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Existence of mild solutions for nonlocal perturbed evolution equations with infinite state-dependent delay in Fréchet spaces

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In this work, we give sufficient conditions to get the existence of mild solutions for two classes of first-order semilinear functional and neutral functional perturbed evolution equations with infinite state-dependent delay when the conditions are nonlocal using Avramescu nonlinear alternative for the sum of compact operators and contraction maps in Fréchet spaces, combined with semigroup theory.

Key words: perturbed evolution equations, neutral problems, mild solutions, state-dependent delay, fixed point, nonlinear alternative, semigroup theory, Fréchet spaces, infinite delay, nonlocal conditions

1. Introduction

In this paper, we give, in a real Banach space $(E, |.|)$, the existence of mild solutions defined on a semi infinite real interval $J := [0, +\infty)$ for two classes of first order partial functional and neutral functional perturbed evolution equations with infinite state dependent delay when the conditions are nonlocal.

We study in Section 3 the following perturbed evolution equations with infinite state-dependent delay when the conditions are nonlocal

$$y'(t) = A(t)y(t) + f\left(t, y_{\rho(t,y_t)}\right) + g\left(t, y_{\rho(t,y_t)}\right), \qquad \text{a.e. } t \in J, \qquad (1)$$

$$y(t) = \phi(t) - h_t(y), \qquad t \in (-\infty, 0], \qquad (2)$$

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where for an abstract phase space $\mathcal{B}$ which will be defined later; $f, g : J\times\mathcal{B} \to E$, $h_t : \mathcal{B} \to E$, $\rho : J \times \mathcal{B} \to \mathbb{R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{t\geqslant 0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generates an unique evolution system of operators $\{U(t, s)\}_{(t,s)\in J\times J}$ for $s \leqslant t$.

For any continuous function y and any $t \leqslant 0$, we denote by y_t the element of $\mathcal{B}$ defined by

$$y_t(\theta) = y(t + \theta) \text{ for } \theta \leqslant 0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t \leqslant 0$ up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$.

Next, in Section 5, we study the nonlocal neutral functional perturbed evolution equations of the form

$$\frac{d}{dt}[y(t) - Q\left(t, y_{\rho(t,y_t)}\right)] = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + g\left(t, y_{\rho(t,y_t)}\right), \qquad \text{a.e. } t \in J, \tag{3}$$

$$y(t) = \phi(t) - h_t(y), \qquad t \in (-\infty, 0], \tag{4}$$

where f, g, h_t, ρ, $A(\cdot)$ and ϕ are as in problem (1)–(2) and $Q : J \times \mathcal{B} \to E$ is a given function. Finally, we give two examples in Section 5 to illustrate the abstract theory.

For many years, the functional differential equations or Differential delay equations have been used in modeling scientific phenomena. It has been supposed that the delay is either a fixed constant or is given as an integral in which case is called distributed delay. If the delay is infinite the notion of the phase space $\mathcal{B}$ has an important role in the study of both qualitative and quantitative theory. An usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato in [20], see also Kappel and Schappacher [24] and Schumacher [27]. For more details and applications on this topic, read the book of Hale and Verduyn Lunel [21].

The nonlocal problem was motivated by physical problems. Indeed, Byszewski demonstrated in [13–16] that the nonlocal condition can be more useful than the classical initial condition to describe some physical phenomena. There are many papers concerning the nonlocal problems, see [17–19].

For infinite intervals, Baghli et al. provide results on the existence, uniqueness, and controllability of many mild solutions, even for various perturbed and non perturbed evolution problems with state-dependent and independent delays in Fréchet and Banach spaces in [2–12] and [25].

To extend the previous results specially in [2] for perturbed evolution problem with nonlocal conditions, we propose in this paper sufficient conditions for the existence of mild solutions on a semi infinite interval $J = [0; +\infty)$ for the two classes of first order nonlocal partial and neutral perturbed evolution equations with infinite state dependent delay (1)–(2) and (3)–(4) using the nonlinear alternative of Avramescu for sum of compact operators and contractions maps in Fréchet spaces [4], combined with semigroup theory [1, 26].

2. Preliminaries

In this section, we introduce some notations, definitions and theorems which are used via the different steps of these paper.

Let $C(\mathbb{R}^+; E)$ be the space of continuous functions from $\mathbb{R}^+$ into E and $B(E)$ be the space of all bounded linear operators from E into E, with the usual supremum norm

$$\|N\|_{B(E)} = \sup \{ \, |N(y)| \; : \; |y| = 1 \, \}, \quad N \in B(E).$$

A measurable function $y\colon \mathbb{R}^+ \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of Yosida [28]).

Let $L^1(\mathbb{R}^+, E)$ denotes the Banach space of measurable functions $y\colon \mathbb{R}^+ \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)| \mathrm{d}t.$$

The nonlocal condition $y(t) + h_t(y) = \phi(t)$ for $t \in (-\infty, 0]$, can be applied in physics with better effect than the classical initial condition $y(0) = y_0$.

For example, $h_t(y)$ may be given by

$$h_t(y) = \sum_{i=1}^{p} c_i y(t_i + t), \qquad t \in (-\infty, 0],$$

where $c_i, i = 1, ..., p$ are given constants and $0 < t_1 < ... < t_p < +\infty$.

At time $t = 0$ we have

$$h_0(y) = \sum_{i=1}^{p} c_i y(t_i).$$

We shall employ an axiomatic definition of the phase space $\mathcal{B}$ presented by Hale and Kato in [20] and follow the terminology used in [23]. Thus, $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ will be a semi-normed linear space of functions mapping $\mathbb{R}^-$ into E, and satisfying the following axioms

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$, the following conditions hold

(i) $y_t \in \mathcal{B}$;

(ii) There exists a positive constant $\mathcal{D}$ such that $|y(t)| \leqslant \mathcal{D}\|y_t\|_{\mathcal{B}}$;

(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that

$$\|y_t\|_{\mathcal{B}} \leqslant K(t) \sup_{0 \leqslant s \leqslant t} |y(s)| + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in axiom (A_1), y_t is a $\mathcal{B}-$valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup_{t \in [0,b]} K(t)$ and $M_b = \sup_{t \in [0,b]} M(t)$.

Remark 1.

1. *(ii) is equivalent to* $|\phi(0)| \leqslant \mathcal{D}\|\phi\|_{\mathcal{B}}$ *for every* $\phi \in \mathcal{B}$.
2. *Since* $\|\cdot\|_{\mathcal{B}}$ *is a seminorm, two elements* $\phi, \psi \in \mathcal{B}$ *can check* $\|\phi - \psi\|_{\mathcal{B}} = 0$ *without necessarily* $\phi(\theta) = \psi(\theta)$ *for all* $\theta \leqslant 0$.
3. *From the equivalence in the first remark, we can see that for all* $\phi, \psi \in \mathcal{B}$ *such that* $\|\phi - \psi\|_{\mathcal{B}} = 0$*: We necessarily have that* $\phi(0) = \psi(0)$.

Here are some examples of phase spaces from the book of Hino *et al.* [23].

Example 1. Let

BC denote the space of bounded continuous functions defined from $\mathbb{R}^-$ to E;

BUC denote the space of bounded uniformly continuous functions defined from $\mathbb{R}^-$ to E;

$$C^\infty := \left\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E\right\};$$

$$C^0 := \left\{\phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0\right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup_{\theta \leqslant 0} |\phi(\theta)|.$$

We have that the spaces BUC, C^∞ and C^0 satisfy conditions $(A_1)-(A_3)$. However, BC satisfies axioms (A_1), (A_3) but axiom (A_2) is not satisfied.

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We suppose that the family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ verifies

$$\|x\|_1 \leqslant \|x\|_2 \leqslant \|x\|_3 \leqslant \ldots \text{ for every } x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leqslant \overline{M}_n \text{ for all } y \in Y.$$

In what follows, for the family $\{A(t)\}_{t\geqslant 0}$ of closed densely defined linear unbounded operators on the Banach space E we assume that it satisfies the following assumptions [1]

(P1) The domain $D(A(t))$ is independent of t and is dense in E.

(P2) For $t \geqslant 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leqslant 0$ and there is a constant M independent of λ and t such that

$$\|R(t, A(t))\| \leqslant M(1 + |\lambda|)^{-1}, \text{ for } Re\lambda \leqslant 0.$$

(P3) There exist constants $L > 0$ and $0 < \alpha \leqslant 1$ such that

$$\|(A(t) - A(\theta))A^{-1}(\tau)\| \leqslant L|t - \tau|^{\alpha}, \text{ for } t, \theta, \tau \in J.$$

Lemma 1. *[1] Under assumptions (P1)–(P3), the Cauchy problem*

$$y'(t) - A(t)y(t) = 0, \ t \in J \quad \text{and} \quad y(0) = y_0,$$

has a unique evolution system $U(t,s)$, $(t,s) \in \Delta := \{(t,s) \in J \times J : 0 \leqslant s \leqslant t < +\infty\}$ satisfying the following properties:

1. *$U(t,t) = I$ where I is the identity operator in E,*
2. *$U(t,s)\, U(s,\tau) = U(t,\tau)$ for $0 \leqslant \tau \leqslant s \leqslant t < +\infty$,*
3. *$U(t,s) \in B(E)$ the space of bounded linear operators on E where for every $(t,s) \in \Delta$ and for each $y \in E$, the mapping $(t,s) \to U(t,s)\, y$ is continuous.*

For more details on evolution systems and their properties, see [1, 26].

For the state-dependent delay notion, let us set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{B}, \rho(s,\phi) \leqslant 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we present the following hypothesis

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to \mathbb{R}_+^\star$ such that for every $t \in \mathcal{R}(\rho^-)$

$$\|\phi_t\|_{\mathcal{B}} \leqslant \mathcal{L}^\phi \|\phi\|_{\mathcal{B}}.$$

Remark 2. *The condition (H_ϕ), is frequently checked by continuous and bounded functions. For more details, see for instance [23].*

Lemma 2. *[22] If $y : (-\infty; b] \longrightarrow E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leqslant (M_b + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + K_b|y(\theta)|; \theta \in [0, max\{0, s\}], s \in \mathcal{R}(\rho^-) \cup J$$

where $\mathcal{L}^\phi = \sup\limits_{t \in \mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$.

Proposition 1. *[3], By (H_ϕ), Lemma 2 and the property $(A1)$, we have for each $t \in [0, n]$ and $n \in \mathbb{N}$*

$$\|y_{\rho(t,y_t)}\| \leqslant K_n|y(t)| + \left(M_n + \mathcal{L}^\phi\right)\|y_0\|_{\mathcal{B}}.$$

Definition 1. *A function $f : J \times \mathcal{B} \longrightarrow E$ is said to be an $L^1_{loc.}$-Carathéodory function if it satisfies*

(i) for each $t \in J$ the function $f(t, .) : \mathcal{B} \longrightarrow E$ is continuous;

(ii) for each $y \in \mathcal{B}$ the function $f(., y) : J \longrightarrow E$ is measurable;

(iii) for every positive integer k there exists $\hbar_k \in L^1_{\mathrm{loc}}(J; \mathbb{R}^+)$ such that

$$|f(t, y)| \leqslant \hbar_k(t)$$

for all $\|y\|_{\mathcal{B}} \leqslant k$ and almost every $t \in J$.

The following definition is the appropriate concept of contraction in X.

Definition 2. *A function $f : X \longrightarrow X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $\alpha_n \in (0, 1)$ such that for all $x, y \in X$*

$$\|f(x) - f(y)\|_n \leqslant \alpha_n \|x - y\|_n.$$

Theorem 1. *(Avramescu's Nonlinear Alternative [4]) Let X be a Fréchet space and let $A, B : X \longrightarrow X$ be two operators satisfying*

(1) A is a compact operator,

(2) B is a contraction.

Then either one of the following statements holds:

(Av1) The operator $A + B$ has a fixed point;

(Av2) The set $\left\{x \in X, x = \lambda A(x) + \lambda B\left(\frac{x}{\lambda}\right)\right\}$ is unbounded for some $\lambda \in (0, 1)$.

3. Existence results for semilinear perturbed evolutions equations

In this section, we give an existence result for the nonlocal perturbed evolution problem (1)–(2). Firstly we define the mild solution for that problem.

Definition 3. *We say that the function $y : \mathbb{R} \to E$ is a mild solution of (1)–(2) if $y(t) = \phi(t) - h_t(y)$ for all $t \in (-\infty, 0]$ and y satisfies the following integral equation*

$$y(t) = U(t,0)[\phi(0) - h_0(y)] + \int_0^t U(t,s) f(s, y_{\rho(s,y_s)}) \mathrm{d}s + \int_0^t U(t,s) g\left(s, y_{\rho(s,y_s)}\right) \mathrm{d}s \quad a.e\ t \in J. \tag{5}$$

It is necessary to introduce the following hypotheses which are assumed thereafter

$(H1)$ There exists a constant $\widehat{M} \geqslant 1$ such that

$$\|U(t,s)\|_{B(E)} \leqslant \widehat{M}$$

and $U(t,s)$ is compact for $t - s > 0$, and for every $(t,s) \in \Delta$.

$(H2)$ There exist a function $p \in L^1_{loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to [0, +\infty)$ and such that

$$|f(t,u)| \leqslant p(t)\, \psi(\|u\|_{\mathcal{B}}).$$

for all $t \in J$ and for each $u \in \mathcal{B}$.

$(H3)$ There exists a function $\eta \in L^1(J;\mathbb{R}_+)$ such that

$$|g(t,u) - g(t,v)| \leqslant \eta(t)\|u - v\|_{\mathcal{B}}$$

for all $t \in J$ and for each $u, v \in \mathcal{B}$.

$(H4)$ There exist a constant $\varpi > 0$ such that

$$\|h_t\|_{\mathcal{B}} \leqslant \varpi$$

for all $t \in (-\infty, 0]$.

Corollary 1. *By (H_ϕ), Lemma 2 and the Proposion 1, we have for each $t \in [0,n]$ and $n \in \mathbb{N}$*

$$\|y_{\rho(t,y_t)}\| \leqslant K_n|y(t)| + \left(M_n + \mathcal{L}_h^\phi\right)(\|\phi\|_{\mathcal{B}} + \varpi).$$

where $\mathcal{L}_h^\phi = \sup_{t \in \mathcal{R}(\rho^-)} \mathcal{L}_h^\phi(t)$.

Consider the following space

$$B_{+\infty} = \left\{ y : \mathbb{R} \to E : y|_{[0,T]} \text{ continuous for } T > 0 \text{ and } y_0 \in \mathcal{B} \right\},$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0, T]$.

Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by

$$\|y\|_n := \sup_{t\in[0,n]} e^{-\tau\, L_n^*(t)}\, |y(t)|,$$

where $L_n^*(t) = \int_0^t \bar{l}_n \,\mathrm{d}s$, $\bar{l}_n(t) = \widehat{M} K_n \eta(t)$ and η is the function from the hypothesis $(H3)$.

Theorem 2. *Assumed (H_ϕ) and $(H1)$–$(H4)$ are satisfied and moreover for all $n > 0$*

$$\int_{\sigma_n}^{+\infty} \frac{ds}{s + \psi(s)} > K_n \widehat{M} \int_0^n \max\left(p(s); \eta(s)\right) \mathrm{d}s \tag{6}$$

with $\sigma_n = (\widehat{M} K_n \mathcal{D} + Mn + \mathcal{L}_h^\phi)\|\phi\|_{\mathcal{B}} + (\widehat{M}\mathcal{D}K_n + Mn + \mathcal{L}_h^\phi)\varpi + \widehat{M} K_n \int_0^n |g(s, 0)| \mathrm{d}s.$

Then, the nonlocal perturbed evolution problem (1)–(2) has at least one mild solution on $\mathbb{R}$.

Proof. We transform the problem (1)–(2) into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by

$$N(y)(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \leqslant 0; \\ U(t,0)[\phi(0) - h_0(y)] + \int_0^t U(t,s) f(s, y_{\rho(s,y_s)}) \mathrm{d}s & \\ \quad + \int_0^t U(t,s) g\left(s, y_{\rho(s,y_s)}\right) \mathrm{d}s, & \text{if } t \geqslant 0. \end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem (1)–(2) for $\phi \in \mathcal{B}$, we will define the function

$$x(t) = \begin{cases} \phi(t) - h_t(y) & \text{for } t \leqslant 0 \\ U(t,0)\phi(0) - U(t,0)h_0(y) & \text{for } t \in J. \end{cases}$$

Then $x_0 = \phi - h_0$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t).$$

Obviously, y satisfies the definition (5), if and only if z satisfies $z_0 = 0$ and

$$z(t) = \int_0^t U(t,s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s + \int_0^t U(t,s) g\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s.$$

Let $B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0 \in \mathcal{B}\}$.

We define for $t \in J$ the operators $F, G\colon\ B^0_{+\infty} \longrightarrow B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t,s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s,$$

and

$$G(z)(t) = \int_0^t U(t,s) g\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s.$$

Obviously the operator N has a fixed points is equivalents to $F + G$ has one, so it turns to prove that $F + G$ has a fixed point. The proof will be given in serval steps.

Step 1: We show the continuity of F. Let$(z_n)_n$ be a sequence in $B^0_{+\infty}$ such that $z_n \longrightarrow z \in B^0_{+\infty}$, by the hypothesis $(H1)$, we obtain

$$\begin{aligned} &|F(z_n)(t) - F(z)(t)| \leqslant \\ &\leqslant \int_0^t \|U(t,s)\|_{B(E)} \left| f\left(s, z_{n\rho(s,z_{ns}+x_s)} + x_{\rho(s,z_{ns}+x_s)}\right) - f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)\right| \mathrm{d}s \\ &\leqslant \widehat{M} \int_0^t \left| f\left(s, z_{n\rho(s,z_{ns}+x_s)} + x_{\rho(s,z_{ns}+x_s)}\right) - f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \right| \mathrm{d}s. \end{aligned}$$

Since f is continuous. By dominated convergence theorem of Lebesgue, we get

$$|F(z_n)(t) - F(z)(t)| \longrightarrow 0 \quad \text{if } \ n \longrightarrow +\infty.$$

So that F is continuous.

Step 2: Show that F transforms any bounded of $B^0_{+\infty}$ in a bounded set. For any $d > 0$, there exists a positive constant ϱ such that for all $z \in B_d = \{z \in B^0_{+\infty} : \|z\|_n \leqslant d\}$ we get $\|F(z)\|_n \leqslant \varrho$, where $B_d = \{z \in B^0_{+\infty} : \|z\|_n \leqslant d\}$. Let $z \in B_d$. By the hypotheses $(H1)$ and $(H2)$, we have for all $t \in [0, n]$

$$\begin{aligned} |F(z)(t)| &\leqslant \int_0^t \|U(t,s)\|_{B(E)} \left| f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)\right| \mathrm{d}s \\ &\leqslant \widehat{M} \int_0^t p(s)\psi\left(\left\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right\|_{\mathcal{B}}\right) \mathrm{d}s. \end{aligned}$$

From (H_ϕ), Corollary 1 and Assumption (A_1), we have for every $t \in [0, n]$

$$\begin{aligned} &\left\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right\|_{\mathcal{B}} \leqslant \\ &\quad \leqslant \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \\ &\quad \leqslant K_n|z(s)| + (M_n + \mathcal{L}^\phi_h)\|z_0\|_{\mathcal{B}} + K_n|x(s)| + (M_n + \mathcal{L}^\phi_h)\|x_0\|_{\mathcal{B}} \\ &\quad \leqslant K_n|z(s)| + K_n\|U(s,0)\|_{B(E)}|\phi(0)| + K_n\|U(s,0)\|_{B(E)}|h_0(y)| \\ &\qquad + (M_n + \mathcal{L}^\phi_h)\|\phi\|_{\mathcal{B}} + (M_n + \mathcal{L}^\phi_h)\varpi \\ &\quad \leqslant K_n|z(s)| + K_n\widehat{M}|\phi(0)| + K_n\widehat{M}|h_0(y)|(M_n + \mathcal{L}^\phi_h)[\|\phi\|_{\mathcal{B}} + \varpi]. \end{aligned}$$

Using (ii), we get

$$\begin{aligned} \left\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right\|_{\mathcal{B}} &\leqslant K_n|z(s)| + K_n\widehat{M}\mathcal{D}\|\phi\|_{\mathcal{B}} + K_n\mathcal{D}\widehat{M}\varpi \\ &\quad + (M_n + \mathcal{L}^\phi_h)(|\phi\|_{\mathcal{B}} + \varpi) \\ &\leqslant K_n|z(s)| + (M_n + \mathcal{L}^\phi_h + K_n\widehat{M}\mathcal{D})[\|\phi\|_{\mathcal{B}} + \varpi]. \end{aligned}$$

Set $c_n := (M_n + \mathcal{L}^\phi_h + K_n\widehat{M}\mathcal{D})\|\phi\|_{\mathcal{B}} + (M_n + \mathcal{L}^\phi_h + K_n\mathcal{D}\widehat{M})\varpi$ and $\varsigma_n := K_n d + c_n$. Then

$$\left\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right\|_{\mathcal{B}} \leqslant K_n d + c_n := \varsigma_n\,. \tag{7}$$

Using the nondecreasing character of ψ, we get for each $t \in [0, n]$

$$\begin{aligned} |F(z)(t)| &\leqslant \widehat{M} \int_0^t p(s)\psi(\varsigma_n)\mathrm{d}s \\ &\leqslant \widehat{M}\psi(\varsigma_n) \int_0^t p(s)\mathrm{d}s \\ &\leqslant \widehat{M}\psi(\varsigma_n)\|p\|_{L^1} := \varrho. \end{aligned}$$

So there exists a positive constant ϱ such that $\|F(z)\|_n \leqslant \varrho$. Hence $F(B_d) \subset B_\varrho$.
Step 3: F maps bounded sets into equicontinuous sets of $B^0_{+\infty}$. We consider B_d as in Step 2 and we show that $F(B_d)$ is equicontinuous. Let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$ and $z \in B_d$.

$$\begin{aligned}
&|F(z)(\tau_2) - F(z)(\tau_1)| \leqslant \\
&\quad \leqslant \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} |f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)| \\
&\qquad + \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)} |f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)| \mathrm{d}s \\
&\quad \leqslant \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} p(s)\psi\left(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\right) \mathrm{d}s \\
&\qquad + \widehat{M} \int_{\tau_1}^{\tau_2} p(s)\psi\left(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)\|_{\mathcal{B}})\mathrm{d}s.
\end{aligned}$$

By (7) and using the nondecreasing character of ψ, we get

$$\begin{aligned}
|F(z)(\tau_2) - F(z)(\tau_1)| \leqslant \psi(\varsigma_n) \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} p(s)\mathrm{d}s \\
+ \widehat{M}\psi(\varsigma_n) \int_{\tau_1}^{\tau_2} p(s)\mathrm{d}s.
\end{aligned}$$

Observing that $|F(z)(\tau_2) - F(z)(\tau_1)|$ tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$ independently of $z \in B_d$. T he right-hand side of the above inequality tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$. Since $U(t, s)$ is a strongly continuous operator and the compactness of $U(t, s)$ for $t > s$ implies the continuity in the uniform operator topology. As a result of Steps 1 to 3 together with the Arzelá-Ascoli theorem it suffices to show that the operator F maps B_d into a pre-compact set in E.

Let $t \in J$ be fixed and let ϵ be a real number such that $0 < \epsilon < t$. For $z \in B_d$ we define

$$F_\epsilon(z)(t) = U(t, t-\epsilon) \int_0^{t-\epsilon} U(t-\epsilon, s) f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s.$$

Since $U(t,s)$ is a compact operator, the set $Z_\epsilon(t) = \{F_\epsilon(z)(t) : z \in B_d\}$ is pre-compact in E for every ϵ sufficiently small, $0 < \epsilon < t$. Moreover, using and the nondecreasing character of ψ, we have

$$|F(z)(t) - F_\epsilon(z)(t)| \leqslant \widehat{M}\psi(\varsigma_n) \int_{t-\epsilon}^{t} p(s)\,\mathrm{d}s.$$

Therefore, the set $\{F(z)(t) : z \in B_d\}$ is pre-compact in E. So we deduce from Steps 1, 2 and 3 that F is a continuous compact operator.

Step 4: We shall show now that the operator G is a contraction.
Indeed, let $z, \overline{z} \in B^0_{+\infty}$. By the hypotheses $(H1)$ and $(H3)$, we get for all $t \in [0,n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leqslant \int_0^t \|U(t,s)\|_{B(E)} \Big| g\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \\
&\quad - g\left(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\right) \Big| \mathrm{d}s \\
&\leqslant \int_0^t \widehat{M}\eta(s) \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} \mathrm{d}s.
\end{aligned}$$

Using (7), to obtain

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leqslant \int_0^t \widehat{M}\eta(s) K_n |z(s) - \overline{z}(s)| \mathrm{d}s \\
&\leqslant \int_0^t \left[\overline{l}_n(s)\, e^{\tau L_n^*(s)}\right] \left[e^{-\tau L_n^*(s)}\, |z(s) - \overline{z}(s)|\right] \mathrm{d}s \\
&\leqslant \int_0^t \left[\frac{e^{\tau\, L_n^*(s)}}{\tau}\right]' \mathrm{d}s\, \|z - \overline{z}\|_n \\
&\leqslant \frac{1}{\tau}\, e^{\tau\, L_n^*(t)} \|z - \overline{z}\|_n.
\end{aligned}$$

Therefore,

$$\|G(z) - G(\overline{z})\|_n \leqslant \frac{1}{\tau}\, \|z - \overline{z}\|_n.$$

So, the operator G is a contraction for all $n \in \mathbb{N}$.

Step 5: To apply Theorem 1, we must check the statement $(Av2)$: i.e. it remains to show that the following set

$$\Gamma = \left\{ z \in B^0_{+\infty} : \ z = \lambda\, F(z) + \lambda\, G\left(\frac{z}{\lambda}\right) \ \text{ for some } 0 < \lambda < 1 \right\}.$$

is bounded. Let $z \in \Gamma$. By $(H1)-(H3)$, Corollary (1) and inequality (7), we have for each $t \in [0, n]$

$$\begin{aligned}
\frac{|z(t)|}{\lambda} &\leqslant \int_0^t \|U(t,s)\|_{B(E)} \left| f\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \right| \mathrm{d}s \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)} \left| g\left(s, \frac{z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}}{\lambda}\right) \right| \mathrm{d}s \\
&\leqslant \widehat{M} \int_0^t p(s) \psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_B) \mathrm{d}s \\
&\quad + \int_0^t \widehat{M} \left| g\left(s, \frac{z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}}{\lambda}\right) - g(s,0) + g(s,0) \right| \mathrm{d}s \\
&\leqslant \widehat{M} \int_0^t p(s) \psi\left(K_n |z(s)| + c_n\right) \mathrm{d}s + \int_0^t \widehat{M} \eta(s) \left\| \frac{z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}}{\lambda} \right\|_B \mathrm{d}s \\
&\quad + \widehat{M} \int_0^n |g(s,0)| \mathrm{d}s \\
&\leqslant \widehat{M} \int_0^t p(s) \psi\left(K_n |z(s)| + c_n\right) \mathrm{d}s + \widehat{M} \int_0^t \eta(s) \left(\frac{K_n}{\lambda} |z(s)| + c_n \right) \mathrm{d}s \\
&\quad + \widehat{M} \int_0^n |g(s,0)| \mathrm{d}s.
\end{aligned}$$

We consider the function $u(t) := \sup |z(\theta|)$ defined for $t \in J$ with the fact that $0 < \lambda < 1$.

$$\begin{aligned}
\frac{K_n}{\lambda} u(t) + c_n \leqslant c_n + K_n \widehat{M} \int_0^t p(s) \psi\left(\frac{K_n}{\lambda} u(s) + c_n \right) \mathrm{d}s \\
+ K_n \widehat{M} \int_0^t \eta(s) \left(\frac{K_n}{\lambda} u(s) + c_n \right) \mathrm{d}s + K_n \widehat{M} \int_0^n |g(s,0)| \mathrm{d}s.
\end{aligned}$$

Set $\sigma_n := c_n + K_n\widehat{M}\int\limits_0^n |g(s,0)|\,\mathrm{d}s$. Then, we have

$$\frac{K_n}{\lambda}u(t) + c_n \leqslant \sigma_n + K_n\widehat{M}\int\limits_0^t p(s)\,\psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s + K_n\widehat{M}\int\limits_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s.$$

We consider the function $\mu(t)$ defined by

$$\mu(t) = \left\{\sup_{s\in[0,t]}\frac{K_n|u(s)|}{\lambda} + c_n\right\} \quad 0 \leqslant t \leqslant +\infty.$$

Let $t^\star \in [0,t]$ be such that $\mu(t^\star) = \dfrac{K_n|u(t^\star)|}{\lambda} + c_n$, by the previous inequality and the nondecreasing character of ψ, we have $t \in [0,n]$

$$\mu(t) \leqslant \sigma_n + K_n\widehat{M}\int\limits_0^t p(s)\psi(\mu(s))\,\mathrm{d}s + K_n\widehat{M}\int\limits_0^t \eta(s)\mu(s)\,\mathrm{d}s.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$\mu(t) \leqslant v(t) \qquad \text{for all } \ t \in [0,n].$$

From the definition of v, we have

$$v(0) = \sigma_n \quad \text{and} \quad v'(t) = K_n\widehat{M}\,[p(t)\psi(v(t)) + \eta(t)v(t)] \ \text{ a.e. } \ t \in [0,n].$$

This implies that for all $t \in [0,n]$ and using theorem(2) we get

$$\begin{aligned}\int\limits_{\sigma_n}^{v(t)} \frac{ds}{s+\psi(s)} &\leqslant K_n\widehat{M}\int\limits_0^t \max\,(p(s);\eta(s))\,\mathrm{d}s\\ &\leqslant K_n\widehat{M}\int\limits_0^n \max\,(p(s);\eta(s))\,\mathrm{d}s\\ &< \int\limits_{\sigma_n}^{+\infty} \frac{ds}{s+\psi(s)}.\end{aligned}$$

Thus, for every $t \in [0, n]$, there exists a constant Φ_n such that $v(t) \leqslant \Phi_n$ and hence $\mu(t) \leqslant \Phi_n$. Since $\|z\|_n \leqslant \mu(t)$, we have$\|z\|_n \leqslant \Phi_n$. This shows that the set Γ is bounded. Then the statement (Av2) in Theorem 1 does not hold. The nonlinear alternative of Avramescu implies that (Av1) is satisfied, we deduce that the operator $F + G$ has a fixed point z^*. Then $y^*(t) = z^*(t) + x(t),\ t \in \mathbb{R}$ is the fixed point of the operator N which is a mild solution of the problem (1)–(2).

4. Existence results for neutral perturbed evolution equations

In this section, we give an existence result for the problem (3)–(4). Firstly we define the concept of the mild solution for that problem.

Definition 4. *We say that the function $y : \mathbb{R} \to E$ is a mild solution of the problem (3)–(4) if $y(t) = \phi(t) - h_t(y)$ for all $t \leqslant 0$ and y satisfies the following integral equation*

$$
\begin{aligned}
y(t) = {} & U(t,0)[\phi(0) - h_0(y) - Q(0,\phi)] + Q(t, y_\rho(t, y_t)) \\
& + \int_0^t U(t,s)A(s)Q(s, y_{\rho(s,y_s)})\,\mathrm{d}s + \int_0^t U(t,s)f(s, y_{\rho(s,y_s)})\,\mathrm{d}s \\
& + \int_0^t U(t,s)g(s, y_{\rho(s,y_s)})\,\mathrm{d}s \quad \textit{for all } t \in J.
\end{aligned} \tag{8}
$$

We consider the hypotheses (H_ϕ), $(H1)-(H4)$ and we will need the following assumptions

$(H5)$ There exists a constant $\overline{M_0} > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leqslant \overline{M_0}$$

for all $t \in J$.

$(H6)$ There exists a constant $0 < L < \dfrac{1}{\overline{M_0}}$ such that

$$|A(t)Q(t,\phi))| \leqslant L(\|\phi\|_{\mathcal{B}} + 1)$$

for all $t \in J$ for every $\phi \in \mathcal{B}$.

$(H7)$ There exists a constant $L_* > 0$ such that

$$|A(s)Q(s,\phi) - A(\overline{s})Q(\overline{s},\overline{\phi})| \leqslant L_* \left(|s - \overline{s}| + (\|\phi - \overline{\phi}\|_{\mathcal{B}}.\right)$$

for all $s, \overline{s} \in J$ for every $\phi, \overline{\phi} \in \mathcal{B}$.

Theorem 3. *Assume that* $(H\phi)$, $(H1)$–$(H7)$ *are satisfied and moreover for all* $n \geqslant 0$

$$\int_{\xi_n}^{+\infty} \frac{ds}{s+\psi(s)} > \frac{K_n \widehat{M}}{1-\overline{M_0}LK_n} \int_0^n \max(p(s), L+\eta(s))\,\mathrm{d}s \tag{9}$$

with

$$\xi_n := c_n + K_n \frac{\chi_n}{1-\overline{M_0}LK_n},$$

and

$$c_n = \left(\widehat{M}K_n\mathcal{D} + M_n + \mathcal{L}_h^{\phi}\right) \|\phi\|_{\mathcal{B}} + \left(\widehat{M}K_n\mathcal{D} + M_n + \mathcal{L}_h^{\phi}\right) \varpi,$$

and

$$\chi_n = \overline{M_0}\, L(\widehat{M}+1) + n\widehat{M}L + \overline{M_0}L\left(M_n + \mathcal{L}_h^{\phi} + \widehat{M}(K_n\mathcal{D}+1)\right) \|\phi\|_{\mathcal{B}}$$

$$+ \overline{M_0}\, L(K_n\mathcal{D}\widehat{M} + M_n + \mathcal{L}_h^{\phi})\varpi + \widehat{M} \int_0^n |g(s,0)|\,\mathrm{d}s.$$

Then, the nonlocal neutral perturbed evolution problem (3)–(4) has at least one mild solution on $\mathbb{R}$.

Proof. We consider the operator $\widetilde{N}\colon B_{+\infty} \to B_{+\infty}$ defined by:

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \leqslant 0; \\ U(t,0)\,[\phi(0) - h_0(y) - Q(0,\phi)] + Q\left(t, y_{\rho(t,y_t)}\right) & \\ \quad + \int_0^t U(t,s)A(s)Q\left(s, y_{\rho(s,y_s)}\right) \mathrm{d}s + \int_0^t U(t,s) f(s, y_{\rho(s,y_s)})\,\mathrm{d}s & \\ \quad + \int_0^t U(t,s) g\left(s, y_{\rho(s,y_s)}\right) \mathrm{d}s, & \text{if } t \in J. \end{cases}$$

Then, fixed points of the operator $\widetilde{N}$ are mild solutions of the problem (3)–(4).

For $\phi \in \mathcal{B}$, we consider the function $x(.)\colon R \to E$ defined as below by

$$x(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \leqslant; \\ U(t,0)\phi(0) - U(t,0)h_0(y), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi(0) - h_0(y)$ for each function $z \in B^0_{+\infty}$ set $y(t) = z(t) + x(t)$. It obvious that y satisfies (8) if and only if $z_0 = 0$ and

$$\begin{aligned} z(t) = {} & Q\left(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\right) - U(t,0)Q(0,\phi) \\ & + \int_0^t U(t,s)A(s)Q\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s \\ & + \int_0^t U(t,s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \mathrm{d}s \\ & + \int_0^t U(t,s)g\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s. \end{aligned}$$

Let $B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0.\}$ Define the operator $F, \widetilde{G} : B^0_{+\infty} \longrightarrow B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t,s)f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \mathrm{d}s,$$

and

$$\begin{aligned} \widetilde{G}(z)(t) = {} & Q\left(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\right) - U(t,0)Q(0,\phi) \\ & + \int_0^t U(t,s)A(s)Q\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s \\ & + \int_0^t U(t,s)g\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \mathrm{d}s. \end{aligned}$$

Obviously the operator $\widetilde{N}$ having a fixed points is equivalent to $F + \widetilde{G}$ having one, so it turns to prove that $F + \widetilde{G}$ has a fixed point. We have shown that the operator F is continuous and compact as in Section 3. It remains to show that the operator $\widetilde{G}$ is a contraction.

Let $z, \overline{z} \in B^{+\infty}_0$. By $(H1)$, $(H3)$ and $(H5)$, $(H7)$, we have for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$
\begin{aligned}
|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| \leqslant{} & \left|Q\left(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\right) - Q\left(t, \overline{z}_{\rho(t,\overline{z}_t+x_t)} + x_{\rho(t,\overline{z}_t+x_t)}\right)\right| \\
& + \int_0^t \|U(t,s)\|_{B(E)} \Big|A(s)\Big[Q\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \\
& \quad - Q\left(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\right)\Big]\Big| \mathrm{d}s \\
& + \int_0^t \|U(t,s)\|_{B(E)} \Big|g\left(r, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \\
& \quad - g\left(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_r+x_s)}\right)\Big| \mathrm{d}s \\
\leqslant{} & \|A^-1(t)\|_{B(E)} \Big|A(t)Q\left(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\right) \\
& - A(t)Q\left(t, \overline{z}_{\rho(t,\overline{z}_t+x_t)} + x_{\rho(t,\overline{z}_t+x_t)}\right)\Big| \\
& + \int_0^t \widehat{M} \Big|A(s)Q\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right) \\
& \quad - A(s)Q\left(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\right)\Big| \mathrm{d}s \\
& + \int_0^t \widehat{M}\eta(s) \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \mathrm{d}s \\
\leqslant{} & \overline{M_0} L_* \|z_{\rho(t,z_t+x_t)} - \overline{z}_{\rho(t,\overline{z}_t+x_t)}\|_B + \int_0^t \widehat{M} L_* \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} \mathrm{d}s \\
& + \int_0^t \widehat{M}\eta(s) \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} \mathrm{d}s.
\end{aligned}
$$

By (7), we have

$$
\begin{aligned}
|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| \leqslant{} & \overline{M_0} L_* K_n |z(t) - \overline{z}(t)| + \int_0^t \widehat{M} L_* K_n |z(s) - \overline{z}(s)| \mathrm{d}s \\
& + \int_0^t \widehat{M}\eta(s) K_n |z(s) - \overline{z}(s)| \mathrm{d}s \\
\leqslant{} & \overline{M_0} L_* K_n |z(t) - \overline{z}(t)| + \int_0^t \widehat{M} K_n \left(L_* + \eta(s)\right) |z(s) - \overline{z}(s)| \mathrm{d}s.
\end{aligned}
$$

Set $\overline{l_n}(t) = \widehat{M} K_n [L_* + \eta(t)]$ for the family of semi norms $\{\|.\|_n\}_{n\in\mathbb{N}}$, then

$$
\begin{aligned}
|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| &\leqslant \overline{M_0} L_* K_n |z(t) - \overline{z}(t)| + \int_0^t \overline{l_n}(s) |z(s) - \overline{z}(s)| \,\mathrm{d}s \\
&\leqslant (\overline{M_0} L_* K_n e^{\tau L_n^*(t)})(e^{-\tau L_n^*(t)} |z(t) - \overline{z}(t)|) \\
&\quad + \int_0^t (\overline{l_n}(s) e^{\tau L_n^*(s)})(e^{-\tau L_n^*(s)} |z(s) - \overline{z}(s)|) \,\mathrm{d}s \\
&\leqslant \overline{M_0} L_* K_n e^{\tau L_n^*(t)} \|z - \overline{z}\|_n + \int_0^t \left[\frac{e^{\tau L_n^*(s)}}{\tau} \right]' \mathrm{d}s \|z - \overline{z}\|_n \\
&\leqslant \left(\overline{M_0} L_* K_n + \frac{1}{\tau} \right) e^{\tau L_n^*(t)} \|z - \overline{z}\|_n .
\end{aligned}
$$

Therefore,

$$
\left\| \widetilde{G}(z) - \widetilde{G}(\overline{z}) \right\|_n \leqslant \left(\overline{M_0} L_* K_n + \frac{1}{\tau} \right) \|z - \overline{z}\|_n .
$$

Let us fix $\tau > 0$ and assume that $\overline{M_0} L_* K_n + \frac{1}{\tau} < 1$. Then the operator $\widetilde{G}$ is a contraction for all $n \in \mathbb{N}$.

For applying Theorem 1, we must check the statement $(Av2)$: i.e. it remains to show that the following set

$$
\widetilde{\Gamma} = \left\{ z \in B^0_{+\infty} : \ z = \lambda\, F(z) + \lambda \widetilde{G}\left(\frac{z}{\lambda}\right) \ \text{ for some } 0 < \lambda < 1 \right\}
$$

is bounded.

Let $z \in \widetilde{\Gamma}$. By the $(H1)$–$(H3)$, $(H5)$–$(H7)$, we have for each $t \in [0, n]$

$$
\begin{aligned}
\frac{|z(t)|}{\lambda} &\leqslant \|A^{-1}(t)\|_{B(E)} \left|A(t)Q\left(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\right)\right| \\
&+ \widehat{M}\|A^{-1}(0)\|\,|A(0)Q(0,\phi)| \\
&+ \widehat{M}\int_0^t |A(s)Q\left(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\right)|\,\mathrm{d}s \\
&+ \widehat{M}\int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\,\mathrm{d}s \\
&+ \widehat{M}\int_0^t |g\left(s, \frac{z_{\rho(s,z_s+x_s)}}{\lambda} + x_{\rho(s,z_s+x_s)}\right) - g(s,0)|\,\mathrm{d}s + \widehat{M}\int_0^n |g(s,0)|\,\mathrm{d}s \\
&\leqslant \overline{M_0}\,L\left(\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + 1\right) + \widehat{M}\overline{M_0}L(\|\phi\|_{\mathcal{B}} + 1) \\
&+ \widehat{M}L\int_0^t \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} + 1\,\mathrm{d}s \\
&+ \widehat{M}\int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\,\mathrm{d}s \\
&+ \widehat{M}\int_0^t \eta(s)\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}\right\|\mathrm{d}s + \widehat{M}\int_0^n |g(s,0)|\,\mathrm{d}s \\
&\leqslant \overline{M_0}\,L(\widehat{M}+1) + \widehat{M}Ln + \widehat{M}\overline{M_0}L\|\phi\|_{\mathcal{B}} + \widehat{M}\int_0^n |g(s,0)|\,\mathrm{d}s \\
&+ \overline{M_0}\,L\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + \widehat{M}L\int_0^t \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\,\mathrm{d}s \\
&+ \widehat{M}\int_0^t p(s)\psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\,\mathrm{d}s \\
&+ \widehat{M}\int_0^t \eta(s)\left\|\frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}}\right\|\mathrm{d}s.
\end{aligned}
$$

Using Corollary 1 and inequality (7)

$$\begin{aligned}
\left\| \frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)} \right\|_{\mathcal{B}} &\leqslant \frac{1}{\lambda} \|z_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,\frac{z_s}{\lambda}+x_s)}\|_{\mathcal{B}} \\
&\leqslant \frac{K_n}{\lambda}|z(s)| + \frac{M_n + \mathcal{L}_h^{\phi}}{\lambda}\|z_0\|_{\mathcal{B}} + K_n|x_s| + (M_n + \mathcal{L}_h^{\phi})\|x_0\|_{\mathcal{B}} \\
&\leqslant \frac{K_n}{\lambda}|z(s)| + K_n\|U(s,0)\|_{B(E)}|\phi(0) - h_0(y))| + (M_n + \mathcal{L}_h^{\phi})\|\phi - h_0\|_{\mathcal{B}} \\
&\leqslant \frac{K_n}{\lambda}|z(s)| + (K_n\widehat{M}\mathcal{D} + M_n + \mathcal{L}_h^{\phi})\|\phi\|_{\mathcal{B}} + (K_n\widehat{M}\mathcal{D} + M_n + \mathcal{L}_h^{\phi})\varpi.
\end{aligned}$$

Hence

$$\left\| \frac{z_{\rho(s,\frac{z_s}{\lambda}+x_s)}}{\lambda} + x_{\rho(s,\frac{z_s}{\lambda}+x_s)} \right\|_{\mathcal{B}} \leqslant \frac{K_n}{\lambda}|z(s)| + c_n \,. \tag{10}$$

Use the function $u(.)$ and the nondecreasing character of ψ to get

$$\begin{aligned}
\frac{u(t)}{\lambda} \leqslant\ & \overline{M_0}L(\widehat{M}+1) + n\widehat{M}L + \widehat{M}\overline{M_0}L\|\phi\|_{\mathcal{B}} + \widehat{M}\int_0^n |g(s,0)|\mathrm{d}s \\
&+ \overline{M_0}\,L\,(K_n u(t) + c_n) + \widehat{M}L\int_0^t (K_n u(s) + c_n)\,\mathrm{d}s \\
&+ \widehat{M}\int_0^t p(s)\psi\,(K_n u(s) + c_n)\,\mathrm{d}s \\
&+ \widehat{M}\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s \\
\leqslant\ & \overline{M_0}L(\widehat{M}+1) + n\widehat{M}L + \overline{M_0}L\left(M_n + \mathcal{L}_h^{\phi} + \widehat{M}(K_n\mathcal{D}+1)\right)\|\phi\|_{\mathcal{B}} \\
&+ \overline{M_0}\,L(K_n\widehat{M}\mathcal{D} + M_n + \mathcal{L}_h^{\phi})\varpi + \widehat{M}\int_0^n |g(s,0)|\mathrm{d}s \\
&+ \overline{M_0}\,L\frac{K_n}{\lambda}u(t) + \widehat{M}L\int_0^t \left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s \\
&+ \widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s + \widehat{M}\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s) + c_n\right)\mathrm{d}s.
\end{aligned}$$

Set

$$\chi_n = \overline{M_0}\,L(\widehat{M}+1)+n\widehat{M}L+\overline{M_0}L\left(M_n+\mathcal{L}_h^{\phi}+\widehat{M}(K_n\mathcal{D}+1)\right)\|\phi\|_{\mathcal{B}}$$

$$+\ \overline{M_0}\,L(K_n\widehat{M}\mathcal{D}+M_n+\mathcal{L}_h^{\phi})\varpi+\widehat{M}\int_0^n |g(s,0)|\,\mathrm{d}s.$$

Then

$$\frac{u(t)}{\lambda}-\overline{M_0}LK_n\frac{u(t)}{\lambda}\leqslant \chi_n+\widehat{M}L\int_0^t\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s$$

$$+\ \widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s$$

$$+\ \widehat{M}\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s.$$

Hence, multiplying by K_n, we obtain

$$\frac{K_n}{\lambda}(1-\overline{M_0}LK_n)u(t)\leqslant K_n\chi_n+K_n\widehat{M}L\int_0^t\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s$$

$$+\ K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s$$

$$+\ K_n\widehat{M}\int_0^t \eta(s)\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s$$

$$\leqslant K_n\chi+K_n\widehat{M}\int_0^t (L+\eta(s))\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s$$

$$+\ K_n\widehat{M}\int_0^t p(s)\psi\left(\frac{K_n}{\lambda}u(s)+c_n\right)\mathrm{d}s.$$

Set $\xi_n := \dfrac{K_n \chi_n}{1 - \overline{M_0}\, L K_n} + c_n$. Then

$$K_n \frac{u(t)}{\lambda} + c_n \leqslant \xi_n + \frac{K_n \widehat{M}}{1 - \overline{M_0}\, L K_n} \int\limits_0^t (L + \eta(s)) \left(\frac{K_n}{\lambda} u(s) + c_n \right) \mathrm{d}s$$

$$\frac{K_n \widehat{M}}{1 - \overline{M_0}\, L K_n} \int\limits_0^t p(s) \psi \left(\frac{K_n}{\lambda} u(s) + c_n \right) \mathrm{d}s.$$

We consider the function μ defined by:

$$\mu(t) = \sup \left\{ \frac{K_n}{\lambda} u(t) + c_n : 0 \leqslant s \leqslant t \right\} \qquad 0 \leqslant t \leqslant +\infty.$$

Let $t^* \in [0, t]$ be such that $\mu(t) = \dfrac{K_n}{\lambda} u(t^*) + c_n$ by the inequality, we have for $t \in [0, n]$

$$\mu(t) \leqslant \xi_n + \frac{K_n \widehat{M}}{1 - \overline{M_0} L K_n} \int\limits_0^t (L + \eta(s))\, \mu(s) \mathrm{d}s$$

$$+ \frac{K_n \widehat{M}}{1 - \overline{M_0} L K_n} \int\limits_0^t p(s) \psi(\mu(s)) \mathrm{d}s.$$

Let us take the right-hand side the above inequality as $v(t)$. Then we have

$$\mu(t) \leqslant v(t) \qquad \text{for all } t \in [0, n].$$

From the definition of v, we get $v(0) = \xi_n$ and

$$v'(t) = \frac{K_n \widehat{M}}{1 - \overline{M_0} L K_n} \left[(L + \eta(t)) v(t) + p(t) \psi(v(t)) \right].$$

Therefore,

$$\int\limits_{\xi_n}^{v(t)} \frac{ds}{s + \psi(s)} \leqslant \frac{K_n \widehat{M}}{1 - \overline{M_0} L K_n} \int\limits_0^t \max \left(p(s), L + \eta(s) \right) \mathrm{d}s$$

$$\leqslant \frac{K_n \widehat{M}}{1 - \overline{M_0}\, L K_n} \int\limits_0^n \max \left(p(s), L + \eta(s) \right) \mathrm{d}s < \int\limits_{\xi_n}^{+\infty} \frac{ds}{s + \psi(s)}$$

Thus, for every $t \in [0,n]$, there exists a constant $\widetilde{\Phi_n}$ such that $v(t) \leqslant \widehat{\Phi_n}$ and hence $\mu(t) \leqslant \widetilde{\Phi}_n$. Since $\|z\|_n \leqslant \mu(t)$, we have$\|z\|_n \leqslant \widetilde{\Phi}_n$. This shows that the set $\widetilde{\Gamma}$ is bounded. Then the statement (Av2) in Theorem 1 does not hold. The nonlinear alternative of Avramescu implies that (Av1) is satisfied, we deduce that the operator $F + \widetilde{G}$ has a fixed point z^*. Then $y^*(t) = z^*(t) + x(t)$, $t \in \mathbb{R}$ is the fixed point of the operator $\widetilde{N}$ which is a mild solution of the problem (3)–(4).

5. Examples

To illustrate the previous results, we give in this section two examples.

5.1. Example 1.

Consider the partial perturbed evolution equations

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\xi) = \dfrac{\partial^2 v}{\partial \xi^2}(t,\xi) + a_0(t,\xi)v(t,\xi) \\ \quad + \displaystyle\int_{-\infty}^{0} a_1(s-t)v\left[s - \rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\theta)|v(t,\theta)|^2 d\theta\right), \xi\right] \mathrm{d}s, \\ \quad + \displaystyle\int_{-\infty}^{0} a_3(s-t)v\left[s - \rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\theta)|v(t,\theta)|^2 d\theta\right), \xi\right] \mathrm{d}s, \\ \hfill t \geqslant 0,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \hfill t \geqslant 0, \\ v(\theta,\xi) + \displaystyle\sum_{j=1}^{p} c_j v(\theta + t_j, \xi) = v_0(\theta,\xi), \quad -\infty < \theta \leqslant 0,\ \xi \in [0,\pi], \end{cases} \tag{11}$$

where $a_0 : \mathbb{R}^+ \times [0,\pi] \longrightarrow \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t; $a_1, a_3 : \mathbb{R}_- \longrightarrow \mathbb{R}$ and $a_2 : [0,\pi] \longrightarrow \mathbb{R}$, $\rho_i : \mathbb{R}^+ \longrightarrow \mathbb{R}$ for $i = 1,2$ c_j, $j = 1,\cdots,p$, are given constants and $0 < t_1 < \cdots < t_p < +\infty$.

Let $E = L^2([0,\pi],\mathbb{R})$, and the operator $A : D(A) \subset E \longrightarrow E$ given by $Aw = w''$ with domain

$$D(A) := \{w \in E : w'' \in E, w(0) = w(\pi) = 0\}.$$

Thus A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geqslant 0}$ on E. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and corresponding normalized eigenfunctions give by

$$y_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi).$$

In addition, $\{y_n : n \in \mathbb{N}\}$ is an orthonormal basis of E and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}(x, y_n)y_n \quad x \in E, t \geqslant 0.$$

It follows from this representation that $T(t)$ is compact for every $t > 0$ and that

$$\|T(t)\| \leqslant e^{-t} \quad \text{for every } t \geqslant 0.$$

On the domain $D(A)$, we define the operators $A(t) : D(A) \subset E \to E$ by

$$A(t)x(\xi) = Ax(\xi) + a_0(t, \xi)x(\xi).$$

By assuming that $a_0(\cdot)$ is continuous and that $a_0(t, \xi) \leqslant -\delta_0$ $(\delta_0 > 0)$ for every $t \in \mathbb{R}$, $\xi \in [0, \pi]$, it follows that the system

$$u'(t) = A(t)u(t) \quad t \geqslant s,$$
$$u(s) = x \in E,$$

has an associated evolution family given by

$$U(t, s)x(\xi) = \left[T(t-s) \exp\left(\int_s^t a_0(\tau, \xi) d\tau \right) x \right] (\xi).$$

From this expression, it follows that $U(t, s)$ is a compact linear operator and that

$$\|U(t, s)\| \leqslant e^{(1+\delta_0)(t-s)} \text{ for every } (t, s) \in \Delta$$

Theorem 4. *Let $\mathcal{B} = BUC(\mathbb{R}_-; E)$ and $\phi - h_t \in \mathcal{B}$. Assume that condition (H_ϕ) holds. Suppose that the functions $a_1, a_3 : \mathbb{R}^- \to \mathbb{R}$, $a_2 : [0, \pi] \to \mathbb{R}^+$ and $\rho_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2$, are continuous. Then there exists a mild solution of partial perturbed evolution equation with state dependent delay (11) is on $\mathbb{R}$.*

Proof. From the assumptions, we have that

$$f(t, \psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s, \xi)\mathrm{d}s, \qquad t \geqslant 0, \ \xi \in [0, \pi],$$

$$\rho(t, \psi)(\xi) = t - \rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\theta)|\psi(0, \xi)|^2 \mathrm{d}\theta \right), \qquad t \geqslant 0, \ \xi \in [0, \pi],$$

$$g(t, \psi)(\xi) = \int_{-\infty}^{0} a_3(s)\psi(s, \xi)\mathrm{d}s, \qquad t \geqslant 0, \ \xi \in [0, \pi],$$

and

$$h_t(v)(\xi) = \sum_{j=1}^{p} c_j v(t+t_j,\xi), \qquad t \leqslant 0,\ \xi \in [0,\pi],$$
$$\phi(t)(\xi) = v_0(t,\xi), \qquad t \leqslant 0,\ \xi \in [0,\pi],$$

are well defined functions, which permit to transform system (11) into the abstract system (1)–(2). Moreover, the functions f and g are bounded and linear. Now, the existence of mild solutions can be deduced from a direct application of Theorem 2. From Remark 3.2, we have the following result

Corollary 2. *Let $\phi - h_t \in \mathcal{B}$ be continuous and bounded. Then the problem (11) has a mild solution on $\mathbb{R}$.*

5.2. Example 2.

Consider the neutral perturbed evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t}\left[v(t,\xi) - \displaystyle\int_{-\infty}^{0} a_4(s-t)v(s-\rho_1(t)\rho 2(\int_0^{\pi} a_2\left(\theta)|v(t,\theta)|^2 d\theta),\xi\right) \mathrm{d}s\right] \\ = \dfrac{\partial^2 v}{\partial \xi^2}(t,\xi) + a_0(t,\xi)v(t,\xi) \\ \quad + \displaystyle\int_{-\infty}^{0} a_1(s-t)v\left[s-\rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\theta)|v(t,\theta)|^2 d\theta\right),\xi\right] \mathrm{d}s \\ \quad + \displaystyle\int_{-\infty}^{0} a_3(s-t)v\left[s-\rho_1(t)\rho_2\left(\int_0^{\pi} a_2(\theta)|v(t,\theta)|^2 d\theta\right),\xi\right] \mathrm{d}s, t \geqslant 0,\ \xi \in [0,\pi] \\ v(t,0) = v(t,\pi) = 0, \qquad t \geqslant 0, \\ v(\theta,\xi) + \displaystyle\sum_{j=1}^{p} c_j v(\theta+t_j,\xi) = v_0(\theta,\xi), \qquad -\infty < \theta \leqslant 0,\ \xi \in [0,\pi], \end{cases} \tag{12}$$

where $a_4 : \mathbb{R}_- \longrightarrow \mathbb{R}$ is a continuous function.

Theorem 5. *Let $\mathcal{B} = BUC(R_-, E)$ and $\phi - h_t \in \mathcal{B}$. Assume that the condition (H_ϕ) holds. Suppose that the functions $a_1, a_3, a_4 : \mathbb{R}_- \longrightarrow \mathbb{R}$, $a_2 : [0,\pi] \longrightarrow \mathbb{R}$ and $\rho_i : \mathbb{R}_- \longrightarrow \mathbb{R}$ for $i = 1,2$ are continuous. Then there exists a mild solution of (12) on $\mathbb{R}$.*

Proof. From the assumptions, we have that

$$
\begin{aligned}
f(t,\psi)(\xi) &= \int_{-\infty}^{0} a_1(s)\psi(s,\xi)\,\mathrm{d}s, && t \geqslant 0,\ \xi \in [0,\pi],\\
\rho(t,\psi)(\xi) &= t - \rho_1(t)\rho_2\left(\int_{0}^{\pi} a_2(\theta)|\psi(0,\xi)|^2\,\mathrm{d}\theta\right), && t \geqslant 0,\ \xi \in [0,\pi],\\
Q(t,\psi)(\xi) &= \int_{0}^{-\infty} a_4(s)\psi(s,\xi)\,\mathrm{d}s, && t \geqslant 0,\ \xi \in [0,\pi],\\
g(t,\psi)(\xi) &= \int_{-\infty}^{0} a_3(s)\psi(s,\xi)\,\mathrm{d}s, && t \geqslant 0,\ \xi \in [0,\pi],
\end{aligned}
$$

and

$$
\begin{aligned}
h_t(v)(\xi) &= \sum_{j=1}^{p} c_j v(t+t_j,\xi), && t \leqslant 0,\ \xi \in [0,\pi],\\
\phi(t)(\xi) &= v_0(t,\xi), && t \leqslant 0,\ \xi \in [0,\pi],
\end{aligned}
$$

are well defined functions, which permit to transform system (12) into the abstract system (3)–(4). Moreover, the functions f, g and h are bounded and linear. Now, the existence of mild solutions can be deduced from a direct application of Theorem 3. We have the following result

Corollary 3. *Let $\phi - h_t \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (12) *on* $\mathbb{R}$.

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