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Reduced-order fractional descriptor observers for fractional descriptor continuous-time linear system

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Abstract. Fractional descriptor reduced-order observers for fractional descriptor continuous-time linear systems are proposed. Necessary and sufficient conditions for the existence of the observers are established. The design procedure of the observers is given and is demonstrated on two numerical examples.

Key words: fractional descriptor linear systems, design, reduced-order, descriptor, fractional, observer.

1. Introduction

The fractional linear systems have been considered in many papers and books [1–5]. Positive linear systems consisting of n subsystems with different fractional orders have been proposed in [4, 6]. Descriptor (singular) linear systems have been investigated in [5, 7–20]. The eigenvalues and invariants assignment by state and input feedbacks have been addressed in [10, 13, 15]. The computation of Kronecker's canonical form of a singular pencil has been analyzed in [19].

A new concept of perfect observers for linear continuous-time systems has been proposed in [5, 21, 22]. Observers for fractional linear systems have been addressed in [12] and for descriptor linear systems in [2]. Fractional descriptor full-order observers for fractional descriptor continuous-time linear systems have been proposed in [23].

In this paper fractional descriptor reduced-order observers for fractional descriptor continuous-time linear systems will be proposed and necessary and sufficient conditions for the existence of the observer will be established.

The paper is organized as follows. In Sec. 2 the basic definitions and theorems of fractional descriptor linear continuous-time systems are recalled and their full-order fractional descriptor observers are presented. In Sec. 3 the reduced-order fractional descriptor observers are proposed and necessary and sufficient conditions for the existence of the observers are established. A design procedure of the reduced-order observer and illustrating examples are given in Sec. 4. Concluding remarks are given in Sec. 5.

2. Fractional descriptor systems and their full-order observers

Consider the fractional descriptor continuous-time linear system

$$E\frac{d^\alpha x}{dt^\alpha} = Ax + Bu, \qquad x_0 = x(0), \qquad \alpha \in (0,2), \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $\frac{d^\alpha x}{dt^\alpha}$ is the fractional α-order derivative defined by Caputo [4, 24]

$$\begin{aligned} {}_0D_t^\alpha x(t) &= \frac{d^\alpha x(t)}{dt^\alpha} \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\frac{d^n x}{dt^n}}{(t-\tau)^{\alpha-n+1}}d\tau, \end{aligned} \tag{2}$$

$$n-1 < \alpha < n \in N = \{1, 2, ...\},$$

$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors, $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$. It is assumed that $\det E = 0$ and

$$\det[E\lambda - A] \neq 0 \quad \text{for some} \quad \lambda \in C \quad \text{(the field of complex number).} \tag{3}$$

Let U be the set of admissible inputs $u(t) \in U \subset \Re^m$ and $X_0 \subset \Re^n$ be the set of consistent initial conditions $x_0 \in X_0$ for which the Eq. (1) has a solution $x(t)$ for $u(t) \in U$.

The solution of the Eq. (1a) for $x_0 \in X_0$ has been derived in [23].

Definition 1. The fractional descriptor linear system (1) is called asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for any finite $x_0 \in X_0$ and $u(t) = 0$.

Theorem 1. The fractional descriptor linear system (1) is asymptotically stable if the zeros (the eigenvalues of (E, A)) $\lambda_1, ..., \lambda_p$ of the equation

$$\det[E\lambda - A] = \lambda^p + a_{p-1}\lambda^{p-1} + ... + a_1\lambda + a_0 = 0 \tag{4}$$

satisfy the condition

$$|\arg \lambda_k| > \alpha\frac{\pi}{2} \qquad \text{for} \quad k = 1, ..., p. \tag{5}$$

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The eigenvalues satisfying the condition (5) are located in the stability region shown in Fig. 1 for $\alpha \in (0,2)$ and denoted by S_r.

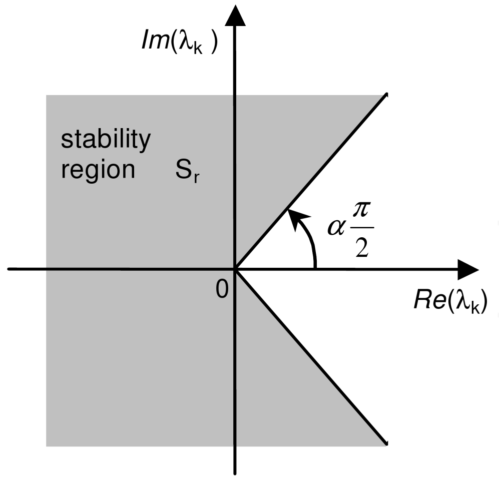

Fig. 1. Stability region

Definition 3. The fractional descriptor continuous-time linear system

$$E\frac{d^\alpha \widehat{x}}{dt^\alpha} = F\widehat{x} + Gu + Hy, \qquad \alpha \in (0,2) \tag{6}$$

$\widehat{x} = \widehat{x}(t) \in \Re^n$ is the estimate of $x(t)$, and $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the same input and output vectors as in (1), $E, F \in \Re^{n\times n}$, $G \in \Re^{n\times m}$, $H \in \Re^{n\times p}$, $\det E = 0$ is called a (full-order) state observer for the system (1) if

$$\lim_{t\to\infty}[x(t) - \widehat{x}(t)] = 0. \tag{7}$$

Theorem 2. The fractional descriptor system (1) has a full state observer (6) if and only if there exists a matrix H such that all eigenvalues of the pair $(E, A - HC)$ are located in the stable region S_r shown on Fig. 1, i.e.

$$\sigma(E, A - HC) \subset S_r, \tag{8}$$

where σ denotes the spectrum of the pair.

The proof is given in [23].

From Theorem 2 it follows that the design of a stable observer (6) of the system (1) has been reduced to finding a matrix H such that the eigenvalues of the pair $(E, A - HC)$ are located in the asymptotic stability region. It is well-known [12, 14] that there exists a matrix H such that the eigenvalues of the pair $(E, A - HC)$ are located in the asymptotic stability region if and only if the fractional descriptor system (1) is detectable [12, 13], i.e.

$$\text{rank}\begin{bmatrix} Es_k - A \\ C \end{bmatrix} = n \\ \text{for} \quad s_k \in \sigma(E, A). \tag{9}$$

The problem of designing of the observer (6) of the system (1) can be reduced to the procedure of designing of a state-feedback $v = -H^T x$ for the dual system [12, 13]

$$E^T\frac{d^\alpha x}{dt^\alpha} = A^T x + C^T v. \tag{10}$$

To guarantee that the descriptor state observer is impulse-free the matrix H must be chosen so that

$$\deg[\det(Es - A + HC)] = \text{rank}E. \tag{11}$$

It is well-known [12, 14] that the finite observers poles (the finite eigenvalues of the pair $(E, A-HC)$ can be arbitrary assigned if and only if the descriptor system (1) is R-observable, i.e.

$$\text{rank}\begin{bmatrix} Es - A \\ C \end{bmatrix} = n \quad \text{for} \quad s \in \mathbb{C}. \tag{12}$$

Therefore, the following theorem has been proved:

Theorem 3. There exists the impulse-free fractional descriptor observer (6) with arbitrary prescribed set of poles of the fractional descriptor system (1) satisfying (3) if and only if the conditions (11) and (12) are met.

3. Reduced-order fractional descriptor observers

Consider the fractional descriptor-system (1) satisfying the assumption (3).

If

$$\text{rank}C = p \tag{13}$$

then there exist a permutation matrix $P \in \Re^{n\times n}$

$$CP = [\, C_1 \quad C_2 \,], \\ C_1 \in \Re^{p\times p}, \qquad \det C_1 \neq 0, \\ C_2 \in \Re^{p\times(n-p)} \tag{14}$$

and the nonsingular matrix

$$Q_1 = \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} \in \Re^{n\times n} \tag{15}$$

such that

$$\overline{C} = CPQ_1 = [\, C_1 \quad C_2 \,]Q_1 \\ = [\, C_1 \quad C_2 \,]\begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} = [\, I_p \quad 0 \,]. \tag{16}$$

Substituting

$$x = PQ_1\overline{x} \tag{17}$$

into (1) we obtain

$$EPQ_1\frac{d^\alpha \overline{x}}{dt^\alpha} = APQ_1\overline{x} + Bu, \tag{18a}$$

$$y = Cx = CPQ_1\overline{x} = [\, I_p \quad 0 \,]\begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} = \overline{x}_1, \\ \overline{x}_1 \in \Re^p, \qquad \overline{x}_2 \in \Re^{(n-p)}. \tag{18b}$$

Premultiplying (18a) by a nonsingular elementary row operations matrix $Q_2 \in \Re^{n\times n}$ we obtain

$$Q_2EPQ_1 = \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix},$$
$$E_{11} \in \Re^{p\times p}, \qquad E_{21} \in \Re^{(n-p)\times p}, \tag{19}$$
$$E_{22} \in \Re^{(n-p)\times(n-p)}$$

and

$$E_{11}\frac{d^\alpha \overline{x}_1}{dt^\alpha} = A_{11}\overline{x}_1 + A_{12}\overline{x}_2 + B_1 u, \tag{20a}$$

$$E_{21}\frac{d^\alpha \overline{x}_1}{dt^\alpha} + E_{22}\frac{d^\alpha \overline{x}_2}{dt^\alpha} = A_{21}\overline{x}_1 + A_{22}\overline{x}_2 + B_2 u, \tag{20b}$$

where

$$Q_2APQ_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$
$$A_{11} \in \Re^{p\times p}, \qquad A_{12} \in \Re^{p\times(n-p)},$$
$$A_{21} \in \Re^{(n-p)\times n}, \qquad A_{22} \in \Re^{(n-p)\times(n-p)}, \tag{20c}$$
$$Q_2B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$B_1 \in \Re^{p\times m}, \qquad B_2 \in \Re^{(n-p)\times m}.$$

From (18b) it follows that $\overline{x}_1 = y$ and for given y the subvector $\overline{x}_1$ is known. Therefore, the reduced-order observer of the fractional descriptor system (1) should reconstruct only the subvector $\overline{x}_2 = \Re^{(n-p)}$.

From (20) we have

$$E_{22}\frac{d^\alpha \overline{x}_2}{dt^\alpha} = A_{22}\overline{x}_2 + \overline{u}, \tag{21a}$$

$$\overline{y} = A_{12}\overline{x}_2, \tag{21b}$$

where

$$\overline{u} = B_2 u - E_{21}\frac{d^\alpha y}{dt^\alpha} + A_{21}y$$
$$\text{and} \quad \overline{y} = E_{11}\frac{d^\alpha y}{dt^\alpha} - A_{11}y - B_1 u \tag{21c}$$

are the new known input and output.

To find the estimate $\widehat{x}_2$ of $\overline{x}_2$ the full-order fractional descriptor observer for the system (21) can be applied [23].

Definition 4. The fractional descriptor continuous-time linear system

$$E_{22}\frac{d^\alpha \widehat{x}_2}{dt^\alpha} = F\widehat{x}_2 + \overline{u} + H\overline{y}, \tag{22}$$

where $\widehat{x}_2 \in \Re^{n-p}$, $F \in \Re^{(n-p)\times(n-p)}$, $H \in \Re^{(n-p)\times p}$, is called the reduced-order fractional descriptor observer for the system (1) if

$$\lim_{t\leftarrow\infty}[\overline{x}_2(t) - \widehat{x}_2(t)] = 0. \tag{23}$$

Applying Theorem 2 to the fractional descriptor system (21) we obtain the following theorem:

Theorem 4. For the fractional descriptor system (1) there exists the reduced-order observer (22) if and only if the system (21) is detectable, i.e.

$$\text{rank}\begin{bmatrix} E_{22}s_k - A_{22} \\ A_{12} \end{bmatrix} = n - p$$
$$\text{for} \quad s_k \in \sigma(E_{22}, A_{22}). \tag{24}$$

It is well known [12, 14] that the eigenvalues of (E_{22}, A_{22}) (the finite poles of the observer) can be arbitrarily assigned if and only if the descriptor system (21) is R-observable, i.e

$$\text{rank}\begin{bmatrix} E_{22}s - A_{22} \\ A_{12} \end{bmatrix} = n - p \qquad \text{for all} \quad s \in \mathbb{C}$$
$$\text{(field of complex numbers).} \tag{25}$$

To guarantee that the descriptor observer (22) is impulse-free the matrix H should be chosen so that [12, 14, 23]

$$\deg\{\det[E_{22}s - A_{22} + HA_{12}]\} = \text{rank}E_{22}. \tag{26}$$

Therefore, the following theorem has been proved:

Theorem 5. There exists the impulse-free reduced-order observer (22) with arbitrary set of poles for the fractional descriptor system (1) satisfying (3) if and only if the conditions (25) and (26) are met.

Remark 1. If $E_{22} = 0$ and $\det A_{22} \neq 0$ then from (21a) we have

$$\overline{x}_2 = A_{22}^{-1}\overline{u} \tag{27}$$

and we can find $\overline{x}_2$ without any observer.

Remark 2. If $\det E_{22} \neq 0$ then from (21a) we have

$$\frac{d^\alpha \overline{x}_2}{dt^\alpha} = E_{22}^{-1}A_{22}\overline{x}_2 + E_{22}^{-1}\overline{u} \tag{28}$$

and the estimate $\widehat{x}_2$ of $\overline{x}_2$ can be found by the use of the classical (standard) fractional observer [22, 23].

4. Procedure and examples

To design the reduced-order observer (22) with arbitrary set of poles for the fractional descriptor system (1) the following procedure can be used.

Procedure 1.

Step 1. Find the permutation matrix P and the nonsingular matrix (15) transferring the matrix C to the form (16).

Step 2. Find the elementary row operations matrix Q_2 such reduced the matrix EPQ_1 to the form (19) and using (19) and (20c) compute the matrices E_{11}, E_{21}, E_{22}, A_{11}, A_{12}, A_{21}, A_{22} and B_1, B_2.

Step 3. Check the conditions (25) and (26) for some $H \in \Re^{(n-p)\times p}$.

Step 4. Using

$$F = A_{22} - HA_{12} \tag{29}$$

find the matrix H such that the pair (E_{22}, F) has the desired eigenvalues located in the stability region S_r.

Step 5. Find the equation (22) of the desired fractional descriptor observer.

Example 1. Consider the fractional descriptor system (1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{30}$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The system satisfies the condition (3) since

$$\det[Es - A] = \begin{vmatrix} s+1 & 0 & -2 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & s \\ 0 & 0 & s & -1 \end{vmatrix} \tag{31}$$

$$= 2(s+1)(1-s^2) \neq 0.$$

Using Procedure 1 we obtain the following:

Step 1. In this case the permutation matrix $P = I_4$ (the identity matrix)

$$CP = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{32}$$

$$Q_1 = \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\overline{C} = CPQ_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Step 2. The matrix of elementary operations is equal $Q_2 = I_4$ since

$$Q_2EPQ_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix}, \tag{34a}$$

$$E_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$E_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$Q_2APQ_1 = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 2 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{34b}$$

$$A_{11} = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$Q_2B = I_4 \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{34c}$$

$$B_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 3. Using (25) and (26) we obtain

$$\operatorname{rank} \begin{bmatrix} E_{22}s - A_{22} \\ A_{12} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -1 & s \\ s & -1 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} = 2 \tag{35}$$

$$\text{for all} \quad s \in \mathbb{C}$$

and for

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix},$$

$$\deg\{\det[E_{22}s - A_{22} + HA_{12}]\}$$

$$= \deg\left\{\det\begin{bmatrix} -1+2h_{11} & s+h_{11}+2h_{12} \\ s+2h_{21} & -1+h_{21}+2h_{22} \end{bmatrix}\right\} \tag{36}$$

$$= 2 = \mathrm{rank}E_{22}.$$

Therefore, the conditions (25) and (26) are satisfied.

Step 4. Using (29) we obtain

$$F = A_{22} - HA_{12} = \begin{bmatrix} 1-2h_{11} & -2h_{12}-h_{11} \\ -2h_{21} & 1-2h_{22}-h_{21} \end{bmatrix}. \tag{37}$$

Let the desired eigenvalues of the pair (E_{22}, F) be $s_{d1} = s_{d2} = -10$. Then

$$\begin{gathered}\det[E_{22}s - F] = \begin{vmatrix} 2h_{11}-1 & s+h_{11}+2h_{12} \\ s+2h_{21} & h_{21}+2h_{22}-1 \end{vmatrix} \\ = -s^2 - (2h_{21}+h_{11}+2h_{12})s \\ + (2h_{11}-1)(h_{21}+2h_{22}-1) - 4h_{12}h_{21} \\ = -(s+10)^2 = -(s^2+20s+100) \\ \text{for} \quad h_{11} = -2h_{12}\end{gathered} \tag{38}$$

and

$$\begin{gathered}-2h_{21} = -20, \\ (2h_{11}-1)(h_{21}+2h_{22}-1) - 4h_{12}h_{21} = -100.\end{gathered} \tag{39}$$

Solving (39) we obtain (for example)

$$\begin{gathered}h_{11} = 5.5, \qquad h_{12} = -2.75, \\ h_{21} = 10, \qquad h_{22} = -9.5.\end{gathered} \tag{40}$$

Step 5. In this case from (21c) we have

$$\begin{gathered}\overline{u} = B_2 u - E_{21}\frac{d^\alpha y}{dt^\alpha} + A_{21}y \\ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \frac{d^\alpha y}{dt^\alpha} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} y = u, \\ \overline{y} = E_{11}\frac{d^\alpha y}{dt^\alpha} - A_{11}y - B_1 u \\ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{d^\alpha y}{dt^\alpha} - \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} y - \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} u \\ = \begin{bmatrix} \frac{d^\alpha y_2}{dt^\alpha} + y_2 - u_1 \\ -2y_1 - u_1 - 2u_2 \end{bmatrix},\end{gathered} \tag{41}$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The desired reduced-order fractional observer of the system is described by the equation

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{d^\alpha \widehat{x}_2}{dt^\alpha} = \begin{bmatrix} -10 & 0 \\ -20 & 10 \end{bmatrix} \widehat{x}_2 + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 5.5 & -2.75 \\ 10 & -9.5 \end{bmatrix} \overline{y} \tag{42a}$$

or

$$\frac{d^\alpha \widehat{x}_2}{dt^\alpha} = \begin{bmatrix} -20 & 10 \\ -10 & 0 \end{bmatrix} \widehat{x}_2 + \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} 10 & -9.5 \\ 5.5 & -2.75 \end{bmatrix} \overline{y}. \tag{42b}$$

Example 2. Consider the fractional descriptor system (1) with the matrices

$$\begin{gathered}E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \\ B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.\end{gathered} \tag{43}$$

The system satisfies the assumption (3) since

$$\det[Es - A] = \begin{vmatrix} 0 & -2 & s \\ 0 & s & -1 \\ -1 & 0 & 2s \end{vmatrix} = s^2 - 2. \tag{44}$$

Using Procedure 1 we obtain the following:

Step 1. In this case the permutation matrix $P = I_3$

$$\begin{gathered}CP = [\ C_1 \quad C_2\ ], \\ C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\end{gathered} \tag{45}$$

$$Q_1 = \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \tag{46}$$

and

$$\begin{gathered}\overline{C} = CPQ_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.\end{gathered} \tag{47}$$

Step 2. For the matrix

$$EPQ_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} \tag{48}$$

the matrix Q_2 has the form

$$Q_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}. \tag{49}$$

Using (19) and (20) we obtain

$$Q_2EPQ_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \tag{50}$$

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad E_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$E_{21} = [0 \quad 0], \qquad E_{22} = [2],$$

$$Q_2APQ_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0.5 & 0 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

$$A_{21} = [0 \quad 1], \quad A_{22} = [-2]$$

$$Q_2B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \qquad B_2 = [1]. \tag{51}$$

Step 3. Using (25) and (26) we obtain

$$\text{rank} \begin{bmatrix} E_{22}s - A_{22} \\ A_{12} \end{bmatrix} = \text{rank} \begin{bmatrix} 2s+2 \\ -1 \\ 0 \end{bmatrix} = 1 \quad \text{for} \quad s \in \mathbb{C} \tag{52}$$

and for $H = [h_1 \quad h_2]$

$$\deg\{\det[E_{22}s - A_{22} + HA_{12}]\} = \deg[2s + 2 - h_1] = 1 = \text{rank}E_{22} = \text{rank}[2]. \tag{53}$$

Therefore, the conditions (25) and (26) are satisfied.

Step 4. Let $s_d = -8$ be the eigenvalues of the pair (E_{22}, F). Then using (29) we obtain

$$\begin{gathered} F = A_{22} - HA_{12} \\ = [-2] - [h_1 \quad h_2] \begin{bmatrix} -1 \\ 0 \end{bmatrix} = h_1 - 2, \\ E_{22}s_d - F = E_{22}s_d + 2 - h_1 \\ = 2s_d + 2 - h_1 = -14 - h_1 = 0 \end{gathered} \tag{54}$$

and $h_1 = 14$ and h_2 arbitrary.

Step 5. In this case from (21c) we have

$$\overline{u} = B_2u - E_{21}\frac{d^\alpha y}{dt^\alpha} + A_{21}y = u + [0 \quad 1]y = u + y_2,$$

$$\begin{aligned} \overline{y} &= E_{11}\frac{d^\alpha y}{dt^\alpha} - A_{11}y - B_1u \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \frac{d^\alpha y}{dt^\alpha} - \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u \\ &= \begin{bmatrix} \dfrac{d^\alpha y_1}{dt^\alpha} - 2y_1 - u_1 \\ \dfrac{d^\alpha y_1}{dt^\alpha} + 0.5y_2 - 0.5u \end{bmatrix}. \end{aligned} \tag{55}$$

The Eq. (22) of the desired reduced-order observer of the system has the form

$$\frac{d^\alpha \widehat{x}_2}{dt^\alpha} = 6\widehat{x}_2 + 0.5u + 0.5y_2 + [7 \quad 0.5h_2]\overline{y}. \tag{56}$$

Note that the observer described by (56) is a classical observer since $\det E_{22} \neq 0$ (Remark 2).

5. Concluding remarks

Fractional descriptor reduced-order observers for fractional descriptor continuous-time linear systems have been proposed. The designing procedure of the fractional descriptor observers has been proposed and illustrated on two numerical examples.

The considerations can be easily extended to fractional descriptor discrete-time linear systems. An open problem is an extension for fractional descriptor 2D continuous-discrete linear systems.

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