BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 58, No. 4, 2010
DOI: 10.2478/v10175-010-0063-x

Stability analysis of a discrete-time system with a variable-, fractional-order controller

P. OSTALCZYK*

Institute of Automatics, Łodź University of Technology, 18/22 Stefanowskiego St., 90-924 Łódź, Poland

Abstract. Variable, fractional-order backward difference is a generalisation of commonly known difference or sum. Equations with these differences can be used to describe a variable-, fractional order digital control strategies. One should mention, that classical tools such as a state-space description and discrete transfer function cannot be used in the analysis and synthesis of such a type of systems. Equations describing a closed-loop system are proposed. They contain square matrices imitating the action of matrices in the system polynomial matrix description. This paper focuses on the stability analysis of a closed-loop SISO linear system with a controller described by the equations mentioned. A stability condition based on a transient denominator matrix condition number is proposed. Investigations are supported by two numerical examples.

Key words: fractional calculus, discrete-time controller, stability.

1. Introduction

The fractional calculus and fractional differential equations [1–4], as a very potential mathematical tool in different areas of engineering and science are applied to create more adequate models of real dynamical systems. In automatics it may be used to create more sophisticated control strategies. A paper is organised as follows. In Sec. 2 a definition of a variable-, fractional-order backward difference (VFOBD) is given [5–7]. With the application of the VFOBD yet some description difficulties are related. A system described by the VFOBD equation cannot be described using the state-space equations. Also the Laplace and Z transform cannot be applied. Hence a new description is proposed. Section 3 presents the main result of the paper – stability conditions of the system described by the VFOBD equation. Next a description of a closed-loop system is discussed. The investigations are explained by two numerical examples.

2. Mathematical preliminaries

2.1. Variable-, fractional-order backward difference of f_k. One defines a VFOBD of a discrete function f_k as a discrete convolution of a function f_k

$$ {}_0\Delta_k^{(\nu_k)} f_k = a_k^{(\nu_k)} * f_k \tag{1}$$

with a discrete function

$$a_k^{(\nu_k)} = \begin{cases} 1 & \text{for} \quad k=0 \\ (-1)^k \frac{\nu_k(\nu_k-1)\ldots(\nu_k-k+1)}{k!} & \text{for} \quad k=1,2,3,\ldots \end{cases} \tag{2}$$

where the term ν_k means the value of a (bounded) order function. One should realise that for a constant order function (i.e. $\nu_k = v = \text{const}$) one obtains in general fractional-order backward difference, and for $\nu_k = n \in \mathbb{Z}_+$ formula (1) defines a classical backward difference of order n. And for $n = 1$ one gets ${}_0\Delta_k^{(\nu_k)} f_k = \Delta f_k = f_k - f_{k-1}$. VFOBD may also be expressed in the form

$$ {}_0\Delta_k^{(\nu_k)} f_k = \sum_{i=0}^{k} a_i^{(\nu_k)} f_{k-1} = $$

$$ = \left[a_0^{(\nu_k)} \; a_1^{(\nu_k)} \; \ldots \; a_{k-1}^{(\nu_k)} \; a_k^{(\nu_k)}\right] \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_1 \\ f_0 \end{bmatrix}. \tag{3}$$

2.2. Variable-, fractional-order linear time-invariant difference equation. Combining VFOBDs with constant coefficients one builds a linear, variable-, fractional-order difference equation

$$\sum_{i=0}^{p} A_i \; {}_0\Delta_k^{(\nu_{i,k})} y_k = \sum_{j=0}^{q} B_i \; {}_0\Delta_k^{(\mu_{j,k})} u_k, \tag{4}$$

where

$\nu_{i+1,k} > \nu_{i,k} > 0$ for $i = 1,\ldots,p-1$ and $k \in \mathbb{Z}_+$,
$\mu_{j+1,k} > \mu_{j,k} > 0$ for $j = 1,\ldots,q-1$ and $k \in \mathbb{Z}_+$
$A_p = 1$,
u_k – the bounded discrete-time function,
$y_{-1}, y_{-2}, \ldots$ the initial conditions.

Such an equation may describe a wide range of dynamical processes or systems. Only casual systems will be considered. This implies the condition

$$\nu_{p,k} \geq \mu_{q,k} \tag{5}$$

*e-mail: postalcz@o.lodz.pl

and one can state that the difference equation (4) is of a variable-, fractional-order $\nu_{p,k}$. It can be equivalently expressed in the form

$$[a_{k,0}\ a_{k,1} \ldots a_{k,k-1}\ a_{k,k}] \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix} = = [b_{k,0}\ b_{k,1} \ldots b_{k,k-1}\ b_{k,k}] \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}, \tag{6}$$

where

$$[a_{k,0}\ a_{k,1} \ldots a_{k,k-1}\ a_{k,k}] = [A_0\ A_1\ \ldots A_{p-1}\ A_p] \times \\ \times \begin{bmatrix} a_0^{(\nu_{0,k})} & a_1^{(\nu_{0,k})} & \ldots & a_{k-1}^{(\nu_{0,k})} & a_k^{(\nu_{0,k})} \\ a_0^{(\nu_{1,k})} & a_1^{(\nu_{1,k})} & \ldots & a_{k-1}^{(\nu_{1,k})} & a_k^{(\nu_{1,k})} \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^{(\nu_{p-1,k})} & a_1^{(\nu_{p-1,k})} & \ldots & a_{k-1}^{(\nu_{p-1,k})} & a_k^{(\nu_{p-1,k})} \\ a_0^{(\nu_{p,k})} & a_1^{(\nu_{p,k})} & \ldots & a_{k-1}^{(\nu_{p,k})} & a_k^{(\nu_{p,k})} \end{bmatrix}, \tag{7}$$

$$[b_{k,0}\ b_{k,1} \ldots b_{k,k-1}\ b_{k,k}] = [B_0\ B_1\ \ldots B_{q-1}\ B_q] \times \\ \times \begin{bmatrix} a_0^{(\mu_{0,k})} & a_1^{(\mu_{0,k})} & \ldots & a_{k-1}^{(\mu_{0,k})} & a_k^{(\mu_{0,k})} \\ a_0^{(\mu_{1,k})} & a_1^{(\mu_{1,k})} & \ldots & a_{k-1}^{(\mu_{1,k})} & a_k^{(\mu_{1,k})} \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^{(\mu_{q-1,k})} & a_1^{(\mu_{q-1,k})} & \ldots & a_{k-1}^{(\mu_{q-1,k})} & a_k^{(\mu_{q-1,k})} \\ a_0^{(\mu_{q,k})} & a_1^{(\mu_{q,k})} & \ldots & a_{k-1}^{(\mu_{q,k})} & a_k^{(\mu_{q,k})} \end{bmatrix}, \tag{8}$$

From (5) it follows that

$$a_{k,0} \neq 0 \quad \text{for} \quad k \in \mathbb{Z}_+. \tag{9}$$

On the contrary, the difference equation (4) would be of order $\nu_{p-1,k} < \nu_{p,k}$. One divides both sides of equation (6) by $a_{k,0}$. In causal systems

$$b_{k,0}, b_{k,1}, \ldots, b_{k,l} = 0 \quad \text{for} \quad k \in \mathbb{Z}_+, \ \ l \in \mathbb{Z}_+. \tag{10}$$

2.3. Discrete-time system description. One considers a linear discrete-time dynamical system described by equation (6) and now one takes into account the initial conditions $y_{-1}, y_{-2}, \ldots$. It should be emphasised that in the case of a variable-, fractional-order discrete-time system (VFODS) one should regard an infinite number of initial conditions. Collecting all such equations evaluated for consecutive time instants k, $k-1$, $k-2, \ldots 1, 0$ in a vector matrix form one obtains

$$\mathbf{D}_k \mathbf{y}_k = \mathbf{N}_k \mathbf{u}_k + \mathbf{M}_{k,\infty} \mathbf{y}_0, \tag{11}$$

where

$$\mathbf{D}_k = \begin{bmatrix} a_{0,k} & a_{1,k} & \ldots & a_{k-1,k} & a_{k,k} \\ 0 & a_{0,k-1} & \ldots & a_{k-2,k-1} & a_{k-1,k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & a_{0,1} & a_{1,1} \\ 0 & 0 & \ldots & 0 & a_{0,0} \end{bmatrix}, \tag{12}$$

where $a_{i,0} = 1$ for $i = 0, 1, \ldots, k$,

$$\mathbf{N}_k = \begin{bmatrix} b_{0,k} & b_{1,k} & \ldots & b_{k-1,k} & b_{k,k} \\ 0 & b_{0,k-1} & \ldots & b_{k-2,k-1} & b_{k-1,k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & b_{0,1} & b_{1,1} \\ 0 & 0 & \ldots & 0 & b_{0,0} \end{bmatrix}, \tag{13}$$

$$\mathbf{M}_{k,\infty} = - \begin{bmatrix} a_{k+1,k} & a_{+2,kk} & a_{k+3,k} & a_{k+4,k} & \ldots \\ a_{k,k-1} & a_{k+1,k-1} & a_{k+2,k-1} & a_{k+3,k-1} & \ldots \\ \vdots & \vdots & & \vdots & \vdots \\ a_{2,1} & a_{3,1} & a_{4,1} & a_{5,1} & \ldots \\ a_{1,0} & a_{2,0} & a_{3,0} & a_{4,0} & \ldots \end{bmatrix}, \tag{14}$$

are transient- denominator, nominator and initial conditions matrices, respectively. Moreover,

$$\mathbf{y}_k = \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_1 \\ y_0 \end{bmatrix}, \ \mathbf{u}_k = \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}, \ \mathbf{y}_0 = \begin{bmatrix} y_{-1} \\ y_{-2} \\ y_{-3} \\ y_{-4} \\ \vdots \end{bmatrix} \tag{15}$$

are output, input and initial condition vectors, respectively. From equality (11) with assumption (9) one obtains

$$\mathbf{y}_k = \mathbf{D}_k^{-1} \mathbf{N}_k \mathbf{u}_k + \mathbf{D}_k^{-1} \mathbf{M}_{k,\infty} \mathbf{y}_0. \tag{16}$$

Assuming zero initial conditionsm, equation (16) simplifies to

$$\mathbf{y}_k = \mathbf{P}_k \mathbf{u}_k, \tag{17}$$

where $(k+1) \times (k+1)$ matrix $\mathbf{P}_k$

$$\mathbf{P}_k = \mathbf{D}_k^{-1} \mathbf{N}_k \tag{18}$$

is defined as a transient transfer matrix. For $\nu_{i,k} = \nu_i = \mathrm{const} \in \mathbb{R}_+ \backslash \mathbb{Z}_+$, $i = 1, \ldots, p$ the transient denominator matrix and its inverse are upper triangular k-band (i.e. having k diagonals containing the same elements) matrices

$$\mathbf{D}_k = \begin{bmatrix} a_0 & a_1 & \dots & a_{k-1} & a_k \\ 0 & a_0 & \dots & a_{k-2} & a_{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & \dots & 0 & a_0 \end{bmatrix},$$
$$\mathbf{D}_k^{-1} = \begin{bmatrix} \overline{a}_0 & \overline{a}_1 & \dots & \overline{a}_{k-1} & \overline{a}_k \\ 0 & \overline{a}_0 & \dots & \overline{a}_{k-2} & \overline{a}_{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \overline{a}_0 & \overline{a}_1 \\ 0 & 0 & \dots & 0 & \overline{a}_0 \end{bmatrix}. \tag{19}$$

For $\nu_{i,k} = n_i = \text{const} \in \mathbb{Z}_+$, $i = 1, \dots, p$ the transient denominator matrix is an upper triangular l-band matrix but its inverse is still an upper triangular k-band matrix

$$\mathbf{D}_k = \begin{bmatrix} a_0 & \dots & a_l & \dots & 0 & 0 & 0 \\ 0 & a_0 & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & a_0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_0 & \vdots & a_l \\ 0 & 0 & 0 & \dots & 0 & a_0 & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & a_0 \end{bmatrix},$$
$$\mathbf{D}_k^{-1} = \begin{bmatrix} \overline{a}_0 & \overline{a}_1 & \dots & \overline{a}_{k-1} & \overline{a}_k \\ 0 & \overline{a}_0 & \dots & \overline{a}_{k-2} & \overline{a}_{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \overline{a}_0 & \overline{a}_1 \\ 0 & 0 & \dots & 0 & \overline{a}_0 \end{bmatrix}. \tag{20}$$

3. Stability analysis of the VFODS

In this Section the VFODS response is analysed. Equality (16) shows that the system response is made up of two terms

$$\mathbf{y}_k = \mathbf{y}_{F,k} + \mathbf{y}_{H,k}. \tag{21}$$

The first term

$$\mathbf{y}_{F,k} = \mathbf{D}_k^{-1}\mathbf{N}_k\mathbf{u}_u \tag{22}$$

denotes a forced system response while the second one

$$\mathbf{y}_{H,k} = \mathbf{D}_k^{-1}\mathbf{M}_{k,\infty}\mathbf{y}_0 \tag{23}$$

is a homogenous system response. Now VFODS (16) stability conditions will be established.

Lemma 1. For $k \to \infty$ elements $a_{k,k}$ of transient denominator and numerator matrices (12) and (13) tend to zero.

Proof. It is proved [8] that for $\nu > 0$, $a_k^{(\nu)} \to 0$ when $k \to \infty$. Elements $a_{i,j}$ of transient denominator matrix (12) are by formula (7) linear combinations of $a_i^{(\nu)}$. This implies $a_{k,k} \to 0$ for $k \to \infty$.

Theorem 1. A SISO system described by transient transfer matrix $\mathbf{P}_k = \mathbf{D}_k^{-1}\mathbf{N}_k$ is BIBO stable if and only if

$$\lim_{k\to\infty} c_k(\mathbf{D}_k) = \lim_{k\to\infty} \frac{\overline{\sigma}_k(\mathbf{D}_k)}{\underline{\sigma}_k(\mathbf{D}_k)} \neq 0, \tag{24}$$

where $\underline{\sigma}_k(\mathbf{D}_k)$ and $\overline{\sigma}_k(\mathbf{D}_k)$ denote the smallest and largest singular value of a matrix $\mathbf{D}_k$, [9], [10] respectively.

Proof. Sufficiency. A necessary and sufficient condition for a linear discrete-, time-invariant system to be BIBO stable is [11], [12]

$$\lim_{k\to\infty} \sum_{i=0}^{k} |y_i| < \infty. \tag{25}$$

Sum (25) can be expressed as a 1-norm

$$\lim_{k\to\infty} \sum_{i=0}^{k} |y_i| = \lim_{k\to\infty} \|\mathbf{y}_k\|_1. \tag{26}$$

There exists $U > 0$, such that

$$\begin{aligned} &\lim_{k\to\infty} \|\mathbf{y}_k\|_1 \le \lim_{k\to\infty} U\|\mathbf{y}_k\|_2 = \\ &= U \lim_{k\to\infty} \|\mathbf{D}_k^{-1}\mathbf{N}_k\mathbf{u}_k\|_2 \le U \lim_{k\to\infty} \}\mathbf{D}_k^{-1}\|_2\|\mathbf{N}_k\mathbf{u}_k\|_2. \end{aligned} \tag{27}$$

The induced 2-norm of a matrix $\mathbf{D}_k$ [10] equals to

$$\|\mathbf{D}_k^{-1}\|_2 = \overline{\sigma}_k(\mathbf{D}_k^{-1}) = \frac{1}{\underline{\sigma}_k(\mathbf{D}_k)}. \tag{28}$$

Substituting expression (28) into (27) one gets

$$U \lim_{k\to\infty} \|\mathbf{D}_k^{-1}\|_2\|\mathbf{N}_k\mathbf{u}_k\|_2 = U \lim_{k\to\infty} \frac{\|\mathbf{N}_k\mathbf{u}_k\|_2}{\underline{\sigma}_k(\mathbf{D}_k)}. \tag{29}$$

One can always chose $\mathbf{u}_k$ such that

$$\|\mathbf{N}_k\mathbf{u}_k\|_2 = \frac{\overline{\sigma}_k(\mathbf{D}_k)}{U} < \infty \tag{30}$$

and

$$\lim_{k\to\infty} \sum_{i=0}^{k} |y_i| \le \lim_{k\to\infty} \frac{\overline{\sigma}_k(\mathbf{D}_k)}{\underline{\sigma}_k(\mathbf{D}_k)} = \lim_{k\to\infty} c_k(\mathbf{D}_k). \tag{31}$$

For

$$\lim_{k\to\infty} \sum_{i=0}^{k} |y_i| \le \lim_{k\to\infty} c_k(\mathbf{D}_k) < \infty \tag{32}$$

the system is stable.

Necessity. From (11) with $\mathbf{y}_0 = 0_k$. There exists $U > 0$, such that

$$\begin{aligned} &\lim_{k\to\infty} \|\mathbf{N}_k\mathbf{u}_k\|_2 = \lim_{k\to\infty} \|\mathbf{D}_k\mathbf{y}_k\|_2 \le \lim_{k\to\infty} \|\mathbf{D}_k\|_2\|\mathbf{y}_k\|_2 \le \\ &\le \lim_{k\to\infty} \|\mathbf{D}_k\|_2\|\mathbf{y}_k\|_1 = \lim_{k\to\infty} \|\mathbf{D}_k\|_2 \lim_{k\to\infty} \|\mathbf{y}_k\|_1. \end{aligned} \tag{33}$$

Further transformations yield

$$\lim_{k\to\infty} \|\mathbf{y}_k\|_1 \ge \frac{\lim\limits_{k\to\infty} \|\mathbf{N}_k\mathbf{u}_k\|_2}{\lim\limits_{k\to\infty} \|\mathbf{D}_k\|_2} = \frac{\lim\limits_{k\to\infty} \|\mathbf{N}_k\mathbf{u}_k\|_2}{\overline{\sigma}_k(\mathbf{D})}. \tag{34}$$

One can always chose $\mathbf{u}_k$ such that

$$\|\mathbf{N}_k\mathbf{u}_k\|_2 = \underline{\sigma}_k(\mathbf{D}_k) < \infty. \tag{35}$$

Then

$$\lim_{k\to\infty} \|\mathbf{y}_k\|_1 \ge \frac{\underline{\sigma}_k(\mathbf{D}_k)}{\overline{\sigma}_k(\mathbf{D}_k)} = \frac{1}{\lim\limits_{k\to\infty} c_k(\mathbf{D}_k)} \tag{36}$$

or equivalently

$$\lim_{k\to\infty} c_k(\mathbf{D}_k) \lim_{k\to\infty} \|\mathbf{y}_k\|_1 = 1. \tag{37}$$

If

$$0 < \lim_{k\to\infty} \sum_{i=0}^{k} |\mathbf{y}_i| < \infty \tag{38}$$

from (37) it follows

$$0 < \lim_{k\to\infty} c_k(\mathbf{D}_k) < \infty. \tag{39}$$

The condition number of a matrix relates the linear algebraic equation (11) solution sensitivity to errors in the data.

3.1. Numerical example. One should check the stability condition of a first-order difference equation.

$$_0\Delta_k^{(1)} y_k + A_0 y_k = B_0 u_{k-1}. \tag{40}$$

Equation (40) can be transformed to the form

$$y_k + a_0 y_{k-1} = b_0 u_{k-1}, \tag{41}$$

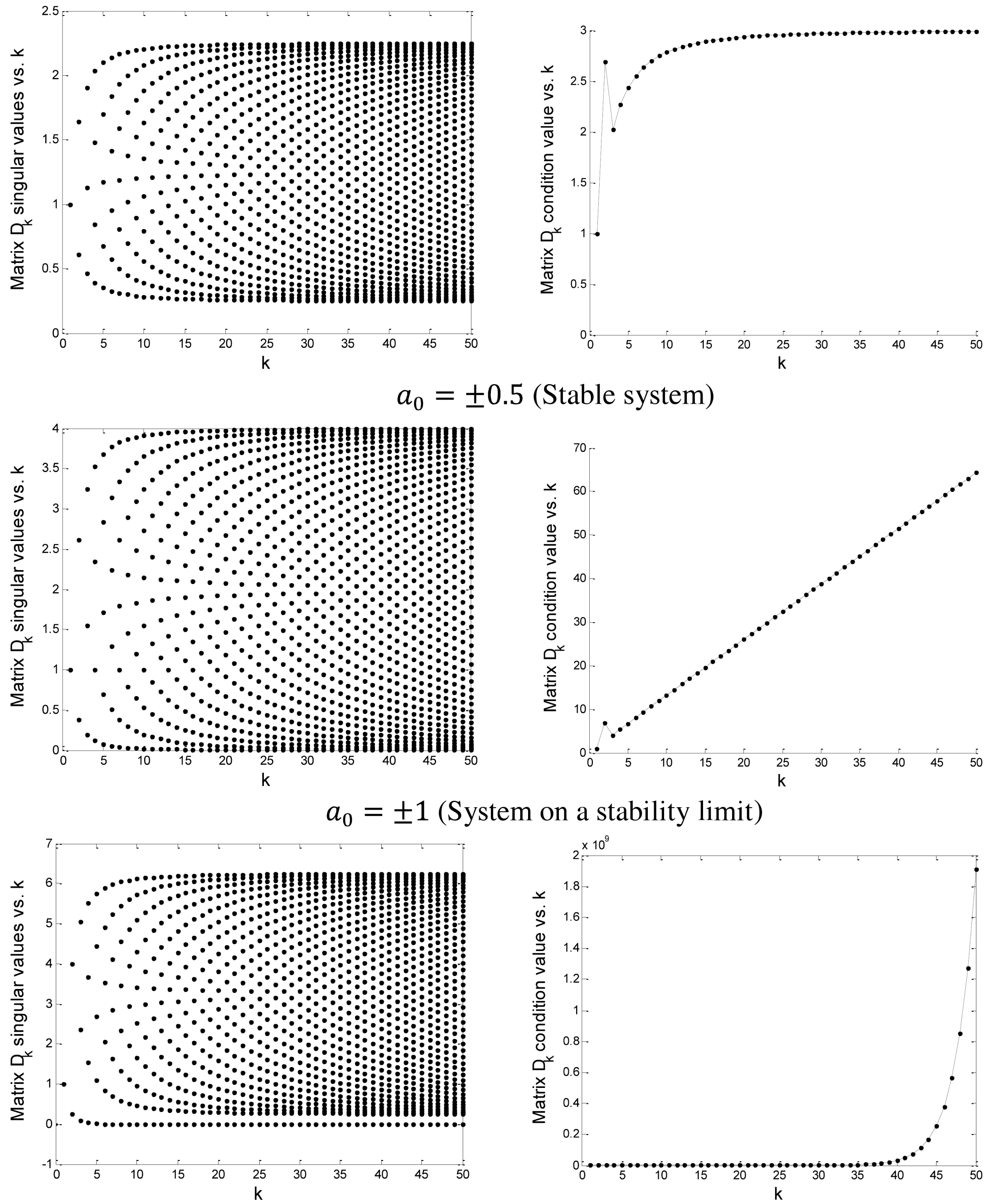

Fig. 1. Singular values and the condition matrix $\mathbf{D}_k$ (34) vs. k for different values of a_0

where $a_0 = b_0 = \dfrac{1}{A_0+1}$. In this example matrix (12) has the form

$$\mathbf{D}_k = \begin{bmatrix} 1 & -a_0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -a_0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -a_0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}, \tag{42}$$

for $k \in 0, 1, \dots$.

In Fig. 1, plots of numerically evaluated singular values and related condition numbers of matrix (42) vs. k are presented for six different values of $a_0 = b_0 = \pm 0.5, \pm 1.0, \pm 1.5$.

4. Stability conditions of a closed-loop system

A feedback system with the block diagram presented in Fig. 2 is considered [10]

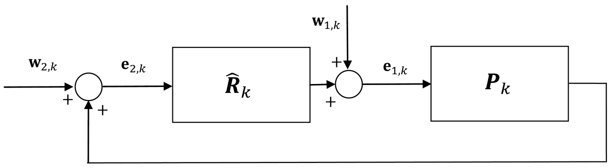

Fig. 2. Block diagram of a closed-loop system

where

$$\mathbf{P}_k = \mathbf{D}_k^{-1}\mathbf{N}_{P,k}, \quad \text{for} \quad k \in 0, 1, \dots \tag{43}$$

are a plant and

$$\begin{aligned} \mathbf{R}_k &= \mathbf{D}_{R,k}^{-1}\mathbf{N}_{R,k}, \quad \text{for} \quad k \in 0, 1, \dots \\ \widehat{\mathbf{R}}_k &= -\mathbf{R}_k \end{aligned} \tag{44}$$

a controller transient transfer matrices, respectively. A block diagram presented in Fig. 2 can be rearranged into an equivalent form shown in Fig. 3. Additional equations describe a closed-loop system.

$$\begin{aligned} \mathbf{e}_{1,k} &= \mathbf{w}_{1,k} + \widehat{\mathbf{R}}_k \mathbf{e}_{2,k} \\ \mathbf{e}_{2,k} &= \mathbf{w}_{2,k} + \mathbf{P}_k \mathbf{e}_{1,k} \end{aligned} \tag{45}$$

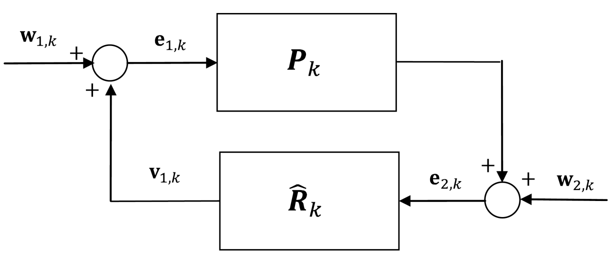

Fig. 3. Transformed block diagram of a closed-loop system

Equations (46) can be represented in a block-matrix form

$$\begin{bmatrix} \mathbf{I}_k & -\widehat{\mathbf{R}}_k \\ -\mathbf{P}_k & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1,k} \\ \mathbf{e}_{2,k} \end{bmatrix} = \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix}. \tag{46}$$

Substitution of (35) and (36) into (38) yields

$$\begin{aligned} &\begin{bmatrix} \mathbf{I}_k & -\mathbf{D}_{R,k}^{-1}\mathbf{N}_{R,k} \\ -\mathbf{D}_{P,k}^{-1}\mathbf{N}_{P,k} & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1,k} \\ \mathbf{e}_{2,k} \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{D}_{R,k}^{-1}\mathbf{D}_{R,k} & \mathbf{D}_{R,k}^{-1}\mathbf{N}_{R,k} \\ -\mathbf{D}_{P,k}^{-1}\mathbf{N}_{P,k} & \mathbf{D}_{P,k}^{-1}\mathbf{D}_{P,k} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1,k} \\ \mathbf{e}_{2,k} \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{D}_{R,k}^{-1} & 0_k \\ 0_k & \mathbf{D}_{P,k}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{R,k} & -\mathbf{N}_{R,k} \\ -\mathbf{N}_{P,k} & \mathbf{D}_{P,k} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1,k} \\ \mathbf{e}_{2,k} \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix} \end{aligned} \tag{47}$$

and further

$$\begin{aligned} &\begin{bmatrix} \mathbf{e}_{1,k} \\ \mathbf{e}_{2,k} \end{bmatrix} = \\ &= \left(\begin{bmatrix} \mathbf{D}_{R,k}^{-1} & 0_k \\ 0_k & \mathbf{D}_{P,k}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{D}_{R,k} & -\mathbf{N}_{R,k} \\ -\mathbf{N}_{P,k} & \mathbf{D}_{P,k} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1,k} \\ \mathbf{e}_{2,k} \end{bmatrix} \right)^{-1} \times \\ &\times \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix} = \\ &= \begin{bmatrix} \mathbf{D}_{R,k} & -\mathbf{N}_{R,k} \\ -\mathbf{N}_{P,k} & \mathbf{D}_{P,k} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{D}_{R,k} & 0_k \\ 0_k & \mathbf{D}_{P,k} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix}. \end{aligned}$$

Hence one obtains a description of the closed-loop system

$$\begin{aligned} &\begin{bmatrix} \mathbf{e}_{1,k} \\ \mathbf{e}_{2,k} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_k & -\widehat{\mathbf{R}}_k \\ -\mathbf{P}_k & \mathbf{I}_k \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix} = \\ &= \begin{bmatrix} (\mathbf{I}_k - \widehat{\mathbf{R}}_k\mathbf{P}_k)^{-1} & \widehat{\mathbf{R}}_k(\mathbf{I}_k - \mathbf{P}_k\widehat{\mathbf{R}}_k)^{-1} \\ \mathbf{P}_k(\mathbf{I}_k - \widehat{\mathbf{R}}_k\mathbf{P}_k)^{-1} & (\mathbf{I}_k - \widehat{\mathbf{R}}_k\mathbf{P}_k)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{w}_{1,k} \\ \mathbf{w}_{2,k} \end{bmatrix}. \end{aligned} \tag{48}$$

The closed-loop system is stable if and only if

$$\lim_{k\to\infty} c_k \left(\begin{bmatrix} \mathbf{D}_{R,k} & -\mathbf{N}_{R,k} \\ -\mathbf{N}_{P,k} & \mathbf{D}_{P,k} \end{bmatrix} \right) < \infty. \tag{49}$$

4.1. Numerical example. One considers a linear discrete-time plant described by a discrete transfer function

$$P(z) = \frac{40}{(z-0.528)(z-0.952)} \tag{50}$$

or by an equivalent description (35) with

$$\mathbf{D}_{P,k} = \begin{bmatrix} 1 & -1.48 & 0.502 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -1.48 & 0.502 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1.48 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1.48 & 0.502 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -1.48 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}, \tag{51}$$

$$\mathbf{N}_{P,k} = \begin{bmatrix} 0 & 0 & 40 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 40 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 40 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}. \tag{52}$$

A variable-, fractional-order controller is described by the VFODE

$$\nu_{1,k} = \frac{1}{T_I} {}_0\Delta_k^{(\nu_{k,1})} e_{2,k} + k_p e_{2,k}, \tag{53}$$

where the order function means [13–16]

$$\nu_{k,1} = \begin{cases} > 0 & \text{PD control strategy} \\ = 0 & \text{P control strategy} \\ < 0 & \text{PI control strategy} \end{cases}. \tag{54}$$

The following controller parameters $k_p = 7.5, T_I = 1.25$ with the VFO

$$\nu_{k,1} = 0.6e^{-\frac{k}{T_I}} - 1 \tag{55}$$

preserve the closed-loop system stability and transient error $e_{k,1}$ presented in Fig. 4.

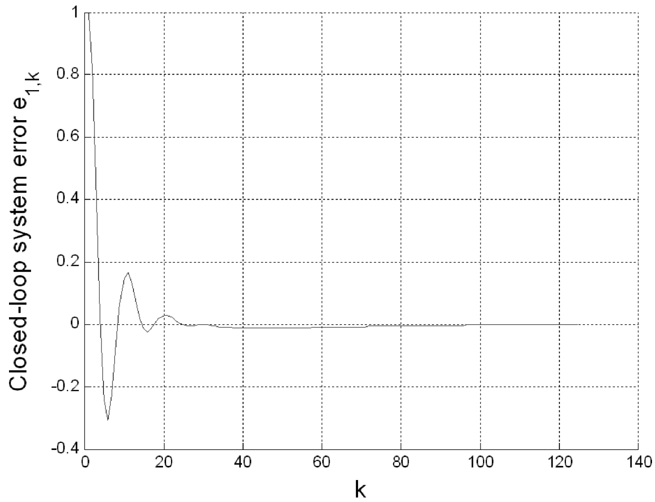

Fig. 4. Plot of the closed-loop system transient error $e_{k,1}$

Below in Fig. 5 the shape of the order function is shown. One can realise that the controller has only an integrating part. The order of integration tends to 1 (to a classical integration). This preserves a zero steady-state error. In Fig. 6 the matrix (41) condition number vs. k is presented.

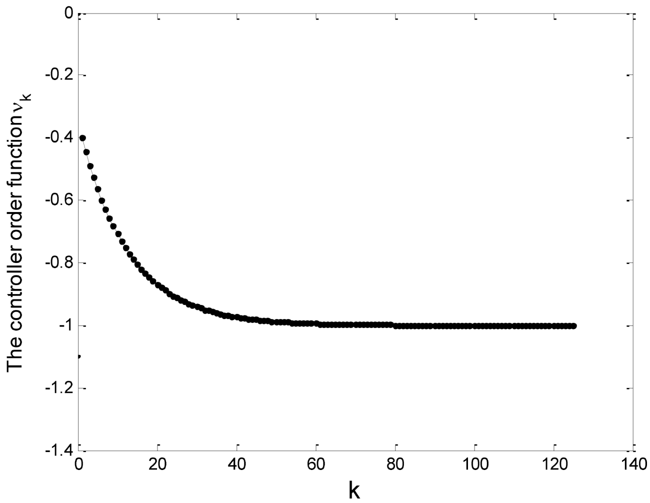

Fig. 5. The controller order function $\nu_{k,1}$

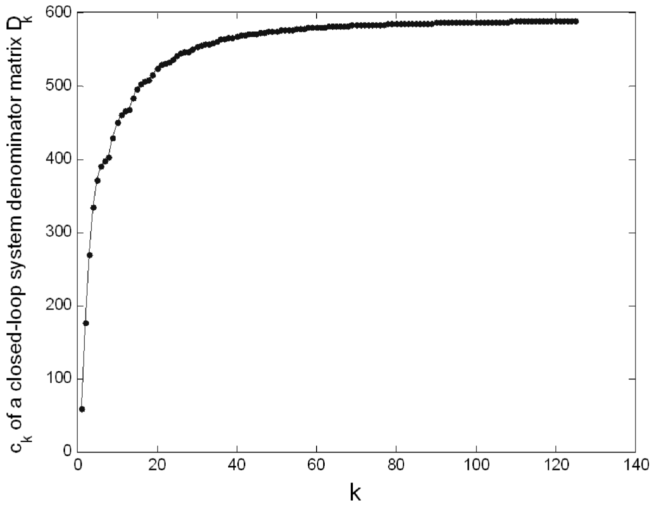

Fig. 6. The closed-loop system matrix (41) condition number vs. k

5. Conclusions

In the considered closed-loop system with the VFO controller there is a wide range of different order functions of the controller. The appropriate choice of the function $\nu_{k,1}$ along with an optimal controller parameters setting gives satisfactory closed-loop transient responses. The choice of an order function is still an open problem. An order function $\nu_{k,1}(\mathbf{e}_{2,k})$ depending on an error function seems to be a promising solution.

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