{-(x/\alpha)^{\beta}}]^{\gamma})^{\theta-1}}{\alpha\beta([1-e^{-(x/\alpha)^{\beta}}]^{\theta\gamma}+(1-[1-e^{-(x/\alpha)^{\beta}}]^{\gamma})^{\theta})^{2}},$$

for $\alpha, \beta, \lambda, \gamma, \theta > 0$.
The pdf of Weibull Lomax (WL) distribution is given by

$$f_{WL}(x;a,b,\alpha,\beta) = \\ = \frac{ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{\alpha b-1}\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^{b-1}\exp\left(-a\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{b}\right),$$

for $a, b, \alpha, \beta > 0$, $x \geq 0$.
The pdf of exponentiated half logistic odd Lindley-Weibull (EHLOL-W) distribution is given by

$$f_{EHLOL-W}(x) = 2\alpha\lambda^{2}ab^{-a}x^{a-1}e^{-(x/b)^{a}} \times \\ \times \frac{\exp\left[\frac{-\lambda(1-e^{-(x/b)^{a}})}{e^{-(x/b)^{a}}}\right]\left\{1-\frac{\lambda+e^{-(x/b)^{a}}}{(1+\lambda)e^{-(x/b)^{a}}}\exp\left[\frac{-\lambda(1-e^{-(x/b)^{a}})}{e^{-(x/b)^{a}}}\right]\right\}^{\alpha-1}}{(1+\lambda)(e^{-(x/b)^{a}})^{3}\left\{1+\frac{\lambda+e^{-(x/b)^{a}}}{(1+\lambda)e^{-(x/b)^{a}}}\exp\left[\frac{-\lambda(1-e^{-(x/b)^{a}})}{e^{-(x/b)^{a}}}\right]\right\}^{\alpha+1}}$$

for $x > 0, \gamma, \alpha, a, b > 0$.

F Datasets used

F.1 Kevlar 49/epoxy strands failure at 90% stress level

The observations are given as:
0.01, 0.01, 0.02, 0.02,0.02, 0.03, 0.03, 0.04, 0.05,0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

F.2 Annual maximum antecedent rainfall measurements

The data are:
264.9, 314.1, 364.6, 379.8, 419.3, 457.4, 459.4, 460, 490.3, 490.6, 502.2, 525.2, 526.8, 528.6, 528.6, 537.7, 539.6, 540.8, 551.0, 573.5, 579.2, 588.2, 588.7, 589.7, 592.1, 592.8, 600.8, 604.4, 608.4, 609.8, 619.2, 626.4, 629.4, 636.4, 645.2, 657.6, 663.5, 664.9, 671.7, 673.0, 682.6, 689.8, 698, 698.6, 698.8, 703.2, 755.9, 786, 787.2, 798.6, 850.4, 895.1.

F.3 Fatigue time of 101 6061-T6 aluminium coupons

The data are:
70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212.