Duality of variable fractional order difference operators and its application in identification

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Abstract. The paper presents a number of definitions of variable order difference and discusses duality among some of them. The duality is used to improve the performance of the least squares estimation when applied to variable order difference fractional systems. It turns out, that by appropriate exploitation of duality one can reduce the estimator variance when system identification is carried out.

Key words: variable order fractional calculus, duality, identification.

1. Introduction

Fractional calculus generalizes traditional integer order integration and differentiation operators onto non-integer order operators. The idea was first mentioned in 1695 by Leibniz and de l'Hôpital. In the end of 19th century, Liouville and Riemann introduced the first definition of fractional derivative. However, only in late 60' of the 20th century, the idea drew attention of engineers. Theoretical background of fractional calculus can be found in, e.g., [1–6]. Fractional calculus has been found a convenient tool to model behavior of many materials and systems, particularly those involving diffusion processes. For example, ultracapacitors can be modeled more efficiently using fractional calculus, as was demonstrated in [7–10].

Recently, the case when the order is time-varying, the begun to be studied extensively. The variable fractional order behavior can be encountered for example in chemistry when system's properties are changing due to chemical reactions. Experimental studies of an electrochemical example of physical fractional variable order system have been presented in [11]. The variable order equations have been used to describe time evolution of drag expression in [12]. Numerical implementations of fractional variable order integrators and differentiators can be found in, e.g., [11, 13, 14]. The fractional variable order calculus can also be used to describe variable order fractional noise [15]. In [16] the variable order interpretation of the analog realization of fractional orders integrators, realized as domino ladders, has been considered. Applications of variable order derivatives and integrals arise also in signal processing [4] and control [17–20].

In [21,22] three general types of variable order derivative definitions have been given, however, without derivation nor interpretation. The switching scheme numerically identical to the 2$^{\text{nd}}$ type of fractional derivative definition has been introduced in [23]. This scheme can be viewed as an interpretation of this type of definition. An alternative definition of variable order derivative was introduced in [24] together with numer-

ical results of comparison to other known definitions. In our article this method is extended in order to obtain another two new definitions. Moreover, the duality of the new definitions, alternative to that known in the literature, is introduced. Finally, the duality properties are used to improve the least-square estimation algorithm.

The discussion of duality property is based on a discrete fractional order state-space (DFOSS) model. The DFOSS model was used to obtain advanced estimation Kalman like algorithms [25]. The fractional Kalman filter (FKF) algorithm has been used for estimation of unknown state variables in the system with an ultracapacitor [26,27] and in a chaotic secure communication scheme [28]. An algorithm similar to FKF was also used [29] for R-wave detection in electrocardiogram signal. Fractional Kalman-like algorithm has also been used to improve measurement results from microelectromechanical sensors [30].

The paper is organized as follows. In Sec. 2 the fractional derivatives of variable order are presented. In Sec. 3 a new definition for variable-order differintegral is proposed and explained. Section 4 contains the discussion of the notion of duality of the definitions considered. Section 5 demonstrates that duality can be used in order to obtain a solution for a discrete variable order state-space system. Numerical examples together with time plots are given in Sec. 6. Finally, Sec. 7 summarizes the main results.

2. Fractional variable order derivatives

The following definition is a starting point for generalization of constant fractional order difference operators onto a variable order case. A constant fractional order difference operator is defined in the following way

$$
{}_0\Delta_q^\alpha x_k \equiv \sum_{j=0}^q (-1)^j \binom{\alpha}{j} x_{k-j}, \tag{1}
$$

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where

$$\binom{\alpha}{j} \equiv \begin{cases} 1 & \text{for} \quad j = 0, \\ \dfrac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for} \quad j > 0. \end{cases}$$

A number of different definitions of variable order difference operator can be found in literature [21, 22]. The first one is obtained by replacing a constant order α by variable order α_k.

Definition 1. The 1st type of fractional variable-order difference is given by

$$_0^A\Delta_k^{\alpha_k} x_k \equiv \sum_{j=0}^k (-1)^j \binom{\alpha_k}{j} x_{k-j}. \tag{2}$$

The second definition assumes that coefficients for past samples are obtained for order that was present for these samples. The particular switching scheme corresponding to this definition was presented in [23].

The second definition reads.

Definition 2. The 2nd type of fractional variable-order difference is given by

$$_0^B\Delta_k^{\alpha_k} x_k \equiv \sum_{j=0}^k (-1)^j \binom{\alpha_{k-j}}{j} x_{k-j}. \tag{3}$$

The third definition is less intuitive and assumes that coefficients for the newest samples are obtained, respectively, for the oldest orders, and it reads.

Definition 3. The 3rd type of fractional variable-order difference is given by

$$_0^C\Delta_k^{\alpha_k} x_k \equiv \sum_{j=0}^k (-1)^j \binom{\alpha_j}{j} x_{k-j}. \tag{4}$$

Example 1. Step responses of systems described by 1st, 2nd and 3rd type of definitions are depicted in Fig. 1 together with step responses of systems of constant order.
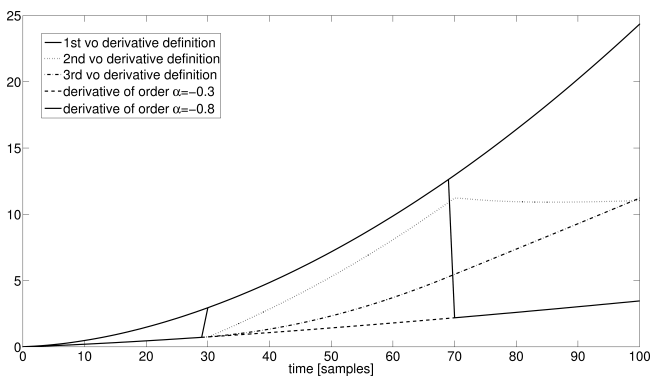

Fig. 1. Step responses for systems described by 1st, 2nd and 3rd type of definitions

3. Other definitions of the variable order difference

Applying the $\mathcal{Z}$ transform to (1) yields

$$\Delta^\alpha(z) = (1 - z^{-1})^\alpha X(z), \tag{5}$$

where $\Delta^\alpha(z)$ denotes the $\mathcal{Z}$ transform of the signal difference of order α of the sequence $\{x_k\}$. Equation (5) can be rewritten as

$$\Delta^\alpha(z)(1 - z^{-1})^{-\alpha} = X(z). \tag{6}$$

Hence, in time domain

$$\sum_{j=0}^k (-1)^j \binom{-\alpha}{j} {_0\Delta_k^\alpha} x_{k-j} = x_k. \tag{7}$$

Finally, one can write

$${_0\Delta_k^\alpha} x_k = x_k - \sum_{j=1}^k (-1)^j \binom{-\alpha}{j} {_0\Delta_{k-j}^\alpha} x_{k-j}. \tag{8}$$

This type of difference is obtained from all values of past differences. For variable-order case one can use the following definition.

Definition 4. [24] The 4th type of fractional variable-order difference is given by

$$_0^D\Delta_k^{\alpha_k} \equiv x_k - \sum_{j=1}^k (-1)^j \binom{-\alpha_k}{j} {_0\Delta_{k-j}^{\alpha_{k-j}}} x_{k-j}. \tag{9}$$

Analogically to the Definition 4, and the method for obtaining coefficients characteristic for Definitions 2 and 3, the following new definitions can be written:

Definition 5. The 5th type of fractional variable-order difference is given by

$$_0^E\Delta_k^{\alpha_k} x_k \equiv x_k - \sum_{j=1}^k (-1)^j \binom{-\alpha_{k-j}}{j} {_0\Delta_{k-j}^{\alpha_{k-j}}} x_{k-j}. \tag{10}$$

Definition 6. The 6th type of fractional variable-order difference is given by

$$_0^F\Delta_k^{\alpha_k} x_k \equiv x_k - \sum_{j=1}^k (-1)^j \binom{-\alpha_j}{j} {_0\Delta_{k-j}^{\alpha_{k-j}}} x_{k-j}. \tag{11}$$

Example 2. Let us consider step responses obtained for composition of switching order difference operators. The composition consists of two operators. The first operator is of order α_k and the second of $-\alpha_k$. The order α_k is equal to 0.5 for $k \leq 50$ and 0.8 for $k > 50$. The output signals obtained in this way are
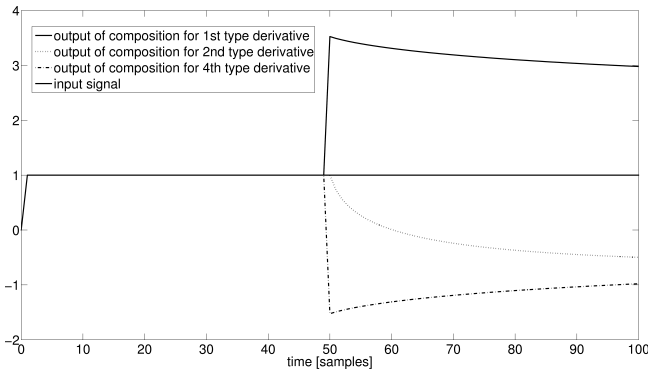
$$y_{A,k} = {_0^A\Delta_k^{\alpha_k}} {_0^A\Delta_k^{-\alpha_k}} u_k,$$

$$y_{B,k} = {_0^B\Delta_k^{\alpha_k}} {_0^B\Delta_k^{-\alpha_k}} u_k,$$

$$y_{D,k} = {_0^D\Delta_k^{\alpha_k}} {_0^D\Delta_k^{-\alpha_k}} u_k.$$

The step responses of operators compositions corresponding to the chosen types of variable order difference definitions are depicted in Fig. 2.

Fig. 2. Step responses of operators compositions for the 1st, 2nd and 4th type of definitions

Remark 1. Contrary to the constant order case, when $\Delta^{\alpha+\beta} = \Delta^\alpha \Delta^\beta$, for variable order difference operators, in general $\Delta^{\alpha+\beta} \neq \Delta^\alpha \Delta^\beta$.

4. Duality among definitions of fractional changing order differences

Let us consider a linear discrete fractional variable order state-space (DFVOSS) system

$$_0\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = A x_k + B u_k, \tag{12}$$

$$x_{k+1} = {_0\Delta_{k+1}^{\alpha_{k+1}}} x_{k+1} - \sum_{j=1}^{k+1}(-1)^j \binom{\alpha_k}{j} x_{k-j+1}, \tag{13}$$

$$y_k = C x_k, \tag{14}$$

where $\alpha_k \in \mathbb{R}$ is the fractional variable-order of the system, $u_k \in \mathbb{R}^d$ is a system input, $y_k \in \mathbb{R}^p$ is a system output, $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times d}$ and $C \in \mathbb{R}^{p \times N}$ are the state system, input, and output matrices, respectively, $x_k \in \mathbb{R}^N$ is a state vector, and N is a number of state equations.

Basic properties of the constant order DFOSS can be found in [31–33].

The following theorem provides solutions to various types of discrete fractional variable order state-space systems.

Theorem 1. With Definitions 1–6 taken into account, the following statements hold true

$$_0^D\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k \Rightarrow x_{k+1} = {_0^A}\Delta_{k+1}^{-\alpha_{k+1}} u_k, \tag{15a}$$

$$_0^A\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k \Rightarrow x_{k+1} = {_0^D}\Delta_{k+1}^{-\alpha_{k+1}} u_k, \tag{15b}$$

$$_0^B\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k \Rightarrow x_{k+1} = {_0^E}\Delta_{k+1}^{-\alpha_{k+1}} u_k, \tag{15c}$$

$$_0^E\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k \Rightarrow x_{k+1} = {_0^B}\Delta_{k+1}^{-\alpha_{k+1}} u_k, \tag{15d}$$

$$_0^C\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k \Rightarrow x_{k+1} = {_0^F}\Delta_{k+1}^{-\alpha_{k+1}} u_k, \tag{15e}$$

$$_0^F\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k \Rightarrow x_{k+1} = {_0^C}\Delta_{k+1}^{-\alpha_{k+1}} u_k. \tag{15f}$$

Proof. In order to prove the point (15a) let us consider the following DFVOSS system

$$_0^A\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k. \tag{16}$$

This can be expanded into

$$\sum_{j=0}^{k+1}(-1)^j \binom{\alpha_{k+1}}{j} x_{k-j+1} = u_k \tag{17}$$

and rewritten as

$$x_{k+1} = u_k - \sum_{j=1}^{k+1}(-1)^j \binom{\alpha_{k+1}}{j} x_{k-j+1}. \tag{18}$$

The solution of the system given by the 1st definition has the structure of 4th definition , namely

$$_0^D\Delta_{k+1}^{\beta_k} w_k = w_k - \sum_{j=1}^{k}(-1)^j \binom{-\beta_k}{j} {_0^D}\Delta_{k+1}^{\beta_{k-j}} u_{k-j}. \tag{19}$$

Comparison of these two relations, along with substitutions

$$w_{k+1} = u_k, \qquad -\alpha_{k+1} = \beta_{k+1},$$

and

$$x_{k+1} = {_0^D}\Delta_{k+1}^{\alpha_{k+1}} w_{k+1}$$

yields

$$x_{k+1} = {_0^D}\Delta_{k+1}^{-\alpha_{k+1}} u_k.$$

In order to prove the point (15b) let us consider the following DFVOSS system

$$_0^D\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k. \tag{20}$$

This can be expanded into

$$x_{k+1} - \sum_{j=1}^{k+1}(-1)^j \binom{-\alpha_{k+1}}{j} {_0^D}\Delta_{k+1-j}^{\alpha_{k+1}} x_{k+1-j} = u_k.$$

Now, by use of

$$x_{k+1} - \sum_{j=1}^{k+1}(-1)^j \binom{-\alpha_{k+1}}{j} u_{k-j+1} = u_k \tag{21}$$

one can see that the solution coincides with the 1st type of difference

$$x_{k+1} = {_0^A}\Delta_{k+1}^{-\alpha_{k+1}} u_k.$$

Thus the point (15a) has been proved. The proof of remaining points follows in a similar way.

Remark 2. By virtue of Theorem 1 the following relations hold true

$$_0^A\Delta_{k+1}^{\alpha_{k+1}} {_0^D}\Delta_{k+1}^{-\alpha_{k+1}} u_k = u_k, \tag{22a}$$

$$_0^D\Delta_{k+1}^{\alpha_{k+1}} {_0^A}\Delta_{k+1}^{-\alpha_{k+1}} u_k = u_k, \tag{22b}$$

$$_0^B\Delta_{k+1}^{\alpha_{k+1}} {_0^E}\Delta_{k+1}^{-\alpha_{k+1}} u_k = u_k, \tag{22c}$$

$$_0^E\Delta_{k+1}^{\alpha_{k+1}} {_0^B}\Delta_{k+1}^{-\alpha_{k+1}} u_k = u_k, \tag{22d}$$

$$_0^C\Delta_{k+1}^{\alpha_{k+1}} {_0^F}\Delta_{k+1}^{-\alpha_{k+1}} u_k = u_k, \tag{22e}$$

$$_0^F\Delta_{k+1}^{\alpha_{k+1}} {_0^C}\Delta_{k+1}^{-\alpha_{k+1}} u_k = u_k. \tag{22f}$$

5. Application of duality in identification

Let us consider the following discrete system

$$
{}_{0}^{A}\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = b u_k.
$$

The parameter b can be retrieved from the following equation

$$
\underbrace{\begin{bmatrix} {}_{0}^{A}\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} \\ {}_{0}^{A}\Delta_{k}^{\alpha_{k}} x_{k} \\ \vdots \\ {}_{0}^{A}\Delta_{1}^{\alpha_{1}} x_{1} \end{bmatrix}}_{\equiv \Delta X} = b \underbrace{\begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}}_{\equiv U}. \tag{23}
$$

Exploiting duality relations discussed in Sec. 4, and taking into account (23), one obtains

$$
\underbrace{\begin{bmatrix} x_{k+1} \\ x_k \\ \vdots \\ x_1 \end{bmatrix}}_{\equiv X} = b \underbrace{\begin{bmatrix} {}_{0}^{D}\Delta_{k+1}^{-\alpha_{k+1}} u_k \\ {}_{0}^{D}\Delta_{k}^{-\alpha_{k}} u_{k-1} \\ \vdots \\ {}_{0}^{D}\Delta_{1}^{-\alpha_{1}} u_0 \end{bmatrix}}_{\equiv \Delta U}. \tag{24}
$$

It turns out that for identification purposes, formulation (24) is more robust with respect to additive output noise. This fact is clearly demonstrated by numerical simulations, which are discussed in the next section.

6. Numerical results

6.1. Step responses of discrete variable fractional order state-space systems (DVFOSS).

Example 3. Let us consider the following DVFOSS system

$$
{}_{0}^{D}\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k. \tag{25}
$$

The order α_k, as it is depicted in Fig. 3, is equal to 0.5 for $k \leq 50$ and 0.8 for $k > 50$.
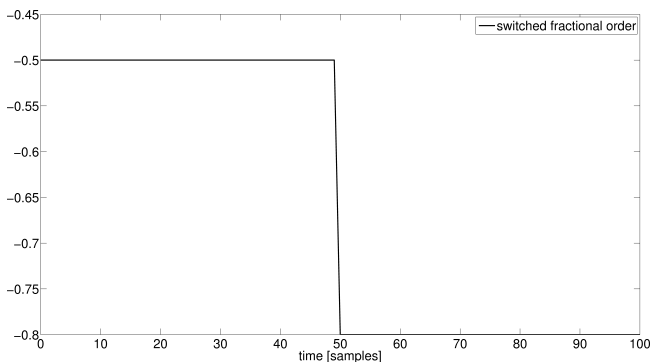

Fig. 3. Switched fractional order

Figure 4 presents comparison of step responses of the considered state-space system based on the 4th definition to a one corresponding to the 1st type of fractional variable order difference. As it can be seen the responses are the same up to

a shift by one sample in time domain, since the solution to (25) is

$$
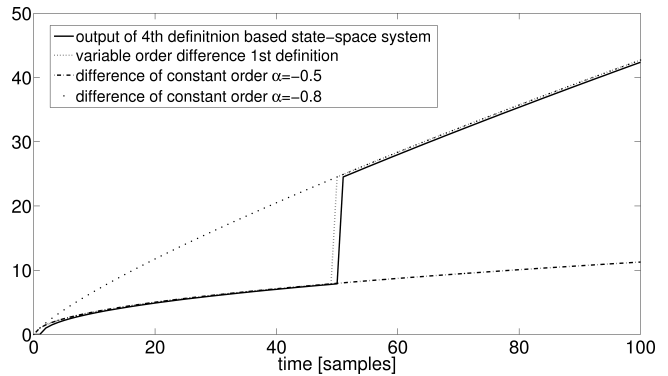x_{k+1} = {}_{0}^{A}\Delta_{k+1}^{-\alpha_{k+1}} u_k. \tag{26}
$$

Fig. 4. Comparison of step responses for 4th definition based DV-FOSS

Example 4. Let us consider the following DVFOSS system

$$
{}_{0}^{A}\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k, \tag{27}
$$

whose order is the same as in Example 3. The comparison of step responses of the considered 1st definition and the 4th definition state-space systems of fractional variable order difference is presented in Fig. 5. Responses have the same character and are only shifted by one sample in time domain since the solution to (27) has the form of

$$
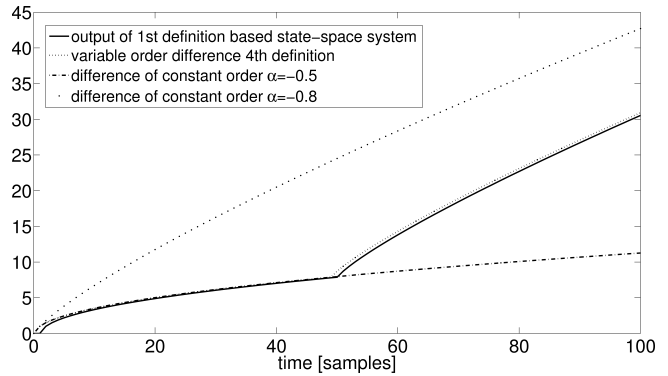x_{k+1} = {}_{0}^{D}\Delta_{k+1}^{-\alpha_{k+1}} u_k. \tag{28}
$$

Fig. 5. Comparison of step responses for 1st definition based DV-FOSS

Remark 3. The Examples 3 and 4 clearly demonstrate the need of an appropriate choice of definitions type of a discrete variable state-space system. For instance, modeling a system with behavior of the 1st definition requires use of the DV-FOSS corresponding to the 4th type of definition, as it has the solution of the type of the 1st definition. In the same vain, when the system has behavior of the 2nd definition (e.g., analog model presented in [23] or [34]) the DVFOSS based on the 5th definition should be applied.

6.2. Identification results. Equations (23) and (24) can be written shortly

$$\Delta X = bU, \qquad (29)$$

where $\Delta X \in \mathbb{R}^{k+1}$, $U \in \mathbb{R}^{k+1}$, $b \in \mathbb{R}$ and

$$X = b\Delta U, \qquad (30)$$

where $X \in \mathbb{R}^{k+1}$, $\Delta U \in \mathbb{R}^{k+1}$. Hence, the estimates of the parameter b are readily obtained, namely

$$b = U^{\dagger} \Delta X, \qquad (31)$$

and

$$b = (\Delta U)^{\dagger} X, \qquad (32)$$

where $(\cdot)^{\dagger}$ denotes Moore-Penrose pseudo-inverse. Numerical experiments suggest, that formula (31) yields better results in terms of both the estimator value and its variance.

Experiment 1. A system of the form

$$_0^A \Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = bu_k,$$

$$y_k = x_k + \nu_k,$$

with $b \in \mathbb{R}$, was simulated. The measurement uncertainty was modeled by an output noise $\nu_k \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 = 1$. The variable order α_k of the system is

$$\alpha_k = M_\alpha \sin(\omega k) + \alpha_0, \quad k \in \mathbb{Z},$$

where magnitude $M_\alpha = 0.2$, frequency $\omega = 0.05$ and offset $\alpha_0 = 0.7$.

The numerical experiment was carried out in the following way. For a given value of b the simulation was run 1000 times. After each run, the value of the estimator $\widehat{b}$ was computed according to formulas (31) and (32). With all runs done, the appropriate means and variances were computed. The whole procedure was repeated for different values of the parameter b. The results obtained in this way are given in Table 1. The first column contains the true values of b, the second and the third – estimator mean $\widehat{b}_{\Delta X}$ and variance $\sigma^2_{\widehat{b}_{\Delta X}}$, respectively, obtained form (31) and finally, the forth and the fifth – estimator mean $\widehat{b}_{\Delta U}$ and variance $\sigma^2_{\widehat{b}_{\Delta U}}$, respectively, obtained form (32). When formula (32) is used, the estimator variance outperforms by order of two the variance value corresponding to formula (31).

Table 1
Numerical results for estimation of the parameter b with noise variance $\sigma^2 = 1$

b	$\widehat{b}_{\Delta X}$	$\sigma^2_{\widehat{b}_{\Delta X}} \times 1.0e-04$	$\widehat{b}_{\Delta U}$	$\sigma^2_{\widehat{b}_{\Delta U}} \times 1.0e-06$
0.0050	0.0048	0.1898	0.0050	0.5316
0.0100	0.0100	0.1757	0.0100	0.5737
0.0500	0.0503	0.1684	0.0500	0.6009
0.1000	0.1013	0.1764	0.1000	0.5792
0.5000	0.5056	0.1894	0.5000	0.5707
1.0000	1.0113	0.1766	1.0000	0.5733
2.0000	2.0230	0.1733	2.0000	0.5488
5.0000	5.0572	0.1702	5.0000	0.5605

Experiment 2. The second experiment was carried in the same vain as the first one, the only difference being in variance of measurement noise, which was set to $\sigma^2 = 10$. The results obtained are given in Table 2 and in Fig. 6–8. Again, as it can be readily seen, the estimator variance corresponding to formula (32) outperforms by order of two the variance value corresponding to formula (31).

Table 2
Numerical results for estimation of the parameter b with noise variance $\sigma^2 = 10$

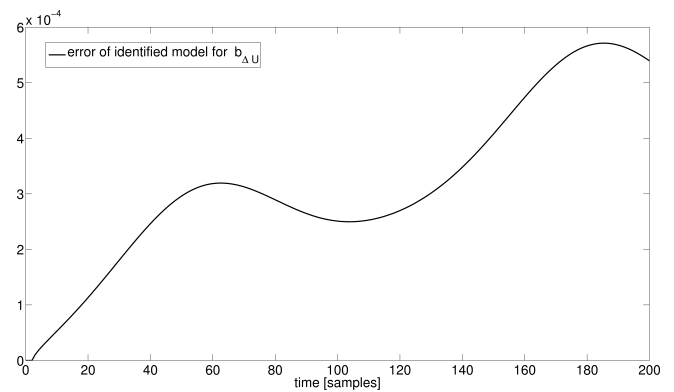| b	$\widehat{b}_{\Delta X}$	$\sigma^2_{\widehat{b}_{\Delta X}} \times 1.0e-03$	$\widehat{b}_{\Delta U}$	$\sigma^2_{\widehat{b}_{\Delta U}} \times 1.0e-05$
0.0050	0.0048	0.1773	0.0051	0.5760
0.0100	0.0114	0.1827	0.0101	0.6100
0.0500	0.0510	0.2003	0.0500	0.5818
0.1000	0.1014	0.1745	0.0999	0.5782
0.5000	0.5050	0.1752	0.5001	0.5566
1.0000	1.0116	0.1830	0.9999	0.5716
2.0000	2.0227	0.1830	2.0000	0.6105
5.0000	5.0577	0.1777	4.9999	0.5584

Fig. 6. The output error between the original and identified systems, with identification carried out in accordance with formula (32). Measurement noise $\nu_k \sim \mathcal{N}(0, 10)$, $b = 5.0000$ (Experiment 2)
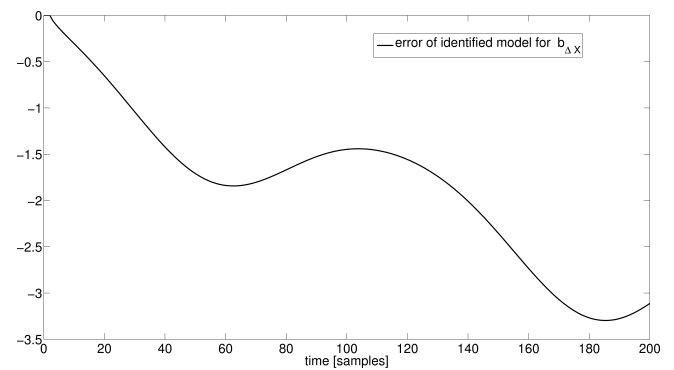

Fig. 7. The output error between the original and identified systems, with identification carried out in accordance with formula (31). Measurement noise $\nu_k \sim \mathcal{N}(0, 10)$, $b = 5.0000$ (Experiment 2)
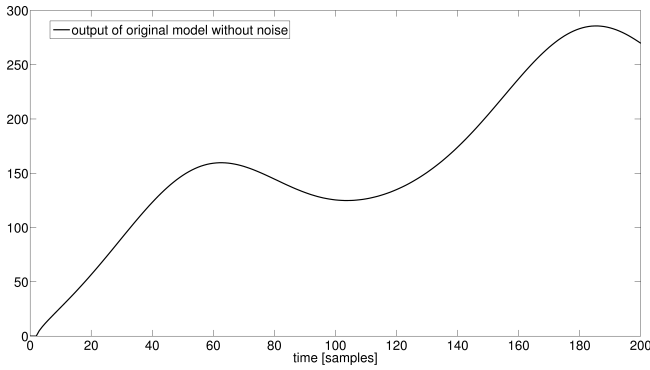
Fig. 8. Output of the original system (without noise), $b = 5.0000$

Experiment 3. The third experiment was also carried in the same vain as the first one, this time the variance of measurement noise set to $\sigma^2 = 100$. The results obtained are given in Table 3 and they clearly demonstrate that the estimator variance corresponding to formula (32), outperforms by order of two the variance value corresponding to formula (31).

Table 3
Numerical results for estimation of the parameter b with noise variance $\sigma^2 = 100$

b	$\widehat{b}_{\Delta X}$	$\sigma^2_{\widehat{b}_{\Delta X}} \times 1.0e-02$	$\widehat{b}_{\Delta U}$	$\sigma^2_{\widehat{b}_{\Delta U}} \times 1.0e-04$
0.0050	0.0041	0.1773	0.0052	0.5760
0.0100	0.0142	0.1827	0.0102	0.6100
0.0500	0.0521	0.2002	0.0500	0.5818
0.1000	0.1020	0.1744	0.0998	0.5782
0.5000	0.5036	0.1751	0.5004	0.5566
1.0000	1.0121	0.1830	0.9997	0.5716
2.0000	2.0225	0.1829	2.0001	0.6105
5.0000	5.0595	0.1777	4.9996	0.5584

7. Conclusions

The definitions of variable order difference have been presented and duality property between some of them has been introduced. Although recursive operators seem to be more complicated, in fact they do not require significantly bigger numerical effort to implement than standard ones. Moreover, because of duality property, they are very useful in modeling and analysis of variable order systems based on standard definitions. The duality relations have been used to improve the performance of the least squares estimation for variable order difference fractional systems. It has been demonstrated, that by exploitation of duality one can reduce the estimator variance when system identification is carried out. Moreover, the duality property may facilitate a better choice of definitions type for a discrete variable state-space system. For modeling of the system with behavior given by the 1st definition the DVFOSS based on the 4th type of definition have to be used, since it has the solution of the type of the 1st definition. In a similar way, for a system with behavior given by the 2nd definition (e.g., an analog model presented in [23] or [34]) the DVFOSS based on the 5th definition ought to be used. The results presented in the paper are valid for SISO systems

and their generalization onto MIMO systems remains an open problem.

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