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On adhesive binding optimization of elastic homogeneous rod to a fixed rigid base as a control problem by coefficient

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The problem of finite, partially glued to a fixed rigid base rod longitudinal vibrations damping by optimizing adhesive structural topology is investigated. Vibrations of the rod are caused by external load, concentrated on free end of the rod, the other end of which is elastically clamped. The problem is mathematically formulated as a boundary-value problem for one-dimensional wave equation with attenuation and variable controlled coefficient. The intensity of adhesion distribution function is taken as optimality criterion to be minimized. Structure of adhesion layer, optimal in that sense, is obtained as a piecewise-constant function. Using Fourier real generalized integral transform, the problem of unknown function determination is reduced to determination of certain switching points from a system of nonlinear, in general, complex equations. Some particular cases are considered.

Key words: topology optimization, optimal design, control by coefficient, nonlinear moments problem, adhesive binding

1. Introduction

Designs and structures in use are made monolith very rarely: the most part of them consists of different elements attached to each other in various ways. The variety of opportunities of practical realization allows us to choose optimal in a certain sense structure of important links between component parts of different designs. Traditionally, optimal design problems are considered in order to optimize some design parameters (weight, volume, load capacity and etc.) for given structure of that design. In monograph [6] a wide range of construction optimization problems of three main classes- optimization of size, form and structure, is investigated. However, so-called structural topology optimization problems have begun to investigate recently, in order to minimize a specific functional describing material distribution in given domain, retaining, or if possible maximizing desirable properties of constructions. Solution of topology optimization

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problems, unlike problems of structural optimization, which generally use necessary conditions of optimality to be solved, are generally reduced to a certain problem of nonlinear programming [1]. In [5] a new, efficient in terms of numerical realization method of topological structure optimization problems investigation is proposed, which is based on genetic algorithm. Nevertheless, explicit analytical form determination for unknown controls in such problems is connected with significant difficulties.

Problems of vibrations forced damping for distributed parameters system play some special role in control theory of systems with distributed parameters. Though it is well known, that vibrations forced damping time can be arbitrarily small via impulsive loads (impacts), mathematically described by generalized functions (for instance Dirac delta function), nevertheless intensities of control impacts may be significantly large [4]. In [7] a problem of longitudinal vibrations forced damping by distributed control impacts in a finite time-interval is investigated for elastic, non-homogeneous finite rod. The problem is mathematically formulated as a boundary-value problem for one-dimensional wave equation with variable coefficients and controlled right hand-side, at that a functional describing linear momentum of control impacts on considered time-interval is taken as control process optimality criterion. Applying Fourier real generalized integral transform, solution of control problem is reduced to minimization procedure of chosen optimality criterion in space of measurable functions L^1 under constraints of equality type on unknown function. Treating that problem of nonlinear programming as a moments problem in functional space L^∞ an explicit form of control impacts is constructed using generalized functions. Intensities and moments of control impulsive impacts application are determined; controllability of system under investigation is achieved for all initial data and system parameters. Convenience of constructed method is that only determination of two solutions of a special Riccati differential equation with different first derivative is required for numerical realization of the algorithm. The same algorithm is used in [8] to solve optimal boundary control problem for non-homogeneous string vibrations caused by impulsive perturbations (discontinuous right-hand side) when control impacts contains constant delay.

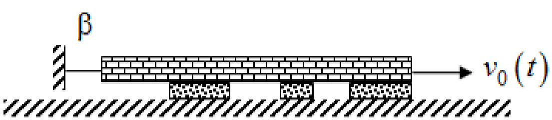

Figure 1. Illustration of the rod.

This investigation is devoted to analytical solution of mixed, in a certain sense, problem of vibration damping and topology optimization. Our aim is homogeneous finite rod elastic longitudinal vibrations damping by optimizing topological structure of adhesion between some part of the rod and a fixed rigid base, when the maximal length of adhesive layer ought to be minimized (Fig. 1). According to glue layer the model of pure shear is taken into consideration. Fourier real generalized integral transform is used for that purpose. The problem is formulated in terms of a boundary-value problem for one-dimensional wave equation with attenuation and variable controlled coefficient.

Throughout all the paper a real-valued function will be called admissible control, if it satisfies existence and uniqueness conditions of given system required solution. Given system will be called fully controllable in a certain space of functions, if there exists an admissible control function from that space, resolving posed control problem [7]– [9].

2. Problem statement

Our main problem is to determine an admissible control function $u^o(x)$ from given set U, consisting of some functions $u(x)$ satisfying:

$$\mathcal{L}[w] \equiv \frac{\partial^2 w(x,t)}{\partial x^2} - \alpha^2 u(x) w(x,t) - \frac{1}{c^2}\frac{\partial^2 w(x,t)}{\partial t^2} = 0, \quad (x,t) \in (-l,l)\times(0,T), \tag{1}$$

$$\left[E\frac{\partial w(x,t)}{\partial x} - \gamma w(x,t)\right]\bigg|_{x=-l} \equiv 0, \quad \frac{\partial w(x,t)}{\partial x}\bigg|_{x=l} = v_0(t), \quad t \in (0,T), \tag{2}$$

which have minimal intensity.

System (1)–(2) describes forced vibrations of elastic rod of $2l$ length, which is in adhesive link with (glued to) a fixed rigid base, at that $\alpha^2 = \frac{G_k}{Ehh_k}$ is elastic characteristic factor of glue layer, where $\{G_k; h_k\}$ are glue layer shear modulus and thickness, which is assumed to be sufficiently small with respect to rod thickness h; E is rod Young modulus, and $c = \sqrt{E\cdot\rho^{-1}}$ is the velocity of elastic wave propagation in the rod, ρ is rod material density. Due to small thickness it is assumed, that the glue layer is deformed in pure shear state.

According to boundary conditions (2), vibrations under study are caused by boundary perturbations $v_0(t)$ (we include rod Young modulus in $v_0(t)$), applied to free end of the rod, while the other end of rod is elastically clamped with stiffness factor γ $(0 < \gamma = E\beta)$, which corresponds to the first boundary condition (2). In particular, when $\gamma = 0$, that boundary condition corresponds to free, and when $\gamma \to \infty$– to rigidly embedded end of the rod. The dimensionless function $u(x)$ in that case, describes adhesion distribution law along contact area. Let us note, that system (1)–(2) can describe also other processes not only in continuum mechanics, but also in many different areas of physics.

The following initial data are supposed to be given:

$$w(x,0) = w_0(x), \quad \frac{\partial w(x,t)}{\partial t}\bigg|_{t=0} = \dot{w}_0(x), \quad x \in [-l,l]. \tag{3}$$

It is assumed, that external perturbations $v_0(t)$ are defined as follows:

$$v_0(t) = [H(t) - H(t-\tau)]v(t), \quad t \in (0,T),$$

where

$$H(t-\tau)=\begin{cases}1, & t>\tau;\\ 0, & t<\tau,\end{cases}$$

is the well-known Heaviside unit step function, and $\tau<T$ $(0<\tau=const)$ is the external perturbations stopping moment.

Let the rod to be glued to a fixed rigid base only partially, namely on the interval $[-a,l]$ $(0\leqslant a<l)$ of his length. This assumption corresponds to investigation of boundary-value problem (1)–(2) only for $x\in[-a,l]$. Otherwise the differential equation (1) will coincide with ordinary wave equation. The aim of the present investigation is the damping of the rod longitudinal vibrations, i.e. the providing of terminal data

$$w(x,T)\equiv 0,\ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=T}\equiv 0,\ \ x\in[-l,l], \tag{4}$$

at any given moment T by appropriate choice of control function $u^o(x)$, $x\in[-a,l]$, having minimal intensity among all admissible control functions $u(x)\in U$. It means, that the functional [9]

$$\kappa[u]=\max_{x\in[-a,l]}|u(x)|,\ \ u\in U, \tag{5}$$

should be minimized.

3. Solution of the problem

Solution of optimal control problem posed above gives the following

Theorem 1 *Resolving control function $u^o(x)$ optimal in the sense of (5) is defined as*

$$u^o(x)=\sum_{j=0}^{m}[H(x-x^o_{2j})-H(x-x^o_{2j+1})],\ x\in[-a,l], \tag{6}$$

and determined by specifying switching points $x^o_{2j}<x^o_{2j+1}$, $j=\overline{0;m}$ $(x^o_0=-a$, and $x^o_{2m+1}=l)$. The switching points are calculated from system of restrictions of equality type:

$$\Gamma[x^o_{2j},x^o_{2j+1},z_k]=M_k,\ \ k\in\mathbb{N}, \tag{7}$$

where, in general, complex numbers z_k are determined from characteristic transcendent equation as follows:

$$\Lambda_\beta(z)\sinh[\lambda_-(z)l]+\cosh[\lambda_-(z)l]=0,\ \ z\in\mathbb{C}^+. \tag{8}$$

All notations are explained in the proof.

The finding of switching points from system (5)–(8) can be interpreted as a problem of nonlinear programming and can be attacked by traditional methods [2].

Proof The natural restriction on admissible control functions $u(x) \in U$, that directly follows from physical interpretation of the problem is they are non-negative: $u(x) \geqslant 0$, when $x \in [-a,l]$. From the other hand, it is obvious from physical considerations that those functions are compactly supported in $[-a,l]$ (identically zero outside it). Taking into account, that the Lebesgue space $L^\infty[-a,l]$ is a Banach space with respect to norm $||u(x)||_{L^\infty[-a,l]} = \kappa[u]$ (see (5)) the set $U \subset L^\infty[-a,l]$, consisting of non-negative compactly supported functions $u(x)$ is taken as set of admissible controls.

Relaying on maximum principle [4, 8, 9] one can prove, that control function $u^o(x)$ optimal in the sense of (5) is piecewise-constant, taking only two values– 1 and 0, and are determined by specifying switching points, where its values jump from one level to another. At that value 1 corresponds to adhesion presence, and 0 – to absence. Unlike to [7, 8], here we write the explicit form of that function, for example, in the form (6).

Now, in order to write the system (1)–(2) for all real $t \in \mathbb{R}$, let us introduce an operator $\mathcal{A}_T[\cdot]$ defined as follows:

$$\mathcal{A}_T[f] = \begin{cases} f(t), \ t \in [0,T]; \\ 0, \ t \notin [0,T]. \end{cases}$$

One may define it with help of characteristic function $\chi_{[0,T]}(t)$, but we define it as follows:

$$\mathcal{A}_T[f] = [H(t) - H(t-T)]f(t) \equiv f_1(t), \quad t \in \mathbb{R},$$

applying which to system (1)–(2) allow us to include initial and terminal data (3), (4) in the right hand-side of homogeneous differential equation (1):

$$\mathcal{A}_T\left[\mathcal{L}[w]\right] = \frac{\partial^2 w_1(x,t)}{\partial x^2} - \alpha^2 u(x) w_1(x,t) - \frac{1}{c^2}\frac{\partial^2 w_1(x,t)}{\partial t^2} = \frac{1}{c^2} W(x,t), \tag{9}$$

$$(x,t) \in (-l,l) \times \mathbb{R},$$

$$W(x,t) = -[w_0(x)\delta'(t) + \dot{w}_0(x)\delta(t)],$$

where $\delta(t)$ is the well-known Dirac delta function, and $\delta'(t)$ is its derivative in generalized sense. As the boundary conditions (2) depend only on variable t, as a result of substitutions made they will retain their form:

$$\left.\left[\frac{\partial w_1(x,t)}{\partial x} - \beta w_1(x,t)\right]\right|_{x=-l} \equiv 0, \quad \left.\frac{\partial w_1(x,t)}{\partial x}\right|_{x=l} = v_0(t), \quad t \in \mathbb{R}. \tag{10}$$

To obtain the second boundary condition the following obvious relation was used:

$$\mathcal{A}_T[v_0] = [H(t) - H(t-T)]v_0(t) = v_0(t).$$

Applying now Fourier real generalized integral transform with respect to t variable to equation (9) and corresponding boundary conditions (10), after some simple algebraic transformations we will respectively obtain:

$$\frac{d^2\overline{w}_1(x,\sigma)}{dx^2}+\left[\frac{\sigma^2}{c^2}-\alpha^2u(x)\right]\overline{w}_1(x,\sigma)=\frac{1}{c^2}\overline{W}(x,\sigma),\ (x,\sigma)\in(-l,l)\times\mathbb{R}, \tag{11}$$

$$\left[\frac{d\overline{w}_1(x,\sigma)}{\partial x}-\beta\overline{w}_1(x,\sigma)\right]\Bigg|_{x=-l}=0,\ \frac{d\overline{w}_1(x,\sigma)}{dx}\Bigg|_{x=l}=\overline{v}_0(\sigma),\ \ \sigma\in\mathbb{R}, \tag{12}$$

where

$$\mathcal{F}[f]\equiv\overline{f}(\sigma)=\int\limits_{-\infty}^{\infty}f(t)e^{i\sigma t}dt,$$

is $f(t)$ function Fourier transform in generalized sense [3], $\mathcal{F}[\cdot]$ is the Fourier operator, $\sigma\in\mathbb{R}$ is the parameter of Fourier transform, at that obviously

$$\mathcal{F}\left[\mathcal{A}_T[f]\right]=\int\limits_0^T f(t)e^{i\sigma t}dt,$$

and

$$\overline{W}(x,\sigma)=i\sigma w_0(x)-\dot{w}_0(x),\ \ \overline{v}_0(\sigma)=\int\limits_0^\tau v(t)e^{i\sigma t}dt.$$

Taking into account restrictions made above on unknown function $u(x)$ for general solution of system (11)–(12) with piecewise constant coefficients we will obtain:

$$\overline{w}_1(x,\sigma)=a(\sigma)\cosh\lambda(x,\sigma)+b(\sigma)\sinh\lambda(x,\sigma)+\Omega(x,\sigma),\ \ (x,\sigma)\in(-l,l)\times\mathbb{R}, \tag{13}$$

where $a=a(\sigma)$ and $b=b(\sigma)$ are constant-valued functions, determining from boundary conditions (12) as follows:

$$\begin{aligned} a(\sigma)&=\frac{\lambda_+\cosh(\lambda_+l)+\beta\sinh(\lambda_+l)}{\beta\cosh(\lambda_+l)+\lambda_+\sinh(\lambda_+l)}\cdot b(\sigma),\\ b(\sigma)&=\frac{\overline{v}_0(\sigma)-\Omega'(l,\sigma)}{\lambda_-\left[\cosh(\lambda_-l)+\frac{\lambda_+\cosh(\lambda_+l)+\beta\sinh(\lambda_+l)}{\beta\cosh(\lambda_+l)+\lambda_+\sinh(\lambda_+l)}\cdot\sinh(\lambda_-l)\right]}, \end{aligned} \tag{14}$$

and

$$\Omega(x,\sigma)=\frac{1}{c^2}\int\limits_{-l}^{x}\frac{i\sigma w_0(\xi)-\dot{w}_0(\xi)}{\lambda'(\xi,\sigma)}\cdot\sinh\left[\lambda(x,\sigma)-\lambda(\xi,\sigma)\right]d\xi,$$

$$\Omega'(x,\sigma)=\frac{\lambda'(x,\sigma)}{c^2}\int\limits_{-l}^{x}\frac{i\sigma w_0(\xi)-\dot{w}_0(\xi)}{\lambda'(\xi,\sigma)}\cdot\cosh\left[\lambda(x,\sigma)-\lambda(\xi,\sigma)\right]d\xi,$$

$$\lambda(x,\sigma)=\begin{cases}\frac{|\sigma|}{c}x, & x\in\mathcal{M}^j;\\ \left[\frac{\sigma^2}{c^2}-\alpha^2\right]^{\frac{1}{2}}x, & x\in\mathcal{M}_j,\end{cases},\quad \lambda'(x,\sigma)=\begin{cases}\frac{|\sigma|}{c}, & x\in\mathcal{M}^j;\\ \left[\frac{\sigma^2}{c^2}-\alpha^2\right]^{\frac{1}{2}}, & x\in\mathcal{M}_j,\end{cases}$$

$$\mathcal{M}_j=[x^o_{2j},x^o_{2j+1}],\ j=\overline{0;m},\quad \mathcal{M}^j=[-l,x^o_0)\cup(x^o_{2j+1},x^o_{2j+2}),\ j=\overline{0;m-1},$$

$$\lambda_+(\sigma)=\frac{|\sigma|}{c},\ \lambda_-(\sigma)=\left[\frac{\sigma^2}{c^2}-\alpha^2\right]^{\frac{1}{2}},\ \sigma\in\mathbb{R}.$$

At that, when $|\sigma|<\alpha c$, $\lambda_-(\sigma)=i\left[\alpha^2-\frac{\sigma^2}{c^2}\right]^{\frac{1}{2}}$ is positive imaginary.

It is easy to see that introduced function $w_1(x,t)$ is determined for all real $t\in\mathbb{R}$ and is compactly supported in rectangle $[-l,l]\times[0,T]$, where it coincides with the main function $w(x,t)$. Then [3], its Fourier generalized transform is an analytical entire function of variable $z=\sigma+i\varsigma$ satisfying inequality

$$|z^\rho\cdot\overline{w}_1(x,z)|\leqslant A_\rho e^{\theta|\varsigma|}$$

for all $x\in[-l,l]$ and $\rho\geqslant 0$, corresponding $A_\rho\geqslant 0$, and some $\theta>0$ (depending only on quantity T). From the other hand, using relation $W(x,t)\in L^1[-l,l]\times[0,T]$ one can prove, that $\Omega(x,z)$, $z\in\mathbb{C}$, function is also analytical entire function, satisfying the same inequality as $\overline{w}_1(x,z)$, $z\in\mathbb{C}$, function does. Therefore, for fulfillment of aforesaid theorem‘s conditions it is necessary and sufficient, that $a(z)$ and $b(z)$ functions be also analytical and entire. It is easy to see from expressions (14) extended for all $z\in\mathbb{C}$ if, for instance, $a(z)$ is entire function of z, then $b(z)$ is also entire. From the other hand, from representation of $\lambda(x,\sigma)$ follows, that it is sufficient to consider only extending for $z\in\mathbb{C}^+$ (the upper half–plane $\sigma,\varsigma\in\mathbb{R}^+$). From (14) extended for all $z\in\mathbb{C}^+$ one can obtain the system of necessary and sufficient conditions

$$\overline{v}_0(z_k)-\Omega'(l,z_k)=0,\ k\in\mathbb{N},\tag{15}$$

when

$$\cosh(\lambda_-l)+\frac{\lambda_+\cosh(\lambda_+l)+\beta\sinh(\lambda_+l)}{\beta\cosh(\lambda_+l)+\lambda_+\sinh(\lambda_+l)}\cdot\sinh(\lambda_-l)=0,\ \ z\in\mathbb{C}^+,\tag{16}$$

for $b(z)$ function to be entire.

According to notations made above, we get

$$\overline{v}_0(z_k)=\int_0^\tau v(t)e^{iz_kt}dt=\int_0^\tau v(t)e^{-\varsigma_kt}\cos(\sigma_kt)dt+i\int_0^\tau v(t)e^{-\varsigma_kt}\sin(\sigma_kt)dt,$$

$$\Omega'(l,z_k)=\frac{\lambda_{+k}J_{1k}+\lambda_{-k}J_{2k}}{c^2\lambda_{+k}},\ J_{1k}=\sum_{j=0}^m\int_{x^o_{2j}}^{x^o_{2j+1}}\overline{W}(\xi,z_k)\cosh[\lambda_{-k}(l-\xi)]d\xi,$$

$$J_{2k} = \int_{-l}^{x_0^o} \overline{W}(\xi, z_k)\cosh[\lambda_{-k}l - \lambda_{+k}\xi]d\xi + \sum_{j=0}^{m-1} \int_{x_{2j+1}^o}^{x_{2j+2}^o} \overline{W}(\xi, z_k)\cosh[\lambda_{-k}l - \lambda_{+k}\xi]d\xi,$$

$$\lambda_{+k} = \lambda_+(z_k), \quad \lambda_{-k} = \lambda_-(z_k), \quad k \in \mathbb{N}.$$

Then from (15) and (16) after some algebraic transformations we will accordingly obtain (7) and (8), where

$$\Gamma[x_{2j}^o, x_{2j+1}^o, z_k] = \lambda_{+k}J_{1k} + \lambda_{-k}J_{2k}, \quad M_k \equiv M(z_k) = c^2\lambda_{+k}\bar{v}_0(z_k), \quad k \in \mathbb{N},$$

$$\Lambda_\beta(z) = \frac{\lambda_+ \cosh(\lambda_+ l) + \beta \sinh(\lambda_+ l)}{\beta \cosh(\lambda_+ l) + \lambda_+ \sinh(\lambda_+ l)}.$$

□

Remark 1 As it is easy to see, if for some k complex number $z_k = \sigma_k + i\varsigma_k$, is a root of characteristic equation (8), then $-z_k = -\sigma_k - i\varsigma_k$ also satisfies that equation. Using properties of Fourier integrals [3] one can prove, that $\Gamma[x_{2j}^o, x_{2j+1}^o, -z_k] = \overline{\Gamma[x_{2j}^o, x_{2j+1}^o, z_k]}$ and $M(-z_k) = \overline{M(z_k)}$, where the line over expressions means their complex adjoint, therefore after separating real and imaginary parts of system (7), consideration may be limited only by roots $z_k = \sigma_k + i\varsigma_k$, $k \in \mathbb{N}$.

So, solution of optimization problem under investigation is reduced to determination of such admissible set of switching points $\{x_{2j}^o, x_{2j+1}^o\}_{j=0}^m$ from countable system of equations (7) that the first switching point x_0^o, which coincides with control parameter a should be minimal. Then, number m of switching points is determined from inclusion conditions $\{x_{2j}^o, x_{2j+1}^o\}_{j=0}^m \subset [-a, l]$ uniquely.

Let us consider now some particular cases.

- When $\beta \to 0$ (according to free end of the rod), from characteristic equation (7) we will obtain $\Lambda_0(z)\sinh(\lambda_- l) + \cosh(\lambda_- l) = 0$, where $\Lambda_0(z) = \coth(\lambda_+ l)$, therefore the characteristic equation will become $\sinh[(\lambda_- + \lambda_+)l] = 0$. So, in this case we have
$$\lambda_{-k} + \lambda_{+k} = i\frac{\pi k}{l}, \quad k \in \mathbb{N}.$$
- In limiting case $\beta \to \infty$, which corresponds to rigidly embedded end of the rod, characteristic equation (7) will derive us to $\Lambda_\infty(z)\sinh(\lambda_- l) + \cosh(\lambda_- l) = 0$, where $\Lambda_\infty(z) = \tanh(\lambda_+ l)$, therefore the characteristic equation will become $\cosh[(\lambda_- + \lambda_+)l] = 0$. So, in this case we have
$$\lambda_{-k} + \lambda_{+k} = i\frac{\pi(2k+1)}{2l}, \quad k \in \mathbb{N}.$$
- When the moment of external perturbations stopping $\tau \to 0$ (quick perturbation) we will obtain $\bar{v}_0(z_k) = 0$, which corresponds to homogeneous system of (7).

It should be added, that another statement of optimization problem can be considered for system under investigation in order to minimize vibration vanishing time T by appropriate choice of control function $u^o(x)$, $x \in [-a,l]$, and parameter a.

At the end let us note, that equations of (9) type arise also in various fields of contemporary physics (the most common name is Klein-Gordon equation, describing also, for instance, motion of a relativistic particle in a quantum scalar or pseudoscalar field) [10]. On the other hand, if as a result of switching points $\{x^o_{2j}, x^o_{2j+1}\}_{j=0}^{m}$ determination it will turn out, that function $u^o(x)$ is periodic, then corresponding ordinary differential equation (11) will be an equation of Hill type [10]. Note also, that in the case when switching points are very close to each other: $x^o_{2j} \to x^o_{2j+1}$, then optimal control function will be reduced to [8]

$$u^o(x) = \sum_{j=0}^{m} \delta(x - x^o_{2j}), \quad x \in [-a,l], \tag{17}$$

which corresponds to discrete contact between rod and base.

4. Numerical results

Let us consider now numerical implementation of obtained results. For that purpose, we first introduce in (9) and corresponding boundary conditions (10) dimensionless variables and functions

$$x_* = \frac{x}{l},\ t_* = \frac{ct}{l},\ \alpha_*^2 = l^2\alpha^2,\ \beta_* = l\beta = \frac{\gamma l}{E},\ w_{1*} = \frac{w_1}{l},\ W_* = \frac{l}{c^2}W,$$

therefore, as $[\sigma] = [t]^{-1}$, then $\sigma_* = \sigma l \cdot c^{-1}$ is dimensionless. Then $\lambda_{+*} = \lambda_+ l$, $\lambda_{-*} = \lambda_- l$, $J_{1k*} = J_{1k} \cdot lc^{-2}$ and $J_{2k*} = J_{2k} \cdot lc^{-2}$ are also dimensionless. $v_0(t)$ function is dimensionless as we include rod Young modulus in it. Further we omit the index $*$.

Obviously, in view of relation $H(ax) = H(x)$ when $a > 0$, optimal control function will retain its form (6).

We consider the case when $\gamma = 0$ (free end) and take in (3) and (10)

$$v(t) = t\sin\left(\frac{\pi}{2}t\right),\ w_0(x) = \cos(2\pi x),\ \ \dot{w}_0(x) = 0,\ \ (x,t) \in [-1,1]\times\mathbb{R},$$

which obviously satisfy corresponding transmission conditions

$$w_0'(-1) = w_0'(1) = v(0) = 0,\ \dot{w}_0'(-1) = \dot{w}_0'(1) = \dot{v}(0) = 0.$$

In that case $\overline{W}(x,z_k) = iz_k\cos(2\pi x)$, all roots $z_k,\ k \in \mathbb{N}$, are determined from relation

$$\lambda_{-k} + \lambda_{+k} = i\pi k,\quad k \in \mathbb{N},$$

as

$$z_k = i\cdot\frac{(\pi k)^2 - \alpha^2}{2\pi k},$$

therefore they are all imaginary ($\sigma_k = 0,\ z_k = i\varsigma_k,\ k \in \mathbb{N}$). Furthermore,

$$\lambda_{+k} = z_k = i \cdot \frac{(\pi k)^2 - \alpha^2}{2\pi k}, \quad \lambda_{-k} = i \cdot \frac{(\pi k)^2 + \alpha^2}{2\pi k}.$$

Thus, in order to consider system (7)–(8) we need

$$\Gamma[x^o_{2j}, x^o_{2j+1}, z_k] = \lambda_{+k} J_{1k} + \lambda_{-k} J_{2k},$$

$$J_{1k} = -\varsigma_k \sum_{j=0}^{m-1} \int_{x^o_{2j}}^{x^o_{2j+1}} \cos\left[\frac{\pi k}{2} + \frac{\alpha^2}{2\pi k}(1-\xi)\right] \cos(2\pi\xi)\, d\xi -$$

$$-\varsigma_k \int_{x^o_{2m}}^{1} \cos\left[\frac{\pi k}{2} + \frac{\alpha^2}{2\pi k}(1-\xi)\right] \cos(2\pi\xi)\, d\xi,$$

$$J_{2k} = -\varsigma_k \int_{-1}^{x^o_0} \cos\left[\frac{\pi k}{2} + \frac{\alpha^2}{2\pi k}(1-\xi)\right] \cos(2\pi\xi)\, d\xi -$$

$$-\varsigma_k \sum_{j=0}^{m-1} \int_{x^o_{2j+1}}^{x^o_{2j+2}} \cos\left[\frac{\pi k}{2} + \frac{\alpha^2}{2\pi k}(1-\xi)\right] \cos(2\pi\xi)\, d\xi,$$

and

$$M_k = \lambda_{+k}\bar{v}_0(z_k), \quad \bar{v}_0(z_k) = \int_0^\tau v(t) e^{-\varsigma_k t} dt = \int_0^\tau t e^{-\varsigma_k t} \sin\left(\frac{\pi}{2} t\right) dt.$$

So, switching point should be determined from the following system of real restrictions:

$$Y_{1k} + Y_{2k} - \frac{\alpha^2}{(\pi k)^2}[Y_{1k} - Y_{2k}] + \frac{2}{\pi k}\bar{v}_0(z_k) = 0, \quad k \in \mathbb{N},$$

where $J_{pk} = -\varsigma_k Y_{pk}, p = 1;2, k \in \mathbb{N}$.

Numerical analysis is done, switching points are found and $u^o(x)$ function is plotted when $\tau = 2, T = \pi, \alpha^2 \in [0.01, 2]$. Analysis also shows, that with increasing k the quantity $2(\pi k)^{-1}\bar{v}_0(z_k)$ decreases very fast for all values of adhesion factor α^2 and for $k = 20$ it is of 10^{-6} order. After integrating one can obtain, that $Y_{pk}, p = 1;2, k \in \mathbb{N}$, are periodic, therefore the consideration may be limited by $k = 20$.

It is obvious from values of switching points, there are almost equal switching points. For example points x^o_0, x^o_1 and x^o_2, x^o_3 of Fig. 5 are close to each other, which corresponds to discrete contact between rod and rigid base (pointwise gluing, riveting).

It is turned out, that with increasing of α^2 in $[0.01, 2]$ switching points are get close to each other, so in this case the form (17) of optimal control function should be used.

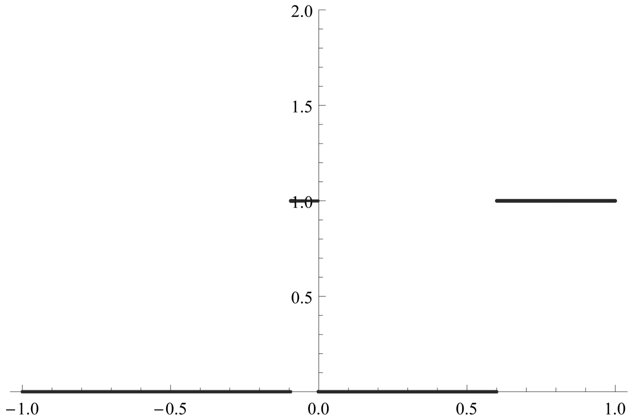

Figure 2. $\alpha^2 = 0.01$, $x_0^o = -0.0953$, $x_1^o = -0.0012$, $x_2^o = 0.0010$, $x_3^o = 0.0012$, $x_4^o = 0.69748$ and $x_5^o = 1$

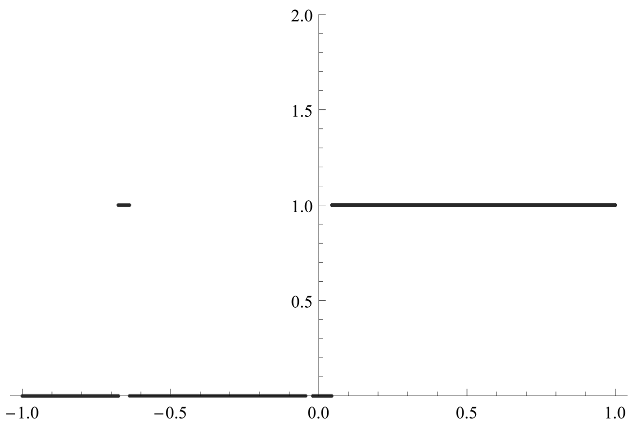

Figure 3. $\alpha^2 = 0.1$, $x_0^o = -0.67606$, $x_1^o = -0.63860$, $x_2^o = -0.04373$, $x_3^o = 0.04354$, $x_4^o = 0.04412$ and $x_5^o = 1$

5. Conclusions

In the present paper, an optimal control problem is considered for partial differential equation with variable controlled coefficient, arising in various fields of theoretical and mathematical physics (see (1)–(2)). An analytical algorithm of solution is constructed which allows reducing solution of coefficient control problem to solution of problem of nonlinear programming (see (7), (8)).

Particularly showed, that the optimal topology of adhesive binding in problems of structural elastic vibration damping when the intensity of adhesion distribution should

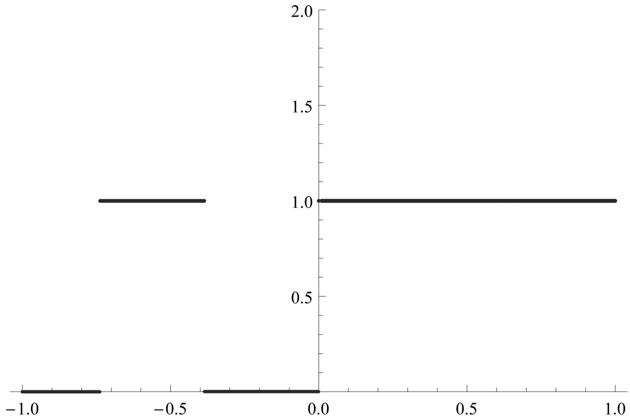

Figure 4. $\alpha^2 = 0.5$, $x_0^o = -0.7387$, $x_1^o = -0.38568$, $x_2^o = -0.00063$, $x_3^o = 0.00003$, $x_4^o = 0.0006$ and $x_5^o = 1$

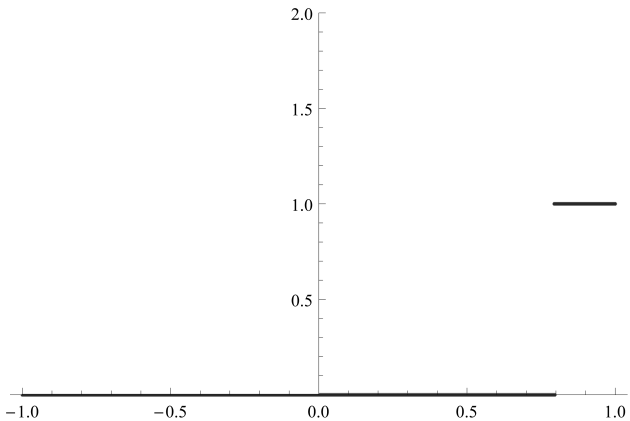

Figure 5. $\alpha^2 = 2$, $x_0^o = -0.793304$, $x_1^o = -0.793302$, $x_2^o = 0.79424$, $x_3^o = 0.79571$, $x_4^o = 0.79668$ and $x_5^o = 1$

be minimized has piecewise realized link. Glue distribution function is obtained explicitly with help of unit step functions, furthermore optimal control function is determined by switching points, corresponding to endpoints of rod glued and free parts (see (6)). The determination of the switching points is reduced to solution of nonlinear system of equalities.

Numerical analysis shows, that with increasing adhesion factor the length of glued parts of the rod decreases and vise versa, with decreasing adhesion factor the length of glued parts of the rod increases, which should be expected.

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