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# Suppression of vibration with optimal actuators and sensors placement

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It is proposed to place the actuators to maximize the mean value of energy transmitted from or dissipated by the actuators, while the sensor location should maximize the mean square value of system output, which also maximizes the signal-to-noise ratio. By using explicit expressions for controllability and observability grammians as well as modal energies, it is shown that the approaches based on the system responses to transient and persistent disturbances are closely related, and are equivalent for structures which damping is small and the natural frequencies of which are well spaced. The method of actuator and sensor optimal location via grammians was proposed and compared it with results given by the method of matrix norms.

**Key words:** optimal location, vibration suppression, actuator, sensor

## 1. Introduction

It is assumed that the controllability and observability measures should depend on properties of the system and sets of actuators/sensors, but should not depend on a particular choice of initial conditions or control laws (which are unknown at this stage). This approach gives actuator/sensor locations based on some measures of controllability/observability grammians. In that situation, it is proposed to place the actuators so as to maximize the mean value of energy transmitted from or dissipated by the actuators, while the sensor location should maximize the mean square value of system output, which also maximizes the signal-to-noise ratio. By using explicit expressions for controllability and observability grammians as well as modal energies, it is shown that the approaches based on the system responses to transient and persistent disturbances are closely related, and are equivalent for structures the damping of which is small and the natural frequencies of which are well spaced.

The majority of authors considered the problem of location of actuators as a problem of transferring the system from some initial state $x(t_0)$ to a terminal state within a

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given time $T$ (not necessary finite) in such a way that energy, usually defined as the time integral of a quadratic form of an input, is minimized. The optimal solution to this problem defines the optimal control energy that explicitly depends on the initial or terminal conditions, and indirectly is a function of the actuator positions. In [1] this technique is applied and was proposed as an optimization criterion effect of initial conditions on the cost function. In [22] a similar approach is used, but the authors included a weighted quadratic form of the state in the cost function. The cost function corresponds to the optimal performance index.

In [6] the search for optimal actuator locations that minimize the control energy under the constraints (which preserve system controllability with a prespecified control law based on a pole placement algorithm). In [16] it is proposed a minimum input energy solution which depends not only on the initial conditions but also on the weighting matrices used in the optimality criterion. Therefore actuator distribution does not depend on the initial condition but depends on the assumed time required to drive the system to the state of equilibrium from a given perturbed state.

Several techniques have been proposed to optimize the sensor locations. In [2] is first proposed to use Kalman filtering in the sensor distribution problem. This method (which accounts for measurement errors) considers as the best location this one, which minimizes some measure of the state estimation control problem. The method is computationally intensive and the criterion used for sensor selection does not have a sound physical basis. In [18] it is proposed to maximize the rate of energy dissipation due to the control action under the output velocity feedback control law. The dissipated energy depends on the locations of actuators, sensors and the feedback gain matrix, and therefore can be used as an optimization criterion to determine all of the above. In both cases, the control law is predetermined and computational requirements are prohibitive for high order systems.

The goals of this contribution is to propose and apply the method of actuator and sensor location via grammians and compare it with with some results given by the method of matrix norms. Both methods will assist to design actuator and sensor optimal placement.

## 2. Actuator and sensor location via grammians

### 2.1. Controllability and observability

Controllability and observability are structural properties that carry information useful for structural testing and control. A structure is controllable, if the installed actuators excite all its structural modes. It is observable if the installed sensors detect the motions of the system dynamics described by the state variable $x$ is excited by the input $u$ and measured by the output $y$. However, the input may not be able to excite all states. In this case we cannot fully control the system. Also, not all states may be represented at the output. In this case we cannot fully observe the system. However, if the input excite all states, the system is controllable, and if all the states are represented in the output, the system is observable.

*Controllability*, as a measure of interaction between the input and states, involves the system matrix $A$ and the input matrix $D$. A linear system, or the pair $(A,D)$, is controllable at $t_0$ if it is possible to find a piecewise continuous input $u(t)$, $t \in [t_0,t_1]$, that will transfer the system from the initial state, $x(t_0)$, to the origin $x(t_1) = 0$ at finite time $t_1 > t_0$. If this is true for all initial moments $t_0$ and all initial states $x(t_0)$ the system is completely controllable. Otherwise, the system, or the pair $(A,D)$ is uncontrollable.

*Observability*, as a measure of interaction between the states and the output, involves the system matrix $A$ and the output matrix $C$. A linear system, or the pair $(A,C)$, is observable at $t_0$ if the state $x(t_0)$ can be determined from the output $y(t)$, $t \in [t_0,t_1]$, where $t_1 > t_0$ is some finite time. If this is true for all initial moments $t_0$ and all initial states $x(t_0)$ the system is completely observable. Otherwise, the system or the pair $(A,C)$, is unobservable.

There are many criteria to determine system controllability and observability.

Linear time-invariant system $(A,D,C)$, with $s$ inputs is completely controllable if and only if the $2n \times 2sn$ matrix

$$O_c = \begin{bmatrix} D & AD & A^2D & \dots & A^{2n-1}D \end{bmatrix} \tag{1}$$

has rank equal to $2n$. A linear time-invariant system $(A,D,C)$ with $r$ outputs is completely observable if and only if $2rn \times 2n$ matrix

$$O_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^{2n-1} \end{bmatrix} \tag{2}$$

has rank equal to $2n$.

The above criteria, although simple, have two serious drawbacks. First, they answer the controllability and observability question in yes and no terms. Second, they are useful only for a system of small dimensions.

An alternative approach uses *grammians to determine the system properties*. Grammians are nonnegative matrices that express the controllability and observability properties qualitatively, and are free of the numerical difficulties mentioned above. The *controllability and observability grammians* are defined as follows

$$W_c(t) = \int_0^t \exp(A\tau)DD^T \exp(A^T\tau)d\tau, \tag{3a}$$

$$W_o(t) = \int_0^t \exp(A^T\tau)C^TC\exp(A\tau)d\tau. \tag{3b}$$

We can determine them alternatively and more conveniently from the following differential equations

$$\dot{W}_c = AW_c + W_cA^T + DD^T, \tag{4a}$$

$$\dot{W}_o = A^TW_o + W_oA + C^TC. \tag{4b}$$

The solutions $W_c(t)$ and $W_o(t)$ are time-varying matrices. At the moment we are interested in the stationary, or time-invariant, solutions. For a stable system, we obtain the stationary solutions of the above equations by assuming $\dot{W}_c = \dot{W}_o = 0$. In this case, the differential equations (4) are replaced with the algebraic equations, called Lyapunov equations,

$$AW_c + W_cA^T + DD^T = 0, \tag{5a}$$

$$A^TW_o + W_oA + C^TC = 0. \tag{5b}$$

### 2.2. Aplication on distributed parameter system

Consider a class of distributed parameter systems described by the partial differential equation (generalized wave equation)

$$M(v)\frac{\partial^2 w(v,t)}{\partial t^2} + 2\zeta[M(v)L]^{1/2}\left[\frac{\partial w(v,t)}{\partial t}\right] + L[w(v,t)] = F(v,t) \tag{6}$$

over a compact domain $D$. In the above, $w(v,t)$ refers to the displacement of the structure with respect to equilibrium position; it is a function of spatial variable $v \in D$ and time $t$. $F(v,t)$ refers to external force distribution. The operator $L$ is a linear homogeneous selfadjoint and nonnegative differential operator consisting of derivatives through order $2q$ with respect to the spatial coordinates $x$ but not with respect to time; it represents the stiffness distribution of the system. The mass density $M(v)$ is a positive definite function of the location $v$. Without loss of generality it can be assumed that $M(v) = 1$, but $M(v)$ is retained in the following development to distinguish between the system mass and stiffness properties. It is further assumed that the control is accomplished by $x$ essentially point actuators acting at locations $v_j$, $(j = 1,2,\ldots,p)$. Hence

$$F(v,t) = \sum_{j=1}^{x} \delta(v - v_j) f_j(t) \tag{7}$$

where $f_j(t)$, $(j = 1,2,\ldots,p)$ are actuator forces and $\delta(v - v_j)$ denotes a spatial Dirac delta function. The structure satisfies $q$ boundary conditions

$$B_k[w(v,t)] = 0 \qquad k = 1,2,\ldots,q, \tag{8}$$

where $B_k$ are linear homogeneous differential operators containing derivatives normal to the boundary and along the boundary of order through $2q-1$. According to the expansion theorem the solution to the equation (6) can be represented as the series

$$w(v,t) = \sum_{i=1}^{\infty} \Phi_i(v)\eta_i(t) \tag{9}$$

where $\eta_i(t)$ are modal coordinates and $\Phi_i(v)$ are eigenfunctions. The eigenfunctions are the solutions of the eigenvalue problem consisting of the differential equation $L[\Phi(v)] = \omega^2 M(v)\Phi(v)$ and satisfy the boundary conditions $B_k[\Phi(v)] = 0$, $k = 1, 2, \ldots, q$. The solution yields an infinite set of eigenfunctions $\Phi_i(v)$ with corresponding natural frequencies $\Omega_i$. The eigenfunctions satisfy the orthogonality condition and can be normalized such that

$$\int_D M(v)\Phi_r(v)\Phi_s(v)dv = \delta_{rs} \tag{10}$$

for any $r, s = 1, 2, \ldots,$ where $\delta_{rs}$ is a Kronecker delta function. In addition,

$$\int_D \Phi_r(v)L[\Phi_s(v)]dv = \Omega_r^2\delta_{rs}. \tag{11}$$

The forcing term on the right side of equation (6) can be expanded into a series in $\Phi_i(v)$:

$$F(v,t) = \sum_{i=1}^{\infty} M(v)\Phi_i(v)Q_i(t) \tag{12}$$

where the generalized force $Q_i(t)$ associated with generalized coordinate $\eta_i(t)$ can be found from

$$Q_i(t) = \int_D \Phi_i(v)F(v,t)dx = \sum_{j=1}^{x} \Phi_i(v_j)f_j(t). \tag{13}$$

Inserting series representations (9) and (12) into equation (6) and using the standard orthogonalization procedure (multiplying equation (6) by $w_j(v)$, integrating over the domain $D$ and using orthogonality relations (10) and (11)), equation (6) can be replaced by an infinite set of ordinary differential equations:

$$\ddot{\eta}_i + 2b_{pi}\Omega_i\dot{\eta}_i + \Omega_i^2\eta_i = Q_i(t) = \sum_{j=1}^{x} \Phi_i(v_j)f_j(t), \qquad i = 1, 2, \ldots \tag{14}$$

Note that actually $b_{pi} = b$ for any $i$ follows from equation (6) but $b_{pi}$, was used in equation (14) to account for different damping models. Since higher order modes (say $i > n$ for some large $n$) are not likely to be excited in practice and typically exhibit higher structural damping, they can be neglected in an approximate analysis. Defining the state and input vectors as contrast to a state vector composed of modal displacements

$$x = [\dot{\eta}_1, \Omega_1\eta_1, \ldots, \dot{\eta}_n, \Omega_n\eta_n]^T, \qquad u = [f_1, \ldots, f_p]^T, \tag{15}$$

yields the state representation of equation (14),

$$\dot{x} = Ax + Du \tag{16}$$

where:

$$A = \mathrm{diag}(A_i), \quad A_i = \begin{bmatrix} -2\zeta_i\Omega_i & -\Omega_i \\ \Omega_i & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \Omega_1(x_1) & \ldots & \Omega_1(x_p) \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \Omega_n(x_1) & \ldots & \Omega_n(x_p) \\ 0 & \ldots & 0 \end{bmatrix} \tag{17}$$

and velocities, the state vector defined above gives both states corresponding to each mode of roughly equal magnitude, irrespective of the units used, which has computational and analytical advantages in the following development. Note the dependence of matrix $D$ on the location of force actuators $v_j$, $j = 1, 2, \ldots, p$. If an actuator is located at the nodal point of a mode, this mode becomes uncontrollable through that actuator. Actuator location in the vicinity of the node would require a large effort to control this mode.

### 2.3. Actuator location under transient and steady state disturbance

*Transient disturbance*

Suppose that due to some disturbance the state vector of the system described by equation (15) was perturbed so that the initial condition $x(0) = x_0$ holds. In placing the actuators it is desirable to minimize the control energy required to bring the system to the desired state $x(T) = x_T$ after some time $T$. This can be accomplished by considering the following minimum energy problem [23], [24]:

$$E(u) = \int_0^T u^T(t)u(t)dt \tag{18}$$

subject to $x(0) = x_0$, $x(T) = x_T$ and the state equation (16). This is a linear quadratic optimal control with fixed terminal time and fixed terminal state. The optimal solution is given by

$$u_0(t) = -D^T e^{A(T-t)} W^{-1}(T)(e^{AT} x_0 - x_T) \tag{19}$$

where $W(\cdot)$ is the controllability grammian, defined by (3a).

One should note different definitions of controllability grammian in literature, but the one given by equation (3a) is the most common. Under the control law (19), the control energy is

$$E=(e^{AT}x_0-x_T)^T W^{-1}(T)(e^{AT}x_0-x_T). \tag{20}$$

Hence, if $W^{-1}(T)$ is 'large' (i.e., $W(T)$ is small) there will be some states $x_0$ and $x_T$ such that the system can be transferred from $x_0$ to $x_T$ only when large input energy is used. More precisely, if any eigenvalue of $W$ is small (all eigenvalues of $W(T)$ are real and nonnegative since $W(T)$ is symmetric and nonnegative definite) there will be at least one structural mode that is difficult to control. One should bear in mind that in expression (20) only $W(T)$ depends on the actuator arrangement through matrix $D$, and the best arrangement should not depend on unknown initial and terminal conditions. It is known [24] that $W(t)$ satisfies

$$\dot{W}(t)=AW(t)+W(t)A^T+DD^T \tag{21}$$

and when $A$ is an asymptotically stable matrix, $W(t)$ reaches a steady state $W_c$, which is the solution of Lyapunov equation

$$AW_c+W_cA^T+DD^T=0. \tag{22}$$

To eliminate dependency of the solution on $T$, we consider a steady state solution with the controllability grammian, $W_c$, satisfying equation (22) for asymptotically stable systems. For structures without damping, equation (22) cannot be applied and the grammian $W(T)$ can be found in closed form from equation (3a).

For systems with distinct natural frequencies, for $i\neq j$, $i,j=1,2,\ldots,n$, we have

$$\beta_{ij}=\sum_{q=1}^{p}\Phi_i(x_q)\Phi_j(x_q). \tag{23}$$

It should be pointed out that the expression (20) for minimum energy can be applied to any initial and terminal conditions. When $x_0=0$ and $x_T\neq 0$ then

$$E=\frac{1}{2}x_T^T W^{-1}(T)x_T. \tag{24}$$

Conversely, if $x_) \neq 0$ and $x_T=0$, then

$$E=\frac{1}{2}x_0^T e^{A^TT}W^{-1}(T)e^{AT}x_0. \tag{25}$$

*Persistent disturbance*

In the case of a persistent disturbance, the distribution of actuators should guarantee that the system steady state behavior is affected to the largest possible degree by actuators in order to suppress the effect of disturbances. In particular, energy transmitted from the actuators to all the structural modes should be as large as possible (under actuator energy constraints). To evaluate these energy contributions, we use covariance analysis. Suppose that the signals generated by the individual actuators are (within the frequency band considered) white noise processes that are mutually uncorrelated and have the covariance matrix.

$$\mathrm{M}[u(t)u^T(\tau)] = U\delta(t-\tau) \tag{26}$$

where $U$ is a positive definite, diagonal matrix (noise intensity). When all actuators have the same power requirements then it can be assumed that $U = I$, where $I$ is the identity matrix. The kinetic and potential (strain) energies of system are given by:

$$E_k = \frac{1}{2}\int\limits_{\Theta} M(x)\dot{w}^2(x,t)dx, \tag{27}$$

$$E_p = \frac{1}{2}\int\limits_{\Theta} w(x,t)L[w(x,t)]dx. \tag{28}$$

Using the solution of the linear partial differential equation which describes bending vibration and using orthogonality relations it is easy to show that

$$E_k = \frac{1}{2}\sum_{i=1}^{\infty}\dot{\eta}_i^2(t) \tag{29}$$

$$E_p = \frac{1}{2}\sum_{i=1}^{\infty}\Omega_i^2\eta_i^2(t) \tag{30}$$

that is, the system energy can be expressed as a sum of contributions from each mode. Considering the truncated system (16) (including $n$ modes) under white noise excitation, the system steady state behavior can be characterized by the state covariance matrix

$$\mathrm{M}[x(t)x^T(t)] = X(t) \tag{31}$$

and is described by equation [5]:

$$AX + XA^T + DUD^T = 0 \tag{32}$$

which is exactly the same as equation (22) when $U = I$.

Taking advantage of the structure of matrix $A$, it is straightforward to show that for $U = I$, the diagonal elements of $X$, $x_{ii}$ are given by

$$x_{2i-1,2i-1} = x_{2i,2i} = \frac{\beta_{ii}}{4b_{p_i}\Omega_i}, \qquad i = 1,2,\ldots,n, \tag{33}$$

where $\beta_{ii} = \sum_{q=1}^{p}\Phi_i^2(x_q)$ follows from equation (23). Hence, the expected values of the kinetic and potential energies are respectively,

$$M\{E_k\} = \frac{1}{2}\sum_{i=1}^{n} E(\dot{\eta}_i^2) = \frac{1}{2}\sum_{i=1}^{n} x_{2i-1,2i-1} = \frac{1}{8}\sum_{i=1}^{n}\frac{\beta_{ii}}{b_{p_i}\Omega_i} \tag{34}$$

$$M\{E_p\} = \frac{1}{2}\sum_{i=1}^{n} E(\Omega_i^2\eta_i^2) = \frac{1}{2}\sum_{i=1}^{n} x_{2i,2i} = \frac{1}{8}\sum_{i=1}^{n}\frac{\beta_{ii}}{b_{p_i}\Omega_i} \tag{35}$$

and the expectation of the total energy is

$$M\{E_k + E_p\} = \frac{1}{4}\sum_{i=1}^{n}\frac{\beta_{ii}}{b_{p_i}\Omega_i}. \tag{36}$$

It can be seen from expression (28), that just like the kinetic and potential energies of the structure, the total energy is the sum of the contributions from each mode, calculated independently.

It follows from the above discussion that considering actuator positioning under conditions of a transient disturbance leads to the requirement of maximizing (in some sense) the norm of the controllability matrix $W_c$, while under the steady state disturbance, the energy transmitted to (or dissipated from) the system at the steady state should be maximized. Both approaches are energy-based and are related through essentially the same Lyapunov equation (equation (5a) or (22)) with $U = I$. Partitioning the grammian $W_c$ according to

$$W_c = \begin{bmatrix} W_{11} & W_{12} & \dots & W_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ W_{n1} & W_{n2} & \dots & W_{nn} \end{bmatrix} \tag{37}$$

where $W_{ij}$, $i = 1,2,\dots,n$, $j = 1,2,\dots,n$ are $2\times 2$ matrices and using particular structures of matrices $A$ and $D$ (equations (17), equation (16)) makes the problem to be solved in a closed form [6].

### 2.4. Proposed criterion for actuator location

To establish a performance criterion for the location of actuators, the following factors have to be considered. Under a persistent disturbance, the expected value of the total energy transmitted to the system from the actuators should be large to allow for effective damping of structural vibration with moderate control effort. Furthermore, the energies (in a mean sense) transmitted to each individual mode that one wishes to control have to be large. The use of a global energy index (based on a sum of all modes) in an optimization procedure may result in an actuator location that gives good control of one or two lower order modes (since the total energy expression may be dominated by these modes for some structures) while some of higher order modes may remain weakly controllable. It is therefore desirable that the value of the performance index to be maximized drops sharply in the vicinity of actuator positions that result in loss of controllability of any of the first n modes. Based on these considerations, the following performance index $PI'$ is proposed:

$$PI' = 2\left(\sum_{i=1}^{n} E_i\right)\sqrt[n]{\prod_{i=1}^{n}(E_i)} \tag{38}$$

in which $E_i$, denotes the expectation of the total energy of the $i$-th mode given by

$$E_i = \frac{\beta_{ii}}{4b_{p_i}\Omega_i}. \tag{39}$$

The particular form of the performance index (38) was chosen after considering a number of alternatives in simulation runs, and is believed to provide a good balance between the importance of all modes. The first term on the right hand side of equation (38) is the total system energy; it will usually be dominated by a few low order modes, since the modal energies typically decrease with the mode number, $i$. The term under the root sign can be interpreted as a volume of an ellipsoid in $n$-dimensional space, the radius of which in each direction is proportional to the energy contributed by each mode. This term vanishes when the controllability of any mode is lost, and ensures that none of the modes is nearly uncontrollable.

On the one hand, it is desirable to keep the size of the controllability matrix, that can be characterized by the sum of the eigenvalues, as large as possible; on the other hand, the individual eigenvalues (corresponding to the various modes) have to remain large because if any of them are close to zero, the inverse of the controllability matrix $W_c$, will be large and there will be some initial states for which it will be difficult to return the system to the required state. Consequently, the proposed criterion is

$$PI = \left(\sum_{j=1}^{n} \lambda_j\right)\sqrt[2n]{\prod_{j=1}^{2n}(\lambda_j)} \tag{40}$$

where $\lambda_j$ denotes the eigenvalue of the controllability grammian $W_c$. It follows from the argument section that when the damping ratios $b_{pi}$ are small and the natural frequencies of the structure are well spaced then

$$\lambda_{2i-1} = \lambda_{2i} = E_i \tag{41}$$

and the above criteria are identical. However, the criterion (40) is more general, since it can be applied to general systems including systems that do not exhibit modal structure.

### 2.5. Sensor location

Consider the system defined in section 2.2. Suppose that $r$ displacements $w(x_q, t)$, $q = 1, 2, \ldots, r$ are measured at $r$ points $x_1, x_2, \ldots, x_r$ of the structure the output equation becomes

$$y(t) = C_d x(t) \tag{42}$$

where $y(t)$ is an $r$-dimensional output vector and $C_d \in R^{r\times 2n}$ is given

$$C_d = \begin{bmatrix} 0 & \Phi_1(x_1)/\Omega_1 & \dots & 0 & \Phi_n(x_1)/\Omega_n \\ 0 & \Phi_2(x_2)/\Omega_1 & \dots & 0 & \Phi_n(x_2)/\Omega_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \Phi_1(x_r)/\Omega_1 & \dots & 0 & \Phi_n(x_r)/\Omega_n \end{bmatrix}. \tag{43}$$

If the $r$ velocities rather than positions are measured, then the output equation is.

$$y(t) = C_v \dot{x}(t) \tag{44}$$

where

$$C_v = \begin{bmatrix} \Phi_1(x_1) & 0 & \dots & \Phi_n(x_1) & 0 \\ \Phi_2(x_2) & 0 & \dots & \Phi_n(x_2) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Phi_1(x_r) & 0 & \dots & \Phi_n(x_r) & 0 \end{bmatrix}. \tag{45}$$

If both displacements and velocities are measured, then matrix $C$ becomes a combination of $C_d$ and $C_v$. Any measure of the observabi1ity of a dynamic system should reflect the amount of information concerning the system states that can be derived from the sensor outputs in the presence of measurement noise. In order to maintain as large as possible a signal-to-noise ratio, the sensor locations should guarantee that under any operating conditions the system output, as well as the contributions of individual modes to the output, be as large as possible. Again, the structure may be either instantaneously perturbed from its desired state or may be subjected to persistent excitation. If the system is released from the initial state $x(0) = x_0$ with $u(t) = 0$, $t \geqslant 0$, then the output energy is [26]

$$\int_0^\infty y^T(t)y(t)dt = x_0^T W_o x_0 \tag{46}$$

where $W_o$ is the observability grammian. It can be seen that when the observability grammian is nearly singular then some initial conditions will have little effect on the output and for asymptotically stable systems it satisfies the Lyapunov equation (5a).

Following the approach used for the controllability grammian, i.e., partitioning $W_c$ and using the structures of matrices $A$ and $C$, closed form solutions can be obtained. In the case of displacement measurements,

$$c_{dij} = \sum_{k=1}^{r} \frac{\Phi_i(x_k)\Phi_j(x_k)}{\Omega_i \Omega_j}, \tag{47}$$

when the velocities are measured,

$$c_{vij} = \sum_{k=1}^{r} \Phi_i(x_k)\Phi_j(x_k). \tag{48}$$

When both positions and velocities are measured, then matrix $C^T = \left[C_d^T C_v^T\right]$, and it can be easily demonstrated that the grammian obtained for this case is the sum of the grammians considered above since $C^T C = C_d^T C_d + C_v^T C_v$.

It is straightforward to show that when the damping is small and all natural frequencies are well-spaced, the grammians are dominated by the diagonal elements which are

$$W_{oii} = diag\left(\frac{c_{dii}}{4b_{p_i}\Omega_i}, \frac{c_{dii}}{4b_{p_i}\Omega_i}\right), \qquad i = 1,2,\ldots,n, \tag{49}$$

when positions are measured and

$$W_{oii} = diag\left(\frac{c_{vii}}{4b_{p_i}\Omega_i}, \frac{c_{vii}}{4b_{p_i}\Omega_i}\right),, \qquad i = 1,2,\ldots,n, \tag{50}$$

when velocities are measured. In that case

$$x_0^T W_o x_0 = \sum_{i=1}^{n} \frac{c_{ii}}{4b_{p_i}\Omega_i}(x_{02i+1}^2 + x_{02i}^2) \tag{51}$$

where $c_{ii} = c_{dii}$, $c_{vii}$ or $c_{dii}, c_{vii}$, depending on whether the displacements, velocities or both are measured and $x_{02i-1}$, $x_{02i}$, refer to the components of the initial state vector $x_0$. In order to make the output energy large for any initial state, all diagonal terms of as well as their sum have to be large. In the general case, the diagonal terms should be replaced by eigenvalues. By comparing equations (50) and (48) with (33) and (23), *it can be seen that in the case of velocity measurements the eigenvalues of the observability grammian and the controllability grammian are the same for lightly damped structures with well separated modes*.

Now consider the system subject to a persistent disturbance. Suppose that this disturbance is the spatially distributed white noise process, that is

$$F(x,t) = f(x)\xi(t) \tag{52}$$

where $\xi(t)$ is a white noise with unitary intensity. Assume further that the spatial distribution $f(x)$ is such that the first $n$ modes are excited with equal strength. This condition is satisfied by the choice

$$f(x) = P(x)\sum_{j=1}^{n} \Phi_j(x) \tag{53}$$

since then

$$Q_i(t) = \xi(t) \tag{54}$$

follows from equations (52), (53), (10) and (13). The system dynamics are now described by equation (14), with $Q_i$ given by equation (54). This is a linear system, driven by a white noise process. The mean square value of the system output is

$$\mathrm{M}[x^T(t)C^TCx(t)] = \mathrm{trace}[(C^TC)X(t)] \tag{55}$$

where $X(t)$ is the covariance matrix for the state vector (equation (31)). At steady state, the matrix $X(t)$ becomes time invariant and satisfies the Lyapunov equation

$$AX + XA^T + dd^T = 0 \tag{56}$$

with $A$ given by equation (17) and $d$ given by

$$d = [\ 1 \quad 0 \quad 1 \quad 0 \quad \ldots \quad 1 \quad 0\ ]^T. \tag{57}$$

If the natural frequencies are well spaced and the damping coefficients are small and since equation of motion has the same structure as equation (14), the matrix $X$ approaches a diagonal matrix with

$$X_{ii} = \mathrm{diag}\left(\frac{1}{4b_{p_i}\Omega_i}, \frac{1}{4b_{p_i}\Omega_i}\right) \tag{58}$$

and, consequently, it follows from equation (55) that

$$\mathrm{M}[y^T(t)y(t)] = \sum_{i=1}^{n} \frac{c_{ii}}{4b_{p_i}\Omega_i} \tag{59}$$

with $c_{ii}$ defined as before. Comparing equation (59) with (49) and (50), it is seen that large diagonal elements of the observability grammian will make the system steady state response to a persistent excitation large, at least for a structure with well spaced natural frequencies and low damping.

### 2.6. Examples of applications

To illustrate the proposed method, consider uniform beam with two types of actuators:

- control is accomplished by essentially point actuator, or
- actuator acting on surface area (for example piezoceramics actuator)

and two types of boundary conditions:

- o simply supported boundary,
- o clamped boundary at $x = 0$ and free boundary at $x = l$.

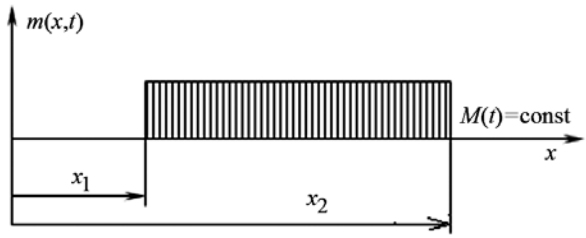


Figure 1. Model of actuator acting on the surface area (for example piezoceramics).

First we derive a mathematical model of actuator acting on the surface area. Actuator acting on the surface area can be presented as distributed moment load Fig. 1 given by

$$m(x,t) = \left\{ \begin{array}{lll} M(t) & \text{for} & x \in \langle x_1, x_2 \rangle \\ 0 & \text{for} & x \notin \langle x_1, x_2 \rangle \end{array} \right\}$$

where

$$m'(x,t) = M(t)[\delta(x - x_1) - \delta(x - x_2)]. \tag{60}$$

If the natural frequencies are well spaced and $b_{pi}$ are small damping coefficients, then after small manipulation for grammian we receive (for application of single piezoceramics)

$$W_{ii} = \text{diag}\left(\frac{\beta_{ii}}{b_{pi}\Omega_i}, \frac{\beta_{ii}}{b_{pi}\Omega_i}\right), \qquad \beta_{ii} = \left[\varphi_i'(x_2) - \varphi_i'(x_1)\right] \tag{61}$$

where $i$ is the number of mode shape and $\phi_i'(x_1)$ is the slope of deflection at the boundary.

**Case 1: Point actuator & simply supported boundary**
*Maximum of the curves is optimal placement.*
*l* is length of beam, *k* is number of modes.

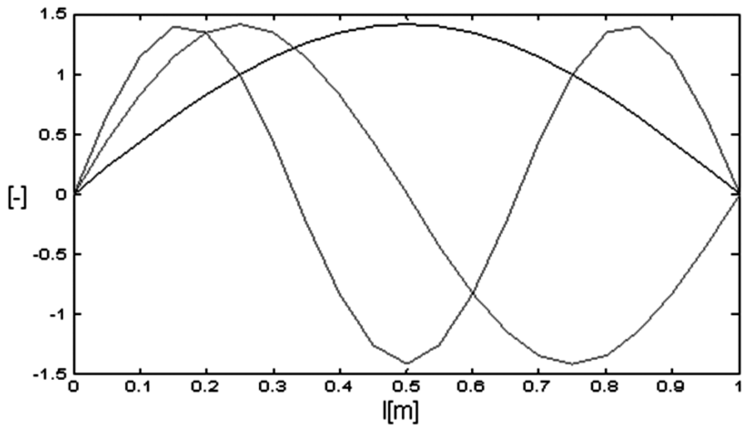


Figure 2. The first three mode shapes for the cases 1 and 3.

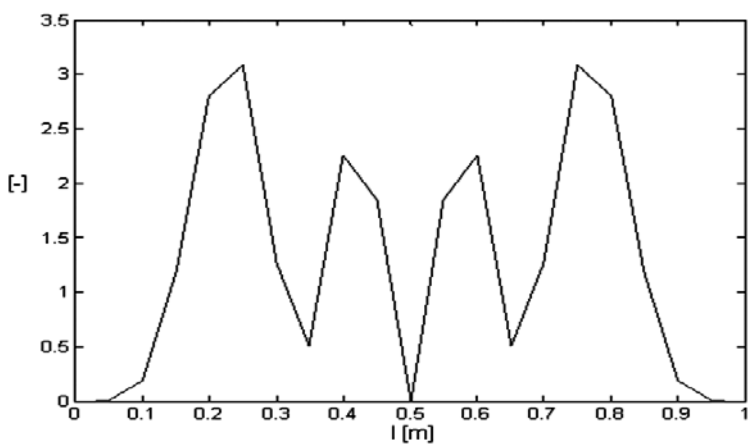


Figure 3. Performance index for the first three mode shape.

**Case 2: Point actuator & clamped boundary at $x = 0$ and free boundary at $x = l$**

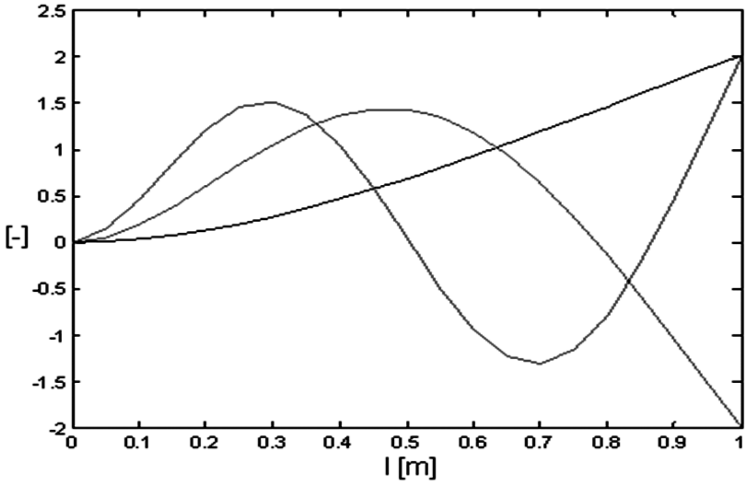


Figure 4. The first three mode shapes for the cases 2 and 4.

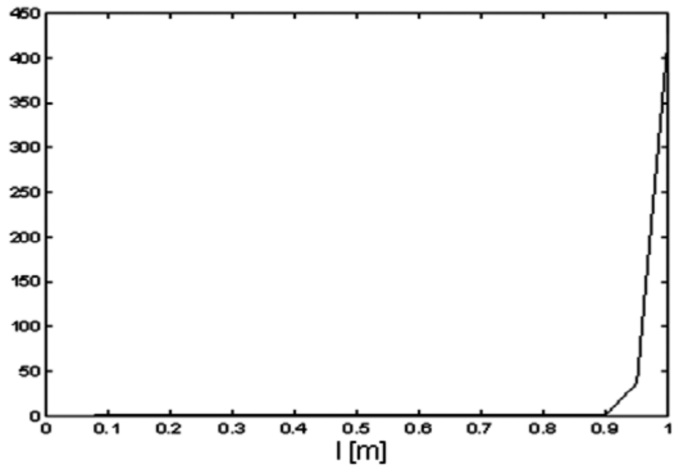


Figure 5. Performance index for the first three mode shapes.

**Case 3: Actuator acting on surface area & simply supported boundary**

$$k = 3$$

The mode shapes for the case 3 are the some as ones for the case 1 (see Fig. 2).

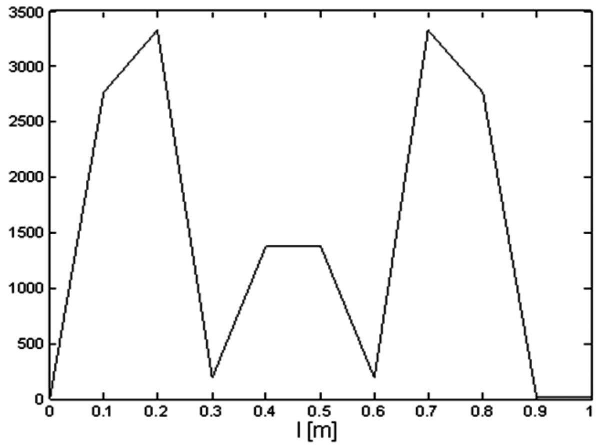


Figure 6. Performance index for the first three mode shapes.

**Case 4: Actuator acting on surface area & clamped boundary at $x = 0$ and free boundary at $x = l$**

$$k = 3$$

The mode shapes for the case 4 are the some as ones for the case 2 (see Fig. 2).

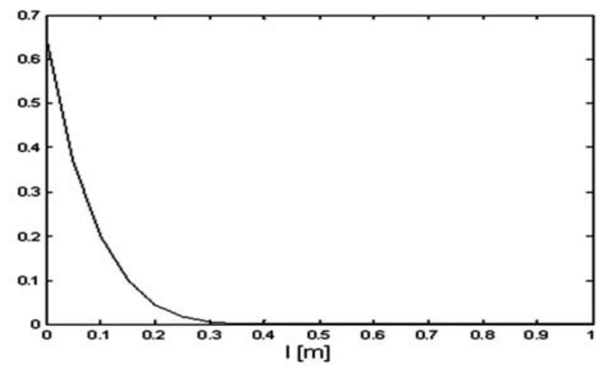


Figure 7. Performance index for the first three mode shapes.

### 2.7. Examples of applications of matrix norms

To illustrate the method, consider uniform beam with two types of boundary conditions:

- simply supported boundary
- clamped boundary at x = 0 and free boundary at x = l

*Maximum of the curves is optimal placement*

**Case 1: Clamped boundary at $x = 0$ and free boundary at $x = l$**

**Evaluation of the actuator (sensor) placement indices via $H_\infty$**

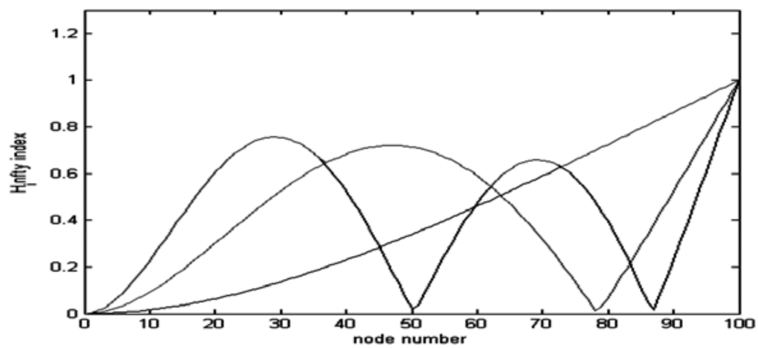


Figure 8. Evaluation of the first, second and third mode shapes simultaneously.

**Evaluation of the actuator (sensor) placement indices via $H_2$**

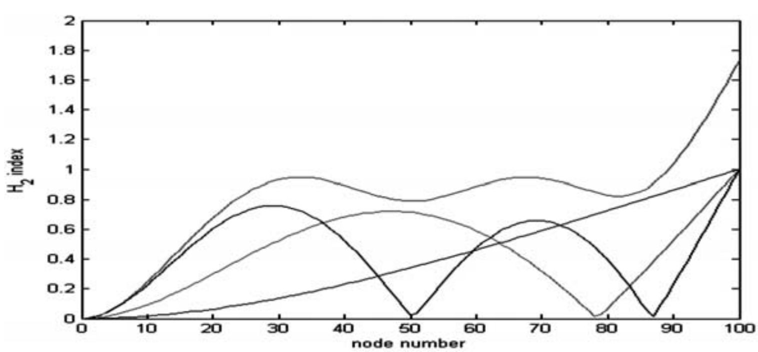


Figure 9. Evaluation of the first, second and third mode shapes simultaneously.

**Case 2: Simply supported boundary**

**Evaluation of the actuator (sensor) placement indices via $H_\infty$**

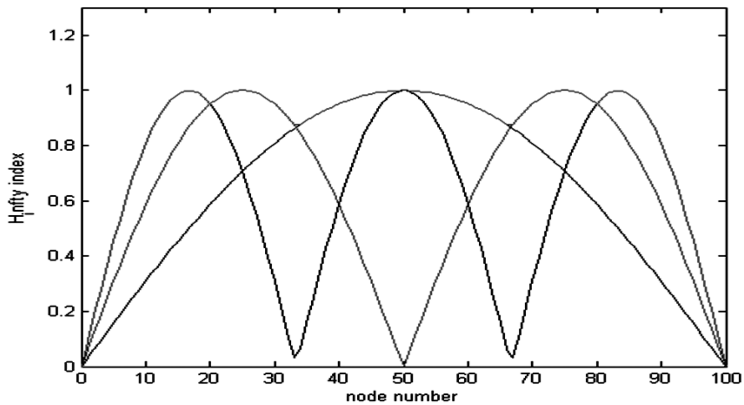


Figure 10. Evaluation of the first and second mode shapes simultaneously.

## 3. Conclusions

In this paper the systematic procedure is outlined that is intended to assist the designer in the selection of sensor and actuator placement in control problems of flexible structures prior to the development of a control strategy. The energy required to transfer the system from the perturbed state to the desired state is minimized by actuator location; the sensor locations are chosen to maximize the output energy integrated over the transient system, which also maximizes the signal-to-noise ratio. It is shown that both transient and perturbed systems are closely related and are equivalent when the structural damping is small and natural frequencies are well separated.

The approach relies on computation of the controllability and observability grammians for which closed form solutions exist for flexible structures. Determination of the objective function for a given actuator/sensor location requires computation of only the eigenvalues of the controllability/observability grammian. For structures with small structural damping and well separated frequencies, these eigenvalues can be approximated by the diagonal elements of the corresponding grammians.

The goals of this paper was to propose and apply the method which is based on grammians for actuator and sensor optimal location and consecutive, compare the results with the results of the method of matrix norms ($H_\infty$ and Hankel are used for definition of the actuator (sensor) index). The comparison both methods look as follow:

- both methods give the same main results,
- the method which used grammians is computational simpler (it required only computation eigenvalues of grammian),
- the method which used grammians can be used only for smaller amount of actuators (sensors), than the method which used matrix norms.

It is advisable to pay a special attention to modeling of actuators, because the results for point actuator and clamped boundary at $x = 0$ and free boundary at $x = l$ and actuator acting on surface area and clamped boundary at $x = 0$ and free boundary at $x = l$ gives significant difference in the optimal placement of sensors and actuators. Both methods will assist to design actuator and sensor optimal placement.

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