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Dead-time compensation in continuous-review perishable inventory systems with a remote supply source

PRZEMYSŁAW IGNACIUK and ANDRZEJ BARTOSZEWICZ

In this paper we address the problem of efficient control of continuous-review perishable inventory systems. In the considered systems the goods at a distribution center used to fulfill unknown, variable demand deteriorate at a constant rate, and are replenished with delay from a remote supply source. We develop a new supply policy which incorporates the Smith predictor for dead-time compensation. A number of properties of the designed policy is formulated, and strictly proved. In particular, we show that the policy guarantees that the assigned storage space at the distribution center is never exceeded which means that the cost of emergency storage is eliminated. Moreover, we show that with appropriately chosen controller parameters all of the demand imposed at the distribution center is realized from the readily available resources, thus ensuring the maximum service level.

Key words: inventory control, perishable inventory systems, time-delay systems, Smith predictor

1. Introduction

It follows from the extensive review papers documenting the research work in the past [4, 9-11, 13, 14] that certain areas of inventory control are not sufficiently addressed at the formal design level. This concerns in particular a large and very important class of problems related to the management of perishable commodities (food, drugs, gasoline, etc.). The main difficulty in developing control schemes for perishable inventories stems from the necessity of conducting exact analysis of product lifetimes. The design problem becomes cumbersome in the situation when the product demand is subject to

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significant uncertainty and inventories are replenished with nonnegligible delay, which frequently happens in modern supply chains. In such circumstances, in order to maintain high service level and at the same time keep stringent cost discipline, when placing an order it is necessary not only to account for the demand during procurement latency but also for the stock deterioration in that time.

Since the stock accumulation of perishables cannot be represented as pure integration, the effects of order procurement delay cannot be adequately accounted for by introducing the notion of work-in-progress or inventory position variables (constituting the sum of the on-hand and on-order goods), as it has been done in a number of successful research works for nondecaying inventories, e.g. [1, 3]. In contrast to our earlier results devoted exclusively to the periodic-review inventory systems with nondeteriorating stock [6]-[8], in this work we analyze continuous-review systems with random lifetime of the stored goods. Since the major obstacle in the considered application is the delay in the feedback loop (see e.g. [5] for a discussion of the influence of lead-time delay on the dynamics of the traditional inventory systems), we explicitly address the issues related to order procurement latency in system modeling and controller design.

In order to solve the stability problems related to nonnegligible delay, we propose to apply the Smith predictor [15]. The designed control strategy is demonstrated to establish nonnegative and bounded ordering signal, which is a crucial requirement for the practical implementation of any replenishment rule. It is also shown that in the inventory system governed by the proposed policy the stock level never exceeds the assigned warehouse capacity. This means that the potential necessity for expensive emergency storage outside the company premises is eliminated. At the same time, we demonstrate that the stock is never depleted, which implies full demand satisfaction from the readily available resources and the 100% service level. It is noted, however, that the designed controller may generate overshoots at the output in response to abrupt demand changes. Therefore, in order to overcome the problems related to the increased storage space due to the stock level overshoots, we propose a modified controller. The improved policy retains all the favorable properties of the original scheme, yet avoids excessive stock growth in the situation of sudden changes in the market trend. The proposed control strategies are compared with the classical ordering rule - order-up-to policy (see e.g. [3] for a comprehensive description of fundamental supply policies). It is shown that in the inventory system with perishable goods our strategy outperforms the classical one in terms of smoother ordering decisions, smaller storage space requirement, and reduced order-to-demand variance ratio.

The paper is organized in the following way. First, in Section 2 we formulate the inventory control problem and provide mathematical description of the relevant system model. Next, in Section 3, we design the first strategy incorporating the Smith predictor for dead-time compensation. In Section 4, we introduce the second, enhanced controller, and prove its properties. The schemes are compared with the traditional ordering policy analytically in Section 5, and in numerical tests in Section 6. Finally, we provide the conclusions in Section 7.

2. Problem formulation

We consider an inventory system in which the goods at a distribution center used to fulfill the customers' (or retailers) demand are acquired with delay from a remote supply source. Such setting, illustrated in Fig. 1, is frequently encountered in production-inventory systems where a common point (distribution center), linked to a factory or an external, strategic supplier, is used to provide goods for another production stage or a distribution network. The task is to design a control strategy which, on one hand, will minimize the holding and shortage costs, and, on the other hand, will ensure smooth flow of goods despite unpredictable changes in the market conditions.

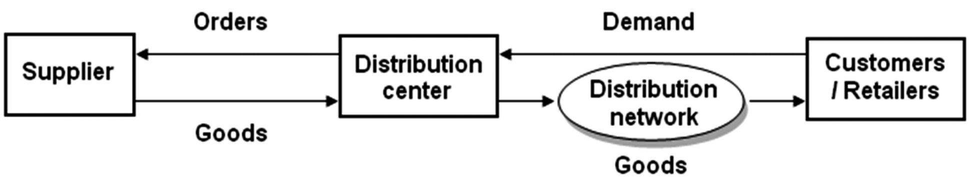

Figure 1. Inventory system with a strategic supplier.

The imposed demand (the requests for goods coming from the market) is modeled as an *a priori* unknown, bounded function of time $d(t)$, where t represents time. We assume that demand can follow any statistical distribution as long as $0 \leqslant d(t) \leqslant d_{max}$, where d_{max} is a positive constant. If there is a sufficient number of items at the distribution center to satisfy the imposed demand, then the actually met demand $h(t)$ (the goods sold to customers or sent to retailers in the distribution network) will be equal to the requested one. Otherwise, the imposed demand is satisfied only from the arriving shipments, and the additional demand is lost (we assume that the sales are not backordered, and the excessive demand is equivalent to a missed business opportunity). Thus,

$$0 \leqslant h(t) \leqslant d(t) \leqslant d_{\max}. \tag{1}$$

The on-hand stock used to fulfill the market demand deteriorates at a constant rate σ, $0 \leqslant \sigma < 1$, when kept in the distribution center warehouse. The stock is replenished with delay $L_p > 0$ from a remote supply source. Denoting the quantity ordered from the supplier at time t by $u(t)$, and the received shipment by $u_R(t)$, we have

$$u_R(t) = u(t - L_p). \tag{2}$$

Consequently, the stock balance equation can be written in the following way

$$\dot{y} = -\sigma y(t) + u_R(t) - h(t) = -\sigma y(t) + u(t - L_p) - h(t). \tag{3}$$

According to the stock balance equation, the on-hand stock decreases due to the realized sales represented by function $h(\cdot)$, and the decay characterized by factor σ. It is refilled from the goods acquired from the supplier $u_R(\cdot)$. For the sake of further analysis it is convenient to represent (3) in an integral form. We assume that initially the warehouse is empty, i.e. $y(0) = 0$, and the first orders are placed at $t = 0$, i.e. $u(t) = 0$ for $t < 0$. Solving (3) for $y(\cdot)$, we obtain (see the Appendix)

$$y(t) = \int_0^t e^{-\sigma(t-\tau)} u_R(\tau)\, d\tau - \int_0^t e^{-\sigma(t-\tau)} h(\tau)\, d\tau. \tag{4}$$

Since $u_R(t) = u(t - L_p)$ and $u(t < 0) = 0$, we can rewrite (4) in the following form

$$\begin{aligned} y(t) &= \int_0^t e^{-\sigma(t-\tau)} u(\tau - L_p)\, d\tau - \int_0^t e^{-\sigma(t-\tau)} h(\tau)\, d\tau \\ &= \int_0^{t-L_p} e^{-\sigma(t-L_p-\tau)} u(\tau)\, d\tau - \int_0^t e^{-\sigma(t-\tau)} h(\tau)\, d\tau. \end{aligned} \tag{5}$$

Note that in order to adequately model the stock accumulation of perishable goods, a saturating integrator needs to be applied, which makes the considered system nonlinear. However, if one can ensure that the control signal is nonnegative for arbitrary t, then by introducing the function representing the actually realized sales, $h(t) \leqslant d(t)$, the stock dynamics is reduced to linear equation (5). In the further part of the paper, we will design a control law which will be shown to satisfy the conditions $u(t) \geqslant 0$ and $h(t) = d(t)$. As a result, the inventory system will stay in the linear region of operation for the whole range of the external disturbance $0 \leqslant d(t) \leqslant d_{max}$. The system block diagram with the linear part represented using transfer functions is shown in Fig. 2. The saturating integrator in the internal loop accounts for accumulating the stock of perishables characterized by decay factor σ. The controller, with transfer function $G_C(s)$, is supposed to steer the on-hand stock level $y(t)$ towards the reference value y_{ref} such that high level of demand satisfaction is achieved.

3. Smith predictor based controller

The primary obstacle in providing efficient control in the considered class of systems is the latency in procuring orders. Indeed, each nonzero order placed at the supplier at instant t will appear at the distribution center with lead-time L_p at instant $t + L_p > t$ which may lead to oscillations, or even cause instability. In order to satisfactorily counteract the adverse effects of delay in the analyzed system with perishable goods, it is not sufficient to introduce the inventory position variables (constituting the sum of on-hand stock and open orders), or the notion of work-in-progress, as it is usually done in the traditional inventory systems with nondeteriorating stock [1, 3]. This is due to the fact that the pure

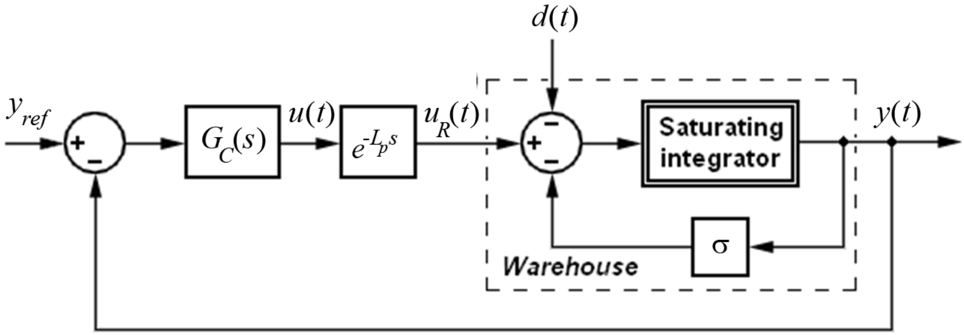

Figure 2. System model.

sum of open orders (or the work-in-progress) does not account for the stock degradation within lead-time. To overcome the delay problem, in this work we propose to apply the Smith predictor [15], which proved a successful method of dead-time compensation in many engineering areas [12]. The basic idea behind the Smith predictor is to simulate the behavior of a remote plant inside the controller structure, thus eliminating the delay from the main feedback loop. The controller incorporating the Smith predictor is described in detail in a latter part of this section.

3.1. Principal control strategy

The schematic diagram of the proposed control strategy employing the Smith predictor for dead-time compensation is illustrated in Fig. 3.

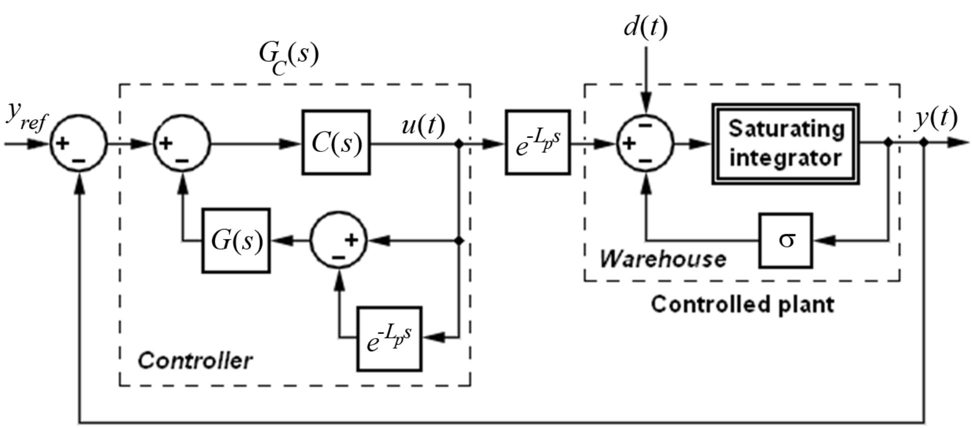

Figure 3. Closed-loop system with a Smith predictor based controller.

The control structure consists of the primary plant controller $C(s)$ and the Smith predictor using the linearized model of the plant $G(s) = 1/(s+\sigma)$. With the primary controller selected as the proportional one $C(s) = K$, where K is a positive constant, we obtain the transfer function of the overall control structure $G_C(s)$,

$$G_C(s) = \frac{C(s)}{1 + C(s)\left[G(s) - e^{-L_p s}G(s)\right]} = \frac{K}{1 + KG(s)\left(1 - e^{-L_p s}\right)}. \tag{6}$$

In the linear region of operation the plant dynamics is fully represented by the transfer function $G(s) = 1/(s+\sigma)$. Then, we can write the closed-loop transfer functions:

a) with respect to the reference input $Y_{ref}(s) = y_{ref}/s$,

$$\frac{Y(s)}{Y_{ref}(s)} = \frac{G_C(s)\,e^{-L_p s}G(s)}{1 + G_C(s)\,e^{-L_p s}G(s)} = \frac{K}{s+\sigma+K}e^{-L_p s}, \tag{7}$$

b) with respect to the disturbance $D(s) = \mathcal{L}(d(t))$,

$$\frac{Y(s)}{D(s)} = -\frac{G(s)}{1 + G_C(s)\,e^{-L_p s}G(s)} = -\frac{s+\sigma+K\left(1 - e^{-L_p s}\right)}{(s+\sigma)(s+\sigma+K)}. \tag{8}$$

It is clear from (7) and (8) that the term related to delay is eliminated from the characteristic equation (the denominator of the closed-loop transfer function). Consequently, since $K > 0$ and $\sigma \geqslant 0$, the closed-loop system under nominal operating conditions is stable for arbitrary lead-time and any bounded disturbance. Moreover, since the closed-loop poles lie on the negative real axis, the oscillations are avoided at the output.

3.2. Properties of the proposed strategy

Before we state the properties of inventory policy (6), it is convenient to present it in time domain. Taking into account the initial conditions, we can write the policy in time domain by direct inspection of the block diagram shown in Fig. 3 in the following form

$$u(t) = K\left[y_{ref} - y(t) - \int_0^t e^{-\sigma(t-\tau)}u(\tau)\,d\tau + \int_0^{t-L_p} e^{-\sigma(t-L_p-\tau)}u(\tau)\,d\tau\right]. \tag{9}$$

This control law can be interpreted as to generate orders in proportion to the difference between the current on-hand stock and its reference value $K[y_{ref} - y(t)]$ decreased by the amount of open orders quantified by the rate of deterioration within the last lead-time (the integrals in formula (9)).

The properties of the proposed control strategy will be given in three theorems, and strictly proved. The first proposition shows that the ordering signal generated by controller (9) is always nonnegative and bounded, which is a crucial prerequisite for the implementation of any cost-efficient inventory management policy. The second theorem specifies the upper bound of the on-hand stock, which constitutes the smallest warehouse capacity required to store all the incoming shipments. Finally, the third proposition shows how to select the stock reference value in order to guarantee that all of the imposed demand will be fulfilled from the readily available resources at the distribution center. As a result the maximum service level is ensured.

Theorem 1 *For any time* $t \geqslant 0$ *the ordering signal generated by controller* (9) *applied to system* (3) *satisfies the following inequalities*

$$K\frac{\sigma y_{ref}}{\sigma+K} \leqslant u(t) \leqslant K y_{ref}. \tag{10}$$

Moreover, there exists a time instant t_0 *such that for any* $t \geqslant t_0$

$$u(t) \leqslant K\frac{\sigma y_{ref}+d_{\max}}{\sigma+K}. \tag{11}$$

Proof Substituting (5) into (9) we get

$$u(t) = K\left[y_{ref} - \int_0^t e^{-\sigma(t-\tau)}u(\tau)\,d\tau + \int_0^t e^{-\sigma(t-\tau)}h(\tau)\,d\tau\right]. \tag{12}$$

Consequently, the derivative

$$\begin{aligned}
\dot{u} &= -K\frac{d}{dt}\left\{e^{-\sigma t}\int_0^t e^{\sigma\tau}\left[u(\tau)-h(\tau)\right]d\tau\right\} \\
&= -K\left\{-\sigma e^{-\sigma t}\int_0^t e^{\sigma\tau}\left[u(\tau)-h(\tau)\right]d\tau + e^{-\sigma t}e^{\sigma t}\left[u(t)-h(t)\right]\right\} \\
&= K\left\{\sigma\int_0^t e^{-\sigma(t-\tau)}\left[u(\tau)-h(\tau)\right]d\tau - \left[u(t)-h(t)\right]\right\}.
\end{aligned} \tag{13}$$

It follows from (12) that

$$\sigma K\int_0^t e^{-\sigma(t-\tau)}\left[u(\tau)-h(\tau)\right]d\tau = \sigma\left[K y_{ref} - u(t)\right]. \tag{14}$$

Hence, we can rewrite (13) as

$$\dot{u} = \sigma\left[K y_{ref} - u(t)\right] - K\left[u(t)-h(t)\right] = \sigma K y_{ref} - (\sigma+K)u(t) + Kh(t). \tag{15}$$

Investigating $\dot{u} = 0$ we get

$$u(t) = K\frac{\sigma y_{ref} + h(t)}{\sigma + K}. \tag{16}$$

According to constraint (1) the minimum satisfied demand equals zero. At the initial time $u(0) = Ky_{ref} > 0$. Therefore, since $0 \leqslant \sigma < 1$ and $h(\cdot) \geqslant 0$, we get from (16) that $u(\cdot)$ decreases as long as it is bigger than $K[\sigma y_{ref} + h(\cdot)]/(\sigma + K)$, and it never falls below $K\sigma y_{ref}/(\sigma + K)$. Moreover, there exists a time instant t_0 when $u(\cdot)$ reaches the level of $K[\sigma y_{ref} + d_{max}]/(\sigma + K)$ for the first time. Since $h(\cdot) \leqslant d_{max}$, we get from (16) that for all $t \geqslant t_0$

$$u(t) \leqslant K\frac{\sigma y_{ref} + d_{\max}}{\sigma + K}.$$

This conclusion ends the proof. □

Theorem 2 *If policy* (9) *is applied to system* (3)*, then the on-hand stock at the distribution center never exceeds the level of* y_{ref} *for* $\sigma = 0$ *and*

$$y_{\max} = \frac{K}{\sigma + K}\left[y_{ref} + \frac{d_{\max}}{\sigma}\left(1 - e^{-\sigma L_p}\right)\right] \quad \text{for} \quad \sigma > 0. \tag{17}$$

Proof Applying (12) to the stock balance equation (3), we get

$$\dot{y} = -\sigma y(t) + Ky_{ref} - K\left[\int_0^{t-L_p} e^{-\sigma(t-L_p-\tau)}u(\tau)\,d\tau - \int_0^{t-L_p} e^{-\sigma(t-L_p-\tau)}h(\tau)\,d\tau\right] - h(t), \tag{18}$$

which can be rewritten as

$$\begin{aligned}\dot{y} = -\sigma y(t) + Ky_{ref} - K&\left[\int_0^{t-L_p} e^{-\sigma(t-L_p-\tau)}u(\tau)\,d\tau - \int_0^{t} e^{-\sigma(t-\tau)}h(\tau)\,d\tau\right] \\ -K\int_0^{t} e^{-\sigma(t-\tau)}h(\tau)\,d\tau &+ K\int_0^{t-L_p} e^{-\sigma(t-L_p-\tau)}h(\tau)\,d\tau - h(t).\end{aligned} \tag{19}$$

Using (5) we can notice that the term in the square brackets in (19) actually equals $y(t)$.

Consequently, we have

$$\dot{y} = Ky_{ref} - (\sigma+K)\,y(t) - K\int_{t-L_p}^{t} e^{-\sigma(t-\tau)}h(\tau)\,d\tau - K\int_{0}^{t-L_p} e^{-\sigma(t-\tau)}h(\tau)\,d\tau$$
$$+K\int_{0}^{t-L_p} e^{-\sigma(t-L_p-\tau)}h(\tau)\,d\tau - h(t) = Ky_{ref} - (\sigma+K)\,y(t) - h(t) \tag{20}$$
$$+K\left(e^{\sigma L_p}-1\right)\int_{0}^{t-L_p} e^{-\sigma(t-\tau)}h(\tau)\,d\tau - K\int_{t-L_p}^{t} e^{-\sigma(t-\tau)}h(\tau)\,d\tau.$$

Investigating $\dot{y} = 0$ leads to

$$y(t) = \frac{Ky_{ref}}{\sigma+K} + \frac{K}{\sigma+K}\left(e^{\sigma L_p}-1\right)\int_{0}^{t-L_p} e^{-\sigma(t-\tau)}h(\tau)\,d\tau$$
$$-\frac{K}{\sigma+K}\left[\int_{t-L_p}^{t} e^{-\sigma(t-\tau)}h(\tau)\,d\tau + \frac{h(t)}{K}\right]. \tag{21}$$

It follows from (21) that since $K > 0$, $\sigma \geqslant 0$, $exp(\sigma L_p) \geqslant 1$, and $h(\cdot) \geqslant 0$, the biggest value of $y(\cdot)$ is expected when $h(\tau) = d_{max}$ for $\tau \leqslant t - L_p$ and $h(\tau) = 0$ in the interval $(t - L_p,\ t]$. We get immediately from (21) that for $\sigma = 0$ (the case of nondeteriorating stock)

$$y(t) = y_{ref} + 0 - \int_{t-L_p}^{t} h(\tau)\,d\tau - \frac{h(t)}{K} \leqslant y_{ref}. \tag{22}$$

Evaluating the first integral in (21) for the case of $\sigma > 0$, we obtain

$$\int_{0}^{t-L_p} e^{-\sigma(t-\tau)}h(\tau)\,d\tau \leqslant d_{\max}\int_{0}^{t-L_p} e^{-\sigma(t-\tau)}d\tau = d_{\max}e^{-\sigma t}\int_{0}^{t-L_p} e^{\sigma\tau}d\tau$$
$$= d_{\max}e^{-\sigma t}\left(\frac{e^{\sigma\tau}}{\sigma}\right)\Bigg|_{0}^{t-L_p} = d_{\max}\frac{e^{-\sigma t}}{\sigma}\left[e^{\sigma(t-L_p)}-1\right] \tag{23}$$
$$= \frac{d_{\max}}{\sigma}\left[e^{-\sigma L_p}-e^{-\sigma t}\right] \leqslant \frac{d_{\max}}{\sigma}e^{-\sigma L_p}.$$

Consequently, applying (23) to (21), we arrive at

$$y(t) \leqslant \frac{K}{\sigma+K}\left[y_{ref} + \left(e^{\sigma L_p} - 1\right)\frac{d_{\max}}{\sigma}e^{-\sigma L_p}\right] = \frac{K}{\sigma+K}\left[y_{ref} + \frac{d_{\max}}{\sigma}\left(1 - e^{-\sigma L_p}\right)\right] = y_{\max}. \tag{24}$$

This ends the proof. □

It follows from Theorem 2 that if the warehouse of size y_{max} specified by (17) is assigned at the distribution center, then all the incoming shipments can be stored locally, and any cost associated with emergency storage is eliminated. Apart from the efficient warehouse space management, a successful inventory control strategy in modern supply chain is expected to achieve high level of demand satisfaction. The proposition formulated below shows how the reference stock level should be selected so that $y(t) > 0$, which implies that all of the demand imposed on the distribution center is satisfied from the readily available resources.

Theorem 3 *If policy* (9) *is applied to system* (3)*, and the reference stock level is selected as*

$$y_{ref} > d_{\max}\left(L_p + 1/K\right) \quad \text{for} \quad \sigma = 0, \tag{25}$$

$$y_{ref} > d_{\max}\left[\left(1 - e^{-\sigma L_p}\right)/\sigma + 1/K\right] \quad \text{for} \quad \sigma > 0, \tag{26}$$

then the on-hand stock level at the distribution center is strictly positive for any $t > L_p$.

Proof : Note that $e^{\sigma L_p} - 1 \geqslant 0$. Hence, considering (1) and (21), we can expect the smallest on-hand stock level in the circumstances when $h(\tau) = 0$ for $\tau \leqslant t - L_p$ and $h(\tau) = d_{max}$ for τ belonging to the interval $(t - L_p, t]$. It follows from the assumed initial conditions that the warehouse is empty for any $t \leqslant L_p$. In the case of the system with nondeteriorating stock $(\sigma = 0)$ we get from (21)

$$y(t) = y_{ref} + 0 - \int_{t-L_p}^{t} h(\tau)\,d\tau - \frac{h(t)}{K} \geqslant y_{ref} - d_{\max}\left(L_p + 1/K\right). \tag{27}$$

Thus, using assumption (25) we have $y(t) > 0$ for $\sigma = 0$. Evaluating the second integral in (21) for $t > L_p$ in the case when $h(t) = d_{max}$ and $\sigma > 0$, we obtain

$$\begin{aligned}\int_{t-L_p}^{t} e^{-\sigma(t-\tau)}h(\tau)\,d\tau &\leqslant d_{\max}\int_{t-L_p}^{t} e^{-\sigma(t-\tau)}d\tau = d_{\max}e^{-\sigma t}\int_{t-L_p}^{t} e^{\sigma\tau}d\tau \\ &= d_{\max}e^{-\sigma t}\left(\frac{e^{\sigma\tau}}{\sigma}\right)\Bigg|_{t-L_p}^{t} = d_{\max}\frac{e^{-\sigma t}}{\sigma}\left[e^{\sigma t} - e^{\sigma(t-L_p)}\right] = \frac{d_{\max}}{\sigma}\left(1 - e^{-\sigma L_p}\right).\end{aligned} \tag{28}$$

Applying (28) to (21), we get the on-hand stock level $y(\cdot)$ at the instant when it is minimum

$$y(t) \geqslant \frac{K}{\sigma + K}\left[y_{ref} - \frac{d_{\max}}{\sigma}\left(1 - e^{-\sigma L_p}\right) - \frac{d_{\max}}{K}\right]. \tag{29}$$

If the reference stock level is adjusted according to (26), then using (29) one may conclude that

$$y(t) \geqslant \frac{K}{\sigma + K}\left\{y_{ref} - d_{\max}\left[\left(1 - e^{-\sigma L_p}\right)/\sigma + 1/K\right]\right\} > 0. \tag{30}$$

This completes the proof. □

4. Modified control law

The Smith predictor based controller designed in the previous section establishes a smooth, nonoscillatory ordering signal. However, an overshoot in the output variable (the stock level) may be generated for an abrupt change in the market trend. This leads to increased warehouse capacity required to accommodate the resulting spike in $y(\cdot)$. Below, we formulate an enhanced control law, which allows for maintaining all the favorable properties of the original strategy, yet with an overshoot-free output, and thus reduced warehouse capacity.

4.1. Proposed control strategy

We propose to apply the following control law

$$u(t) = K\left[y_{ref} - e^{-\sigma L_p} y(t) - \int_{t-L_p}^{t} e^{-\sigma(t-\tau)} u(\tau)\, d\tau\right]. \tag{31}$$

Therefore, in the modified scheme, the order quantity is established in proportion to the reference stock level minus the scaled output (the term $exp(-\sigma L_p)y(t)$), decreased by the amount of goods on-route quantified by the rate of deterioration within lead-time (the integral in (31)). The structure of the improved controller is illustrated in Fig. 4. We can notice from Fig. 4 that the constant delay offset, $exp(-\sigma L_p)$, is introduced in the feedback and compensating loops.

The transfer function of the modified controller is determined as

$$G_C(s) = \frac{C(s)}{1 + C(s)\left[G(s) - e^{-L_p s} e^{-\sigma L_p} G(s)\right]} = \frac{K}{1 + KG(s)\left(1 - e^{-L_p(s+\sigma)}\right)}, \tag{32}$$

and the closed-loop transfer functions:

a) with respect to the reference input $Y_{ref}(s)$ as given by (7),

b) with respect to the disturbance $D(s)$ as

$$\frac{Y(s)}{D(s)} = -\frac{s+\sigma+K\left(1-e^{-L_p(s+\sigma)}\right)}{(s+\sigma)(s+\sigma+K)}. \tag{33}$$

Similarly as in the case of controller (9), the term related to delay is eliminated from the characteristic equation. Therefore, since $K > 0$ and $\sigma \geqslant 0$, the closed-loop system with controller (31) implemented is stable for arbitrary lead-time and any bounded disturbance, and no oscillations appear at the output.

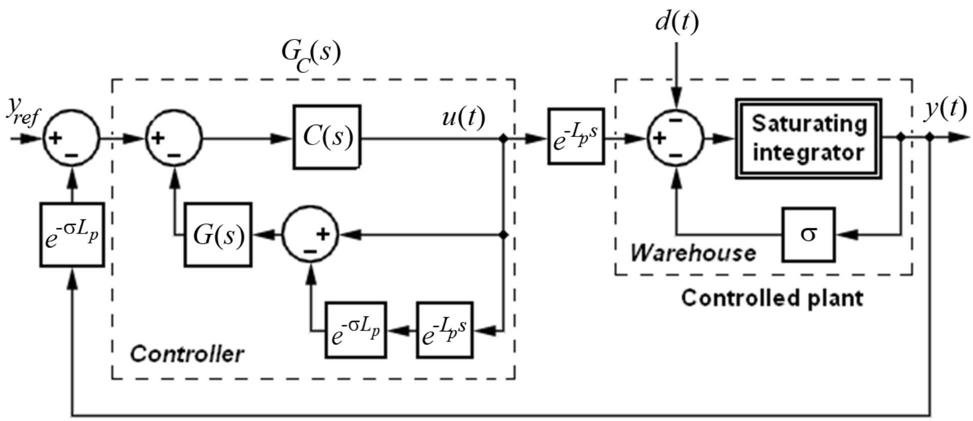

Figure 4. Closed-loop system with the improved controller (31).

4.2. Properties of improved strategy

The properties of controller (31) will be formulated as three theorems, and strictly proved. The first proposition shows that the ordering signal generated by the proposed controller is non-negative and bounded. The second theorem specifies the upper limit of the stock level. Finally, the third proposition shows how the reference stock level should be selected so that all of the demand is realized, and the maximum service level is obtained.

Theorem 4 *For any time $t \geqslant 0$ the ordering signal generated by controller* (31) *applied to system* (3) *satisfies inequalities* (10). *Moreover, there exists a time instant t_0 such that for any $t \geqslant t_0$*

$$u(t) \leqslant K\frac{\sigma y_{ref} + e^{-\sigma L_p} d_{\max}}{\sigma + K}. \tag{34}$$

Proof Substituting (5) into (31) we get

$$u(t)=K\left\{y_{ref}-e^{-\sigma L_p}\left[\int_0^{t-L_p} e^{-\sigma(t-L_p-\tau)}u(\tau)\,d\tau-\int_0^t e^{-\sigma(t-\tau)}h(\tau)\,d\tau\right]-\int_{t-L_p}^t e^{-\sigma(t-\tau)}u(\tau)\,d\tau\right\}$$
$$=K\left[y_{ref}-\int_0^t e^{-\sigma(t-\tau)}u(\tau)\,d\tau+e^{-\sigma L_p}\int_0^t e^{-\sigma(t-\tau)}h(\tau)\,d\tau\right]. \tag{35}$$

Consequently,

$$\dot{u}=-K\frac{d}{dt}\left\{e^{-\sigma t}\int_0^t e^{\sigma\tau}\left[u(\tau)-e^{-\sigma L_p}h(\tau)\right]d\tau\right\}$$
$$=-K\left\{-\sigma e^{-\sigma t}\int_0^t e^{\sigma\tau}\left[u(\tau)-e^{-\sigma L_p}h(\tau)\right]d\tau+e^{-\sigma t}e^{\sigma t}\left[u(t)-e^{-\sigma L_p}h(t)\right]\right\} \tag{36}$$
$$=K\left\{\sigma\int_0^t e^{-\sigma(t-\tau)}\left[u(\tau)-e^{-\sigma L_p}h(\tau)\right]d\tau-\left[u(t)-e^{-\sigma L_p}h(t)\right]\right\}.$$

It follows from (35) that

$$\sigma K\int_0^t e^{-\sigma(t-\tau)}\left[u(\tau)-e^{-\sigma L_p}h(\tau)\right]d\tau=\sigma\left[Ky_{ref}-u(t)\right]. \tag{37}$$

Hence, we can rewrite (36) as

$$\dot{u}=\sigma\left[Ky_{ref}-u(t)\right]-K\left[u(t)-e^{-\sigma L_p}h(t)\right]=\sigma Ky_{ref}-(\sigma+K)\,u(t)+Ke^{-\sigma L_p}h(t). \tag{38}$$

Investigating $\dot{u}=0$ we get

$$u(t)=K\frac{\sigma y_{ref}+e^{-\sigma L_p}h(t)}{\sigma+K}. \tag{39}$$

According to constraint (1) the minimum satisfied demand equals zero. At the initial time $u(0)=Ky_{ref}>0$. Therefore, since $0\leqslant\sigma<1$ and $h(\cdot)\geqslant 0$, we get from (39) that $u(\cdot)$ decreases as long as it is bigger than $K[\sigma y_{ref}+exp(-\sigma L_p)h(\cdot)]/(\sigma+K)$, and it never falls below $K\sigma y_{ref}/(\sigma+K)$. Moreover, there exists a time instant t_0 when $u(\cdot)$ reaches the level of $K[\sigma y_{ref}+exp(-\sigma L_p)d_{max}]/(\sigma+K)$ for the first time. Since $h(\cdot)\leqslant d_{max}$, we get from (39) that for all $t\geqslant t_0$

$$u(t)\leqslant K\frac{\sigma y_{ref}+e^{-\sigma L_p}d_{\max}}{\sigma+K}.$$

This conclusion ends the proof. □

Theorem 5 *If policy* (31) *is applied to system* (3)*, then the on-hand stock at the distribution center never exceeds the level of* $Ky_{ref}/(\sigma+K)$.

Proof Applying (35) to the stock balance equation (3), we get

$$\dot{y} = -\sigma y(t) + Ky_{ref} - K\left[\int_0^{t-L_p} e^{-\sigma(t-L_p-\tau)}u(\tau)\,d\tau - e^{-\sigma L_p}\int_0^{t-L_p} e^{-\sigma(t-L_p-\tau)}h(\tau)\,d\tau\right] - h(t). \tag{40}$$

Using (5), we can represent (40) in the alternative form

$$\dot{y} = -\sigma y(t) + Ky_{ref} - Ky(t) - K\int_{t-L_p}^{t} e^{-\sigma(t-\tau)}h(\tau)\,d\tau - h(t). \tag{41}$$

Investigating $\dot{y}=0$, we obtain

$$y(t) = \frac{K}{\sigma+K}\left[y_{ref} - \int_{t-L_p}^{t} e^{-\sigma(t-\tau)}h(\tau)\,d\tau - \frac{h(t)}{K}\right]. \tag{42}$$

Therefore, since $K>0$, $\sigma \geqslant 0$, and $h(\cdot)\geqslant 0$, it follows from (42) that the biggest value of $y(\cdot)$ is expected when $h(\tau)=0$, i.e. $y(t) \leqslant Ky_{ref}/(\sigma+K)$. This ends the proof. □

Theorem 6 *If policy* (31) *is applied to system* (3)*, and the reference stock level is selected as* (26)*, then the on-hand stock level at the distribution center is strictly positive for any* $t > L_p$.

Proof It follows from the assumed initial conditions that $y(t)=0$ for $t \leqslant L_p$. On the other hand, we get from (1) that the maximum realized demand equals d_{max}. Consequently, using (42) and (28), we get

$$\begin{aligned} y(t) &\geqslant \frac{K}{\sigma+K}\left[y_{ref} - \frac{d_{\max}}{\sigma}\left(1-e^{-\sigma L_p}\right) - \frac{d_{\max}}{K}\right] \\ &= \frac{K}{\sigma+K}\left\{y_{ref} - d_{\max}\left[\left(1-e^{-\sigma L_p}\right)/\sigma + 1/K\right]\right\}. \end{aligned} \tag{43}$$

Applying the theorem assumption, we obtain $y(t) > 0$. This completes the proof. □

Remark It follows from Theorems 3 and 6 that the proposed controllers (9) and (31) use the same reference stock value to ensure full demand utilization. Theorem 5 states that the on-hand stock in the system regulated by controller (31) never exceeds the level of $Ky_{ref}/(\sigma+K)$, which is smaller than the maximum stock level expected in the system regulated by controller (9) stated in Theorem 2. Consequently, the improved controller (31) requires less storage space and imposes smaller holding costs than policy (9). Consequently, it offers a less costly solution to the goods flow control problem in the analyzed class of systems with deteriorating stock. We will show in the simulation example considered in a latter part of the paper that it also outperforms the Smith predictor based strategy in throttling the demand variations, thus providing a viable solution to eliminating the bullwhip effect (amplification of demand variations translated to the ordering signal) in perishable inventory systems.

5. Relation to the classical inventory policies

In this section, we compare the proposed inventory management policies with the classical ones - order-up-to (OUT) policy, and heuristically determined proportional OUT (POUT) policy.

5.1. Classical inventory policies

In the case when demand forecasting is not used, the classical OUT policy can be synthesized in the following way

$$u_{\mathrm{OUT}}(t) = y_{\mathrm{OUT}} - y(t) - \mathrm{WIP}(t), \tag{44}$$

where y_{OUT} is the order-up-to level, $y(t)$ is the current stock value, and WIP(t) represents the pending order (order placed but not yet realized due to lead-time). Notice that in the considered system with delay the pending order can be calculated by summing orders $u(\cdot)$ generated within lead-time L_p. Therefore, $\mathrm{WIP}(t) = \int_{t-L_p}^{t} u_{\mathrm{OUT}}(\tau)d\tau$, and the OUT policy can be rewritten as

$$u_{\mathrm{OUT}}(t) = y_{\mathrm{OUT}} - y(t) - \int_{t-L_p}^{t} u_{\mathrm{OUT}}(\tau)\, d\tau. \tag{45}$$

On the other hand, in the systems where the bullwhip effect is of significant concern, a different ordering rule typically needs to be applied. A successful modification of the classical OUT policy aimed at smoothening the order variations and thus counteracting the bullwhip effect in the traditional inventory systems is the POUT policy [3]. When demand forecasting is not applied, the POUT policy can be presented as

$$u_{\mathrm{POUT}}(t) = \frac{1}{T_n}\left[y_{\mathrm{POUT}} - y(t)\right] - \frac{1}{T_w}\int_{t-L_p}^{t} u_{\mathrm{POUT}}(\tau)\, d\tau, \tag{46}$$

where T_n and T_w are positive constants used for tuning purposes. Note that for $T_n = T_w = 1$ the POUT policy actually reduces to one given by (45). The structure of policy (46) represented by means of transfer functions is illustrated in Fig. 5.

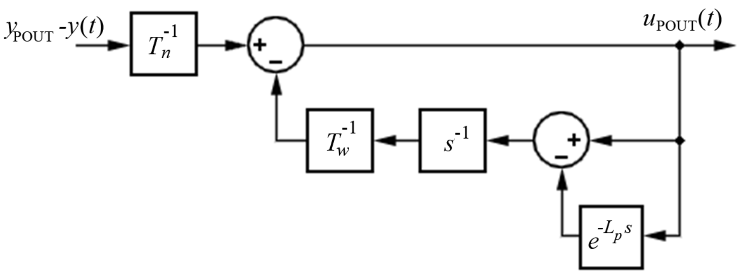

Figure 5. Transfer function realization of policy (46).

Comparing the proposed strategy (31) with policy (46), we can recognize a similar control structure which involves the measurement of the current stock level and the calculations performed on the order history. However, policy (31) explicitly accounts for the effects caused by deteriorating stock, and thus it allows for avoiding the oscillations both in the output variable and in the ordering signal which is not guaranteed by the POUT policy for arbitrary system parameters (delay and decay factor). Moreover, our scheme requires less tuning effort than controller (46) as it relies on one gain coefficient. Notice also that both of the proposed policies (9) and (31) when applied to the system without perishables ($\sigma = 0$) become equivalent the POUT one with $T_n = T_w = 1/K$. As a result, all the properties defined in the theorems will be valid for the POUT policy with $\sigma = 0$ and $T_n = T_w = 1/K$ when applied to the standard inventory system with nondeteriorating stock.

5.2. Stability analysis

The transfer function of the POUT policy (46) is determined in the following way

$$G_C(s) = \frac{1/T_n}{1 + \frac{1}{T_w s}\left(1 - e^{-L_p s}\right)} = \frac{T_w s}{T_n\left(1 + T_w s - e^{-L_p s}\right)}, \tag{47}$$

and the closed-loop transfer functions are given by

a) with respect to the reference input $Y_{\rm POUT}(s) = y_{\rm POUT}/s$,

$$\frac{Y(s)}{Y_{\rm POUT}(s)} = \frac{T_w s e^{-L_p s}}{T_n(s+\sigma)(1+T_w s) + [(T_w - T_n)s - T_n\sigma]\, e^{-L_p s}}, \tag{48}$$

b) with respect to the disturbance $D(s)$,

$$\frac{Y(s)}{D(s)} = -\frac{T_n\left(1 + T_w s - e^{-L_p s}\right)}{T_n(s+\sigma)(1+T_w s) + [(T_w - T_n)s - T_n\sigma]\, e^{-L_p s}}. \tag{49}$$

In order to comment on the closed-loop stability we need to study the roots of the characteristic polynomial

$$P(s,e^{-L_ps}) = T_n\,(s+\sigma)\,(1+T_ws) + [(T_w-T_n)\,s - T_n\sigma]\,e^{-L_ps}. \tag{50}$$

Due to the transcendental form of the characteristic equation $P(s,exp(-L_ps)) = 0$, the standard stability tests, e.g. Routh-Hurwitz test, are not applicable. Note that $s = 0$ is the root of the numerator and the denominator of (48) and (49). Hence, to ascertain stability one needs to ensure that there are no roots of (50) with positive real parts. For the case of systems with non-deteriorating stock ($\sigma = 0$) we get immediately from (50) that for $T_n = T_w$ the delay term is eliminated from the characteristic equation, and the system is stable independent of delay. In the nontrivial case $\sigma > 0$, the conditions for stability independent of delay will be established using the two-variable criterion approach discussed in [16, ch. 2] which originates from the works of Kamen [17]-[19].

It follows from [17] that the closed-loop system with the characteristic equation $P(s,exp(-L_ps)) = 0$ is stable independent of delay if and only if

$$P(s,e^{-L_ps}) \neq 0 \quad \text{Re}s \geqslant 0, \quad L_p \geqslant 0. \tag{51}$$

Alternatively to solving (51), one can seek for the critical delay values at which the stable poles of the system with no delay cross the imaginary axis. Following the reasoning presented in [16, ch. 2], verification of the stability of the system with single delay (or with multiple commensurate delays) amounts to checking if $P(s,z) = 0$ admits an imaginary solution $s_0 = j\omega$, or a unitary solution $|z_0| = 1$, where $z = exp(-L_ps)$. On the other hand, by the complex conjugate property of polynomials with real coefficients, in order to find the roots of the bivariate polynomial $P(s,z)$, it is sufficient to obtain the simultaneous solution (for positive ω) of two equations

$$P(s,z) = 0 \quad \text{and} \quad \bar{P}(s,z) = 0, \tag{52}$$

where $\bar{P}(s,z) = zP(-s,z^{-1})$ is the conjugate polynomial of $P(s,z)$. By eliminating either s or z from set (52), the problem of finding the roots of the bivariate polynomial $P(s,z)$ may be reduced to calculating the roots of a one-variable polynomial. The parameter range for the stability independent of delay of system (48)-(49) established using the discussed approach is given in Theorem 7.

Theorem 7 *The closed-loop system* (48)-(49) *is stable independently of delay for arbitrary* $0 < \sigma < 1$, *if either*

$$T_w > 0 \quad \text{for} \quad T_n \geqslant 1/\sigma, \tag{53}$$

or

$$0 < T_w < 2T_n/\left(1 - T_n^2\sigma^2\right) \quad \text{for} \quad 0 < T_n < 1/\sigma. \tag{54}$$

Proof First, notice that $P(s,1) = T_w s(T_n s + T_n\sigma + 1)$ has no root with positive real part for positive gain constants T_n and T_w. Therefore, the system is stable in the delay-free case.

With $z = exp(-L_p s)$ we get the characteristic and conjugate polynomials of the analyzed system,

$$\begin{aligned} P(s,z) &= T_n(s+\sigma)(1+T_w s) + [(T_w - T_n)s - T_n\sigma]z = 0, \\ \bar{P}(s,z) &= zT_n(\sigma - s)(1 - T_w s) + [(T_n - T_w)s - T_n\sigma] = 0. \end{aligned} \tag{55}$$

Eliminating z, we obtain

$$\frac{T_w s^2\left[T_n^2 T_w\left(s^2-\sigma^2\right) - 2T_n + T_w\right]}{T_n(s-\sigma)(T_w s - 1)} = 0, \tag{56}$$

which has two nonzero roots for $T_n \neq 1/\sigma$ and $T_w \neq 2T_n/(1 - T_n^2\sigma^2)$,

$$s_0^{\pm} = \pm\sqrt{\frac{T_n^2 T_w \sigma^2 + 2T_n - T_w}{T_n^2 T_w}}. \tag{57}$$

The system is stable independent of delay if there are no imaginary solutions to (56). We get from (57) that no imaginary root exists if

$$T_n^2 T_w \sigma^2 + 2T_n - T_w = T_w\left(T_n^2\sigma^2 - 1\right) + 2T_n > 0. \tag{58}$$

In the case when $T_n \geqslant 1/\sigma$ condition (58) is fulfilled for any $T_w > 0$. On the other hand, when $T_n < 1/\sigma$, (58) is equivalent to $T_w < 2T_n/(1 - T_n^2\sigma^2)$. Thus, whenever the gains are chosen as either (53) or (54), the stable poles of the system with no delay do not cross the imaginary axis when $L_p > 0$, and system (48)-(49) is stable independent of delay for arbitrary $0 < \sigma < 1$. This ends the proof. □

6. Numerical example

The properties of the designed policies (9) and (31) are verified in simulations conducted for the model of perishable inventory system described in Section 2. The system parameters are set in the following way: lead-time $L_p = 7$ days, inventory decay factor $\sigma = 0.08$ day^{-1}, and the maximum daily demand at the distribution center $d_{max} = 20$ items/day. Two tests are run: one for the piecewise constant demand subject to seasonal changes illustrated in Fig. 6, and the other test for the highly variable demand following the normal distribution with mean 10 items/day and variance 36 (items/day)^2. The controller performance is compared with the classical inventory policy (46).

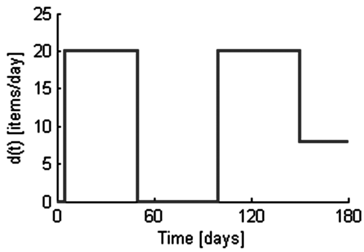

Figure 6. Demand at the distribution center.

Test 1 In the first simulation example we assume that the market demand follows the pattern illustrated in Fig. 6 which reflects abrupt seasonal changes in a half-year trend. Since $d(\cdot)$ does not fluctuate within the respective seasons, variance reduction (and the bullwhip effect) is of little concern. However, in order to gain competitive advantage in the analyzed case, it is imperative to quickly react to the sudden trend changes. Consequently, the controller gain is adjusted to obtain fast reaction to $d(\cdot)$ transitions as $K = 10$ day^{-1}, and the POUT tuning coefficients as $T_n = T_w = 1/K = 0.1$ day. The reference stock level for policy (9) and (31) is adjusted according to (26) as $y_{ref} = 115 > 109$ items so that high level of demand satisfaction is achieved. This results in the required storage space: for policy (9) determined from Theorem 2 as 221 items, and for policy (31) obtained from Theorem 5 as 115 items. The order-up-to level for the POUT policy is adjusted so that it generates the same holding costs as the improved policy (31). We set $y_{\text{POUT}} = 160$ items.

The orders generated by the controllers are shown in Fig. 7, and the resulting on-hand stock level in Fig. 8: policy (9) - curve a), policy (31) - curve b), and policy (46) - curve c). We can see from the graphs depicted in Fig. 7 that the proposed controllers (9) and (31) quickly respond to the sudden changes in the demand trend without oscillations or overshoots in the ordering signal. Policy (31) establishes smaller order quantities than controller (9), which leads to smaller purchase costs while maintaining a given service level. The ordering signal generated by the POUT policy is contaminated by overshoots and oscillations, thus being more difficult to follow by the supplier and requiring bigger safety stock from the supply source. We can see from Fig. 8 that the stock level resulting from the application of policies (9) and (31) does not increase beyond the maximum level calculated from Theorems 2 and 5, which means that the assigned warehouse capacity is sufficient to store the goods at the distribution center at all times. Moreover, the on-hand stock never falls to zero after the initial phase which implies full demand satisfaction and the 100% service level. The $y(t)$ curve obtained from policy (9) exhibits large overshoots after sudden changes in the demand trend, which leads to increased holding and warehouse maintenance costs. The improved policy (31) is free from overshoots, and it offers the least costly solution of the three investigated controllers.

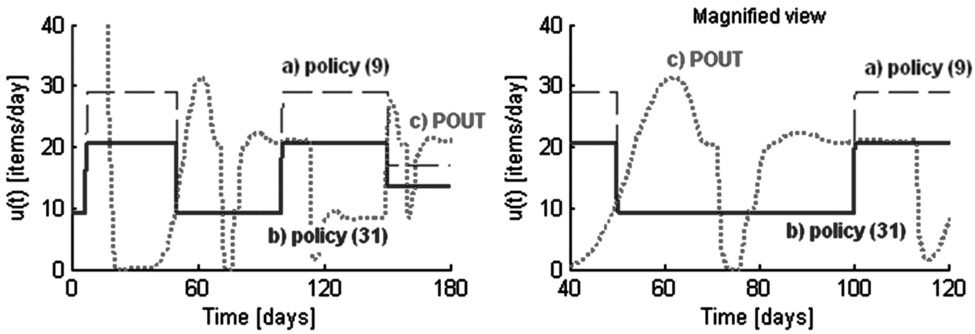

Figure 7. Ordering signal: a) policy (9), policy (31), POUT policy (46).

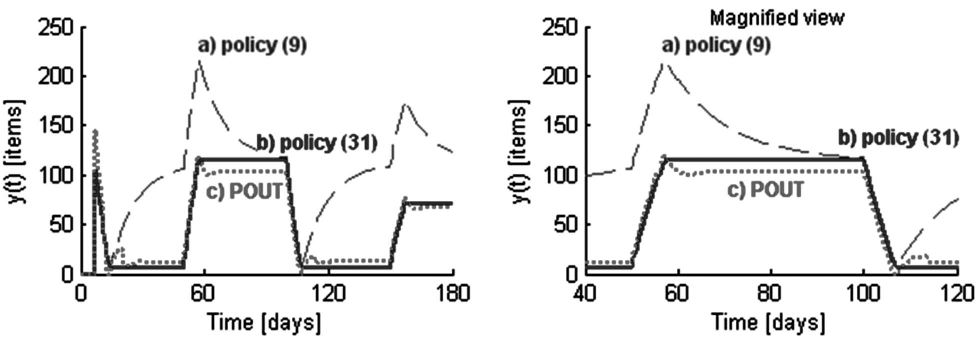

Figure 8. On-hand stock level: a) policy (9), policy (31), POUT policy (46).

Test 2 In the second test we verify the controller performance in the presence of highly variable demand following the normal distribution with mean 10 items/day and variance 36 (items/day)^2. Since the demand exhibits rapid fluctuations (coefficient of variation CV - standard deviation over mean - amounts to 0.6), we decrease the controller gains to smoothen the ordering variations. We set $K = 0.5$ day^{-1} for policy (9) and (31), and $T_n = T_w = 1/K = 2$ days for policy (46). In order to ensure full demand satisfaction, the stock reference value is set according to (26) as $y_{ref} = 150 > 147$ items. The order-up-to level for policy (46) $y_{\mathrm{POUT}} = 205$ items so that the same holding costs are generated as in the case of policy (31). The first simulation (a) is run for the nominal system, whereas in the second one (b), the orders are realized with delay varying randomly between 6 and 9 days, and the decay factor fluctuates randomly within the interval [0, 0.2]. The statistics of the ordering decisions and the bullwhip indicator (order-to-demand variance ratio, [2]) are given in Table 1. The data listed in Table 1 demonstrates that with proper gain adjustment, all controllers eliminate the risk of the bullwhip effect as the bullwhip indicator is smaller than one. This is achieved even when the precise values of the system parameters are not known to the controllers. In that case (b) however, the or-

Table 3. Variable statistics.

System	Ordering rule	Order variance [items^2/day^2]	Bullwhip indicator
a) nominal	controller (9)	19	0.53
	controller (31)	10	0.28
	controller (46)	15	0.42
b) perturbed	controller (9)	28	0.78
	controller (31)	12	0.33
	controller (46)	18	0.5

der smoothening property degrades resulting in a bigger order-to-demand variance ratio. The POUT policy performs better than controller (9), but it results in a bigger value of the bullwhip indicator than the improved policy (31).

7. Conclusions

In this work we addressed the problem of goods flow control in continuous-review inventory systems with deteriorating stock. The focus was placed on the effects related to delay in realizing the stock replenishment orders. A new supply policy employing the Smith predictor for dead-time compensation was proposed. The closed-loop system stability with the designed controller implemented is guaranteed for arbitrary delay and any bounded demand pattern. The ordering signal smoothly adapts to the demand changes, and thus it is easy to follow by the supplier. It is also demonstrated that the stock level resulting from the application of the proposed policy does not increase beyond the precisely determined warehouse capacity, which eliminates the need for costly emergency storage and facilitates capacity planning at the goods distribution center. It is shown how to select the controller parameters to achieve full satisfaction of the unknown market demand. Since the basic Smith predictor based strategy may lead to overshoots in the stock level in response to abrupt demand changes, we introduce a modified controller. The improved control law retains all the favorable properties of the original scheme but ensures exponential convergence of the stock level to steady state without overshoots and oscillations

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Appendix

We solve differential equation (3) with the initial conditions: $y(0) = 0$, and $u_R(t) = u(t - L_p) = 0$ for $t < L_p$. First we consider the homogeneous equation

$$\dot{y} + \sigma y(t) = 0, \tag{59}$$

which leads to

$$y(t) = y(0)\, e^{-\sigma t}. \tag{60}$$

In order to determine the nonhomogeneous solution we assume $y(t)$ in the following form

$$y(t) = q(t)\, e^{-\sigma t}, \tag{61}$$

where $q(t)$ is a function differentiable with respect to time. Differentiating both sides of (61) we obtain

$$\dot{y} = \dot{q} e^{-\sigma t} - \sigma q(t)\, e^{-\sigma t}. \tag{62}$$

Substituting (61) and (62) into (3), we get

$$\dot{q}e^{-\sigma t} = u_R(t) - h(t). \tag{63}$$

Solving (63) for $q(t)$ yields

$$q(t) = \int_0^t e^{\sigma\tau}\left[u_R(t) - h(t)\right] d\tau + C, \tag{64}$$

where C is the constant of integration. Substituting (64) into (61), we arrive at

$$y(t) = \left\{ \int_0^t e^{\sigma\tau}\left[u_R(t) - h(t)\right] d\tau + C \right\} e^{-\sigma t} = Ce^{-\sigma t} + \int_0^t e^{-\sigma(t-\tau)}\left[u_R(t) - h(t)\right] d\tau. \tag{65}$$

Applying the initial condition $y(0) = 0$, we have $C = 0$, and

$$y(t) = \int_0^t e^{-\sigma(t-\tau)}\left[u_R(t) - h(t)\right] d\tau. \tag{66}$$