Numerical solution of fractional neutron point kinetics model in nuclear reactor

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This paper presents results concerning solutions of the fractional neutron point kinetics model for a nuclear reactor. Proposed model consists of a bilinear system of fractional and ordinary differential equations. Three methods to solve the model are presented and compared. The first one entails application of discrete Grünwald-Letnikov definition of the fractional derivative in the model. Second involves building an analog scheme in the FOMCON Toolbox in MATLAB environment. Third is the method proposed by Edwards. The impact of selected parameters on the model's response was examined. The results for typical input were discussed and compared.

Key words: fractional differential equation, fractional derivatives and integrals, system of equations, nuclear reactor, neutron point kinetics

1. Introduction

The main processes taking place in a nuclear reactor is neutron kinetics. With respect to time of appearance, neutrons are divided into immediate and delayed neurons. The immediate neutrons (more than 99% of all fission neutrons) are emitted directly in the fission act, while the delayed neutrons (less than 1%) are produced during the radioactive decay of certain fission products - delayed neutron precursors.

By taking into account averaged values of neutrons and fractional derivative a fractional neutron point kinetics (FNPK) model can be built. This issue based on ordinary derivative has been discussed for many years [1, 2, 5]. In [5] a FNPK model in a nuclear reactor was introduced. In its simplest form it consists of a single neutron density equation, while in the full form this equation is complemented by six equations of delayed neutron precursor densities. An intermediate form of this model makes use of one averaged group of delayed neutron precursors.

In recent years, however, the FNPK model has become the subject of research [8, 11]. In [8] the fractional point kinetics model with one group of delayed neutron precursors was proposed. In this paper, the authors extended the model for six groups.

Point models for nuclear reactors have been commonly used for many years [1, 2, 5, 6, 8, 15]. In order to improve control better methods of approximating processes taking place in the nuclear reactor have been researched. The use of fractional point models may prove to be such an improvement. Authors' research concentrated on nuclear reactor analysis. Their interests revolved around 1D [3] and recently 3D models.

The paper is organized as follows. First introduced in the paper was a form of the fractional derivative. Then the FNPK model with six groups of delayed neutron precursors was described and normalized, along with the assumed values of particular constant parameters and the adopted initial conditions. After that methods to solve the FNPK model were proposed and described. Finally, tests results, including comparison of methods used were presented and conclusions were drawn.

2. Definition of fractional derivative

There are several definitions and methods for calculating the fractional derivative. The most recognized definitions were given by: Caputo, Riemann-Liouville and Grünwald-Letnikov [13]. In the paper the Grünwald-Letnikov and Caputo approaches are used. Grünwald-Letnikov's approach, which is used in the first and the second method, has the following form [14]:

$$_aD_t^k f(t) = \lim_{h \to 0} h^{-k} \sum_{p=0}^{\left[\frac{t-a}{h}\right]} (-1)^p \binom{k}{p} f(t - ph) \qquad (1)$$

where k is the order of fractional derivative, h is step-size, a and t are the limits relating to the above operation.

Caputo's approach, used in the third method, has the following form [14]:

$$_aD_t^k f(t) = \frac{1}{\Gamma(m-k)} \int_a^t \frac{f^{(m)}(\xi)}{(t-\xi)^{(k+1-m)}} d\xi \qquad (2)$$

for $m+1 > k > m$, where m is the ordinary derivative. Γ is the gamma function.

The interpretation of the operator $_aD_t^k$ (hereinafter referred to as D^k) may be presented as:

$$D^k = \begin{cases} \frac{d^k}{dt^k} & : & k > 0 \\ 1 & : & k = 0 \\ \int_a^t (d\tau)^k & : & k < 0. \end{cases} \qquad (3)$$

Notice that for $k<0$ the operation D^k takes the form of the integrator. For integer values of k the operation takes the form of ordinary derivative or ordinary integrator.

3. The fractional neutron point kinetics model

The FNPK model with six groups of delayed neutron precursors, which was initially described for one group in [8] and extended to six groups in [11], may be given by fractional differential equations:

$$\tau^k \frac{d^{k+1}}{dt^{k+1}} n(t) + \frac{d}{dt} n(t) + \tau^k \left(\frac{1}{l} + \frac{1-\beta}{\Lambda} \right) \frac{d^k}{dt^k} n(t) + \frac{\beta - \rho(t)}{\Lambda} n(t)$$

$$= \sum_{j=1}^{6} \tau^k \frac{d^k}{dt^k} C_j(t) + \sum_{j=1}^{6} \lambda_j C_j(t) \tag{4}$$

$$\frac{d}{dt} C_j(t) = \frac{\beta_j}{\Lambda} n(t) - \lambda_j C_j(t) \quad \text{for} \quad j = 1, ..., 6 \tag{5}$$

where the initial conditions are: $n_0 = n(0)$ and $C_{j0} = C_j(0)$ for $j = 1, 2, ..., 6$, $n(t)$ is the density of neutrons, $C_j(t)$ is the concentration of delayed neutron precursor nuclei of the j-th group, β_j is the fraction of the delayed neutrons of the j-th group in the population of neutrons, λ_j is the decay constant of delayed-neutron precursors, l is the immediate neutron lifetime, $\rho(t)$ is the reactivity, τ^k is the relaxation time, and Λ is the effective lifetime of the neutrons.

Neutron density was examined in the paper [3,6]. It is proportional to nuclear reactor's power. Therefore, the neutron density will be normalized further in the paper for purposes of testing nuclear reactor's power.

Reactivity is the relative deviation of the relation between the sizes of successive generations of neutrons. It determines increase and decrease of nuclear reactor's power. If nuclear reactor's power is greater than 0 and if reactivity is negative - the power decreases (and also neutron density), if positive - the power increases and if equal to 0 - the power is at a constant level.

Relaxation time is the parameter specifying the rate of system return to the state of equilibrium and is time-constant. Greater values of this parameter mean that the system (in this case: the density of neutrons in the nuclear reactor core) returns to the state of equilibrium slower. The impact of relaxation time on the neutron density of the FNPK model was examined in [11].

Tab. 1 shows the assumed fractions of the delayed-neutron fractions of the population of neutrons and decay constant of delayed-neutron precursors.

The following parameters were adopted in Eqs. (4)-(5): $\beta = \sum_{j=1}^{6} \beta_j = 0.007$, $\Lambda = l = 0.00003$ s. The effect of k and τ^k on the obtained numerical solutions for proposed methods will be discussed further in the paper.

The analysis was performed using the normalized values of $n(t)$ and $C_j(t)$. The normalized values of neutron density were denoted by n, while the normalized concentration of delayed neutron precursor nuclei of the j-th group by C_j. In order to perform normalization, the normalized quantities must be defined first:

$$n = \frac{n(t)}{n_B} \tag{6}$$

Table 1. Delayed neutron fractions and precursor decay constants for U-235 [10].

j	β_j	λ_j
		$1/s$
1	0.000231	0.0124
2	0.001533	0.0305
3	0.001327	0.1110
4	0.002765	0.3010
5	0.000805	1.1300
6	0.000294	3.0000

$$C_j = \frac{C_j(t)}{C_{j,B}} \quad \text{for } j = 1,2,...,6 \tag{7}$$

where n_B and $C_{j,B}$ are base values of above quantities. The relation between these values is defined by:

$$C_{j,B} = \frac{\beta_j}{\Lambda\lambda_j}n_B \quad \text{for } j = 1,2,...,6. \tag{8}$$

The FNPK model after normalization may be given by formula:

$$\tau^k \frac{d^{k+1}}{dt^{k+1}}n(t) + +\frac{d}{dt}n(t) + \tau^k \left(\frac{1}{l} + \frac{1-\beta}{\Lambda}\right)\frac{d^k}{dt^k}n(t) + \frac{\beta - \rho(t)}{\Lambda}n(t)$$
$$= \sum_{j=1}^{6} \tau^k \frac{d^k}{dt^k}\frac{\beta_j}{\Lambda\lambda_j}C_j(t) + \sum_{j=1}^{6} \frac{\beta_j}{\Lambda}C_j(t) \tag{9}$$

$$\frac{d}{dt}C_j(t) = \lambda_j n(t) - \lambda_j C_j(t) \quad \text{for } j = 1,2,...,6. \tag{10}$$

The critical state of equilibrium was adopted as the initial state. The analysis was performed for the critical state of equilibrium which was different than the switch-off state of the reactor, therefore the initial conditions of this state were: $n_0 = C_{j0} = 1$.

4. Numerical methods to solve the nonlinear fractional differential system of equations

There are methods to solve system of fractional differential equations. One of them is Adomian decomposition method [9]. The method turned out to be ineffective. That is why other methods to solve the FNPK model were examined. The first entails substituting the discrete Grünwald-Letnikov definition of fractional derivative into the NFPK

derivatives [12]. The second is to use the FOMCON Toolbox in MATLAB to build ana-log scheme in Simulink, where the system of equations could be solved. The third is method proposed by Edwards, which was used in previous paper [11].

The following symbols were introduced to simplify the notation: D^k instead of $\frac{d^k}{dt^k}$, D instead of $\frac{d}{dt}$, D^{k+1} instead of $\frac{d^{k+1}}{dt^{k+1}}$. After applying the new nomenclature, Eqs. (9)-(10) become as follows:

$$D^{k+1}n + a_3 Dn + a_2 D^k n + a_1 n = \sum_{j=1}^{6} b_{2j} D^k C_j + \sum_{j=1}^{6} b_{1j} C_j \tag{11}$$

$$DC_j = a_{0j}n - a_{0j}C_j \quad \text{for } j = 1,2,...,6. \tag{12}$$

According to Eqs. (9)-(10) the coefficients take the following form: $a_{0j} = \lambda_j$, $a_1 = \frac{1}{\tau^k}\frac{\beta-\rho}{\Lambda}$, $a_2 = \frac{1}{l} + \frac{1-\beta}{\Lambda}$, $a_3 = \frac{1}{\tau^k}$, $b_{1j} = \frac{\beta_j}{\tau^k \Lambda}$, $b_{2j} = \frac{\beta_j}{\Lambda \lambda_j}$. The above symbols will be used in all proposed methods.

4.1. Application of discrete Grünwald-Letnikov definition of the fractional derivative to the FNPK model

It is necessary to consider all values of the function from a to t and every step-size h to solve fractional derivative. Solving fractional derivative is iterative. The value of current (i-th) iteration depends on all previous iterations.

To numerically solve linear fractional derivative equation in i-th iteration one can use the discrete expression of Eq. (1) [12]:

$$D^k f(t) \approx h^{-k} \sum_{p=0}^{i} (-1)^p \binom{k}{p} f(t_{i-p}) = h^{-k} \sum_{p=0}^{i} c_p{}^{(k)} f(t_{i-p}) \tag{13}$$

where $t_i = ih$ and $c_p{}^{(k)}$ $(p = 0,2,...,i)$ - binomial coefficients that can be written as [12]:

$$c_0{}^{(k)} = 1, c_p{}^{(k)} = \left(1 - \frac{1+k}{p}\right) c_{p-1}{}^{(k)}. \tag{14}$$

By substituting the right hand side of the Eq. (13) into $D^{k+1}n$, Dn and $D^k n$ in Eq. (11):

$$\frac{1}{h^{1+k}} \sum_{p=0}^{i} c_p{}^{(1+k)} n_{i-p} + \frac{a_3}{h} \sum_{p=0}^{i} c_p{}^{(1)} n_{i-p} + \frac{a_2}{h^k} \sum_{p=0}^{i} c_p{}^{(k)} n_{i-p} + (a_{11} + a_{12}\rho_i)n_i =$$

$$= \sum_{j=1}^{6} \frac{b_{2j}}{h^k} \sum_{p=0}^{i} c_p{}^{(k)} C_{j,i-p} + \sum_{j=1}^{6} \frac{b_{1j}}{C_{j,i}}. \tag{15}$$

To get the neutron density in the i-th iteration one can write it on the left hand side of the equation, while the remaining part of the equation on the right hand side. It results in:

$$n_i = \frac{\sum_{j=1}^{6} \frac{b_{2j}}{h^k} \sum_{p=0}^{i} c_p^{(k)} C_{j,i-p} + \sum_{j=1}^{6} b_{1j} C_{j,i}}{\frac{1}{h^{1+k}} + \frac{a_3}{h} + \frac{a_2}{h^k} + a_{12}\rho_i + a_{11}} + \tag{16}$$

$$- \frac{\frac{1}{h^{1+k}} \sum_{p=1}^{i} c_p^{(1+k)} n_{i-p} + \frac{a_3}{h} \sum_{p=1}^{i} c_p^{(1)} n_{i-p}}{\frac{1}{h^{1+k}} + \frac{a_3}{h} + \frac{a_2}{h^k} + a_{12}\rho_i + a_{11}} - \frac{\frac{a_2}{h^k} \sum_{p=1}^{i} c_p^{(k)} n_{i-p}}{\frac{1}{h^{1+k}} + \frac{a_3}{h} + \frac{a_2}{h^k} + a_{12}\rho_i + a_{11}}.$$

Using the same transformation $C_{j,i}$ can be calculated from:

$$C_{j,i} = \frac{a_{0j} n_i - \frac{1}{h} \sum_{p=1}^{i} c_p^{(1)} C_{j,i-p}}{\frac{1}{h} + a_{0j}} \quad \text{for } j = 1, 2, ..., 6. \tag{17}$$

Note that both on the left and right hand side of the above equations are quantities in i-th iteration. Because it is impossible to calculate the value of one quantity in i-th iteration while not having the value of another quantity in this iteration, the mentioned quantities on the right hand side of the equations in i-th iteration were replaced by $i - 1$-th iteration.

The application of discrete Grünwald-Letnikov definition of the fractional derivative is hereinafter referred to as the first method. In figures it appears as GL.

4.2. Application of FOMCON Toolbox

In order to make a clear presentation of an analog scheme, the auxiliary variables were defined:

$$x_1 = n \tag{18}$$

$$x_2 = D^k n = D^k x_1 \tag{19}$$

$$x_3 = Dn = Dx_1 \tag{20}$$

$$x_4 = D^{k+1} n = D^{k+1} x_1 = D^k x_3 \tag{21}$$

$$y_{1j} = C_j \quad \text{for } j = 1, 2, ..., 6 \tag{22}$$

$$y_{2j} = D^k C_j = D^k y_{1j} \quad \text{for } j = 1, 2, ..., 6 \tag{23}$$

$$y_{3j} = DC_j = Dy_{1j} \quad \text{for } j = 1, 2, ..., 6. \tag{24}$$

It results from the Eqs. (11)-(12) and Eqs. (18)-(24) that:

$$x_4 = -\sum_{i=2}^{3} a_i x_i - a_{11} x_1 - a_{12} z_3 x_1 + \sum_{i=1}^{2} \sum_{j=1}^{6} b_{ij} y_{ij} \tag{25}$$

$$y_{3j} = a_{0j} x_1 - b_{2j} y_{1j} \quad \text{for } j = 1, 2, ..., 6. \tag{26}$$

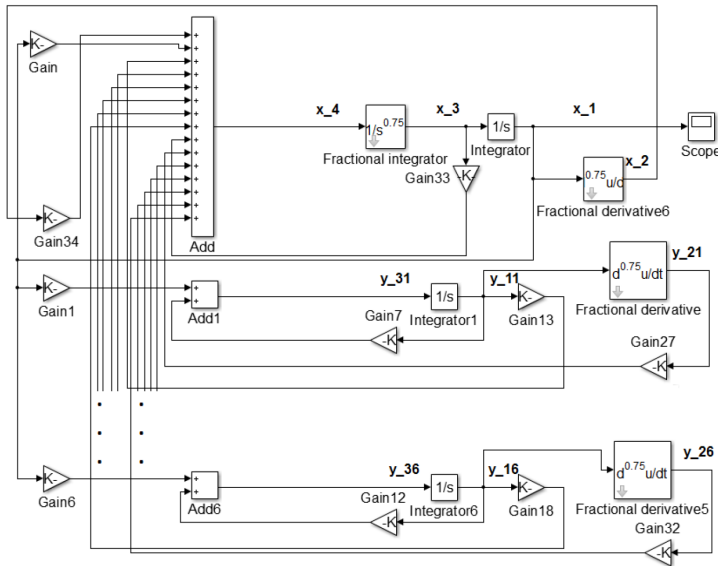The Eqs. (18)-(26) were modeled as an analog scheme (see Fig. 1).

Figure 1. Fractional neutron point kinetics model analog scheme.

The presented analog scheme was built using blocks from the FOMCON Toolbox. Those blocks are fractional derivative s^k and fractional integrator s^{-k}. They consist of the Oustaloup recursive filter that gives a very good approximation of fractional derivative and integrator. For fractional derivative s^k where $0<k<1$ it is defined by [16]:

$$s^k = G_f(s) = K \prod_{l=-N}^{N} \frac{s + \omega'_k}{s + \omega_k} \tag{27}$$

where N is the approximation order and the ω'_k, ω_k, K are obtained from:

$$\omega'_k = \omega_b \left(\frac{\omega_h}{\omega_b} \right)^{\frac{l+N+\frac{1}{2}(1-k)}{2N+1}} \tag{28}$$

$$\omega_k = \omega_b \left(\frac{\omega_h}{\omega_b} \right)^{\frac{l+N+\frac{1}{2}(1+k)}{2N+1}} \tag{29}$$

$$K = \omega_h^k \tag{30}$$

where (ω_b, ω_h) is specified frequency range.

The default frequency range used in toolbox blocks is $(\omega_b, \omega_h) = (10^{-3}, 10^3)$, whilst the default approximation order is $N = 5$. Default values returned results inconsistent with basic laws of physics taking place in a nuclear reactor. For this reason the frequency range was changed to $(\omega_b, \omega_h) = (10^{-1}, 1)$.

The transfer function for the Oustaloup recursive filter was calculated for different values of k (0.25, 0.5, 0.75) for which the tests were performed. For all mentioned k:

$$K = 1. \tag{31}$$

The other quantities are shown in Tab. 2.

Table 2. Quantities to calculate transfer function of Oustaloup recursive filter.

l	$\omega'_{0.25}$ $1/s$	$\omega_{0.25}$ $1/s$	$\omega'_{0.5}$ $1/s$	$\omega_{0.5}$ $1/s$	$\omega'_{0.75}$ $1/s$	$\omega_{0.75}$ $1/s$
-5	0.1082	0.1140	0.1054	0.1170	0.1027	0.1201
-4	0.1334	0.1405	0.1299	0.1442	0.1266	0.1481
-3	0.1644	0.1732	0.1602	0.1778	0.1560	0.1825
-2	0.2027	0.2136	0.1974	0.2192	0.1923	0.2250
-1	0.2499	0.2633	0.2434	0.2703	0.2371	0.2774
0	0.3081	0.3246	0.3001	0.3332	0.2924	0.3421
1	0.3798	0.4002	0.3700	0.4108	0.3604	0.4217
2	0.4682	0.4934	0.4561	0.5065	0.4444	0.5199
3	0.5772	0.6083	0.5623	0.6244	0.5478	0.6409
4	0.7117	0.7499	0.6933	0.7698	0.6754	0.7902
5	0.8774	0.9245	0.8546	0.9490	0.8326	0.9742

From the values presented in Tab. 2 and Eq. (31) the transfer function of Oustaloup recursive filter for s^k where $k = 0.25$:

$$s^{0.25} = \frac{(s+0.1082)(s+0.1334)...(s+0.8774)}{(s+0.1140)(s+0.1405)...(s+0.9245)} \tag{32}$$

where $k = 0.5$:

$$s^{0.5} = \frac{(s+0.1054)(s+0.1299)...(s+0.8546)}{(s+0.1170)(s+0.1442)...(s+0.9490)} \tag{33}$$

where $k = 0.75$:

$$s^{0.75} = \frac{(s+0.1027)(s+0.1266)...(s+0.8326)}{(s+0.1201)(s+0.1481)...(s+0.9742)}. \tag{34}$$

The Oustaloup recursive filter for s^{-k}, which is the fractional integration of order k, can be obtained from:

$$s^{-k} = \frac{1}{s^k} = \frac{1}{G_f(s)}. \tag{35}$$

It is suggested to use ode23tb solver [16] which is the implicit Runge-Kutta method. It is a two-stage method. Its first stage is a trapezoidal rule. The second is a second-order backward differentiation formula.

The application of FOMCON Toolbox was hereinafter referred to as the second method. In figures it appears as FOMCON.

4.3. Method proposed by Edwards

The above methods are suitable for solving linear and nonlinear systems of fractional differential equations. Method proposed by Edwards is suited for linear systems. Because the reactivity step at the beginning of the simulation is constant, the bilinear model (see Eq. (5)-(5)) becomes linear. That is why this method can be applied for the considered model.

In order to calculate the current (i-th) values of quantities in Edward's method, system of linear equations has to be solved. From Eqs. (18)-(26) it can be presented as:

$$
\begin{bmatrix}
D^k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & D^k & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & D^k & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \vdots & \ddots & \vdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & D^k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & D & \cdots & 0 \\
0 & 0 & 0 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & D
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ y_{11} \\ \vdots \\ y_{16} \\ y_{21} \\ \vdots \\ y_{26}
\end{bmatrix}
=
\begin{bmatrix}
x_2 \\ x_3 \\ x_4 \\ y_{21} \\ \vdots \\ y_{26} \\ y_{31} \\ \vdots \\ y_{36}
\end{bmatrix}
\tag{36}
$$

The system of equation takes form:

$$
\boldsymbol{D} = \boldsymbol{uz}
\tag{37}
$$

where:

$$
\boldsymbol{D} =
\begin{bmatrix}
D^k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & D^k & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & D^k & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \vdots & \ddots & \vdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & D^k & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & D & \cdots & 0 \\
0 & 0 & 0 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & D
\end{bmatrix}
\tag{38}
$$

$$\boldsymbol{u} = [x_1, x_2, x_3, y_{11}, \ldots, y_{16}, y_{21}, \ldots, y_{26}]^T \tag{39}$$

$$\boldsymbol{z} = [x_2, x_3, x_4, y_{21}, \ldots, y_{26}, y_{31}, \ldots, y_{36}]^T. \tag{40}$$

To solve system of differential equations (see Eq. (36)) using numerical method proposed by Edwards, it is necessary to discretize the model. A discrete approximation of the fractional derivative of the k-th row in the i-th step can be written as:

$$D^k x_i = \frac{1}{\gamma_i} \left(\sum_{p=0}^{i} \omega_{p,i} x_{i-p} + \frac{x_0}{k} \right) \tag{41}$$

where particular components have the following form [4, 10]:

$$\gamma_i = (ih)^k \Gamma(-k) \tag{42}$$

$$\omega_{p,i} k (1-k) i^{-k} = \begin{cases} -1 & : \quad p = 0 \\ 2p^{1-k} - (p-1)^{1-k} - (p+1)^{1-k} & : \quad p = 1, \ldots, i-1 \\ (k-1)p^k - (p-1)^{1-k} + p^{1+k} & : \quad p = i. \end{cases} \tag{43}$$

The above relations show that due to the presence of the Γ function (see Eq. (42)), which is discontinuous for negative integer arguments, the discrete model can be built only for $0 < k < 1$.

Using Eqs. (18)-(26) and Eqs. (41)-(43) the discrete model can be derived. Calculating the value of the variable x_2, which is the first variable in the z vector in the i-th step, we obtain:

$$x_{2,i} = D^k x_{1,i} = \frac{1}{\gamma_i} \left(\sum_{p=0}^{i} \omega_{p,i} x_{1,i-p} + \frac{x_{1,0}}{k} \right). \tag{44}$$

Then introducing discrete variable $S_{1,i}$ [7]:

$$S_{1,i} = \sum_{p=1}^{i} \omega_{p,i} x_{1,i-p} + \frac{x_{1,0}}{k} \tag{45}$$

and putting Eq. (45) into Eq. (44), obtaining:

$$x_{2,i} = \frac{1}{\gamma_i} \left(\omega_{0,i} x_{1,i} + S_{1,i} \right). \tag{46}$$

Eq. (44) can be written as:

$$-\omega_{0,i} x_{1,i} + \gamma_i x_{2,i} = \sum_{p=1}^{i} \omega_{p,i} x_{1,i-p} + \frac{x_{1,0}}{k}. \tag{47}$$

In a similar way as for x_2, the relation between the variables y_{1j} and y_{2j} where j is the precursor group number can be derived. The way of deriving the relations between the

variables x_1 and x_3, as well as between x_3 and x_4, will be described further in the paper. Like in Eq. (47), for the variables y_{1j} and y_{2j}, we obtain:

$$-\omega_{0,i}y_{1j,i} + \gamma_i y_{2j,i} = \sum_{p=1}^{i} \omega_{p,i}y_{1j,i-p} + \frac{y_{1j,0}}{k}. \tag{48}$$

The variable x_3 is the first-order derivative of the variable x_1 (see Eq. (20)). It can be calculated using the trapezoidal method:

$$x_{1,i} = x_{1,i-1} + \frac{h}{2}(x_{3,i} + x_{3,i-1}). \tag{49}$$

By putting the values of variables obtained in the same step on one hand side of the equation we have:

$$x_{1,i} - \frac{h}{2}x_{3,i} = x_{1,i-1} + \frac{h}{2}x_{3,i-1} \tag{50}$$

and introducing the new variable $S_{2,i}$ [7]:

$$x_{1,i} - \frac{h}{2}x_{3,i} = S_{2,i} \tag{51}$$

we have:

$$S_{2,i} = x_{1,i-1} + \frac{h}{2}x_{3,i-1}. \tag{52}$$

For y_{1j}, introducing a new variable, we obtain:

$$R_{1j,i} = \sum_{p=1}^{i} \omega_{p,i}y_{1j,i-p} + \frac{y_{1j,0}}{k}. \tag{53}$$

A discrete equation of the third variable in vector z is the following:

$$x_{4,i} = D^k x_{3,i} = \frac{1}{\gamma_i}\left(\sum_{p=0}^{i} \omega_{p,i}x_{3,i-p} + \frac{x_{3,0}}{k}\right). \tag{54}$$

By putting Eq. (25) into Eq. (54) we have:

$$-a_1 x_{1,i} - a_2 x_{2,i} - a_3 x_{3,i} + \sum_{j=1}^{6} b_{1j}y_{1j,i} + \sum_{j=1}^{6} = \frac{1}{\gamma_i}\left(\sum_{p=0}^{i} \omega_{p,i}x_{3,i-p} + \frac{x_{3,0}}{k}\right). \tag{55}$$

After transformation and putting the i-th step variables to the left hand side of the equation, we obtain:

$$-a_1 x_{1,i} - a_2 x_{2,i} - (a_3 \gamma_i + \omega_{0,i}) x_{3,i} + \sum_{j=1}^{6} b_{1j} y_{1j,i} + \sum_{j=1}^{6} b_{2j} y_{2j,i}$$

$$= \frac{1}{\gamma_i} \left(\sum_{p=1}^{i} \omega_{p,i} x_{3,i-p} + \frac{x_{3,0}}{k} \right). \tag{56}$$

Like in case of Eq. (50), the last equation can be derived using the trapezoidal method:

$$y_{1j,i} = y_{1j,i-1} + \frac{h}{2}(y_{3j,i} + y_{3j,i-1}) \quad \text{for} \ \ j = 1,2,...,6. \tag{57}$$

Putting Eq. (26) into Eq. (57) gives:

$$y_{1j,i} = y_{1j,i-1} + \frac{h}{2}(a_{0j} x_{1,i} - b_{2j} y_{2j,i} + a_{0j} x_{1,i-1} - b_{2j} y_{2j,i-1}) \ \ \text{for} \ \ j = 1,...,6. \tag{58}$$

Eq. (58) can be written as:

$$-a_{0j}\frac{h}{2} x_{1,i} + (1 + b_{2j}\frac{h}{2}) y_{1j,i} = a_{0j}\frac{h}{2} x_{1,i-1} + (1 - b_{2j}\frac{h}{2}) y_{1j,i-1} \quad \text{for} \ \ j = 1,...,6. \tag{59}$$

Finally, the discrete model gets the following form:

$$\begin{bmatrix} -\omega_{0,i} & \gamma_i & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & -\frac{h}{2} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -\gamma_i a_1 & -\gamma_i a_2 & -(\gamma_i a_3 + \omega_{0,i}) & \gamma_i b_{11} & \cdots & \gamma_i b_{16} & \gamma_i b_{21} & \cdots & \gamma_i b_{26} \\ 0 & 0 & 0 & -\omega_{0,i} & \cdots & 0 & \gamma_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\omega_{0,i} & 0 & \cdots & \gamma_i \\ -a_{01}\frac{h}{2} & 0 & 0 & 1+b_{21}\frac{h}{2} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ -a_{06}\frac{h}{2} & 0 & 0 & 0 & \cdots & 1+b_{26}\frac{h}{2} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_{11} \\ \vdots \\ y_{16} \\ y_{21} \\ \vdots \\ y_{26} \end{bmatrix}$$

$$= [S_{1,i},\ S_{2,i},\ S_{3,i},\ R_{11,i},\ ...,\ R_{16,i},\ R_{21,i},\ ...,R_{26,i}]^T \tag{60}$$

where $S_{1,i}$, $S_{2,i}$ and $R_{1j,i}$ are given, respectively, by Eqs. (45) and (52)-(53). Based on Eqs. (55) and (59) the variables $S_{3,i}$ and $R_{2j,i}$ can be written as:

$$S_{3,i} = \sum_{p=1}^{i} \omega_{p,i} x_{3,i-p} + \frac{x_{3,0}}{k} \tag{61}$$

$$R_{2j,i} = a_{0j}\frac{h}{2} x_{1,i-1} + (1 - b_{2j}\frac{h}{2}) y_{1j,i-1} \quad \text{for} \ \ j = 1,...,6. \tag{62}$$

In order to determine the density of neutrons, represented by the variable x_1, it is necessary to inverse the square matrix (Eq. (60)) and multiply it by R and S. The result in the first row is the normalized neutron density in the i-th step.

The application of method proposed by Edwards was hereinafter referred to as the third method. In figures it appears as Edwards.

5. Results

The following results were obtained for positive input reactivity $\rho = 0.0005$. Input step was set at the beginning of simulation. For this reason the bilinear model can be treated as linear model. Neutron density is the quantity shown in the figures. The difference between results obtained using the first, second and the third method was presented and discussed. The impact of relaxation time τ^k, fractional order of derivative k and step-size h on results was examined. τ^k was at the level of $10^{-4}s^k$ and $10^{-5}s^k$, k was equal to 0.25, 0.5 and 0.75, whilst h was equal to 0.1 s nad 0.05 s. Step-size was changed only in the first and the third method due to the fact these are a fixed-step methods. The second method is variable-step method.

The tests were performed for each τ^k, k and h. The initial state of the reactor was assumed by the critical state. This means that the reactor operated at a constant power output and with fixed physical quantities before testings.

The simulation time was equal to 150 s, but the results were shown also for 5 s. to show the biggest changes taking place in the first seconds after reactivity step.

Normalized neutron density for $\tau^k = 10^{-4}s^k$ is shown in Figs 2-7.
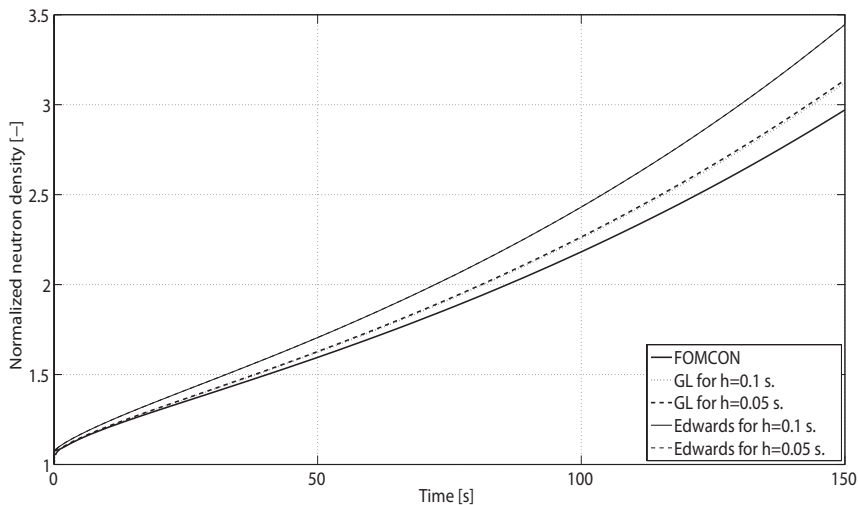
Figure 2. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-4}s^k$ and $k = 0.25$.
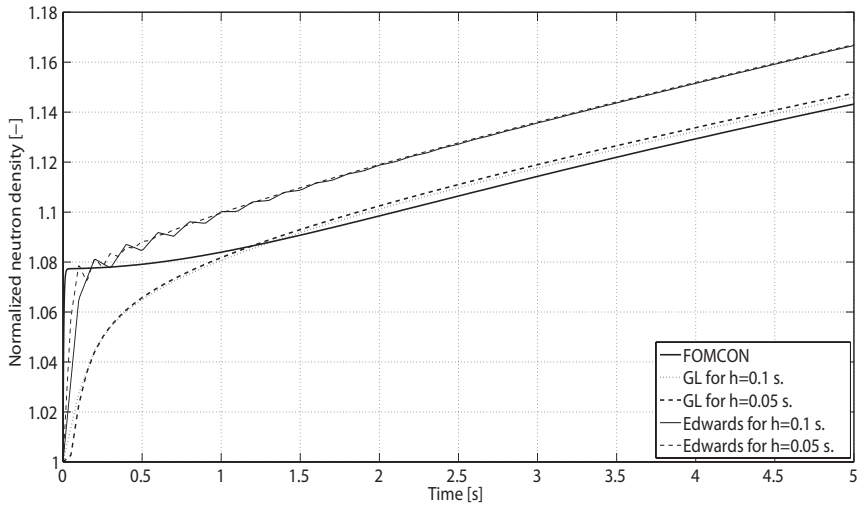
Figure 3. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-4} s^k$ and $k = 0.25$.
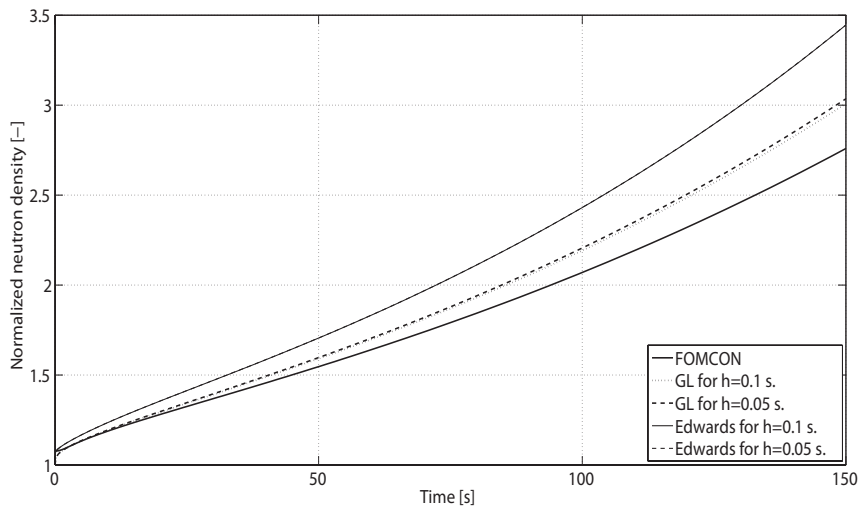

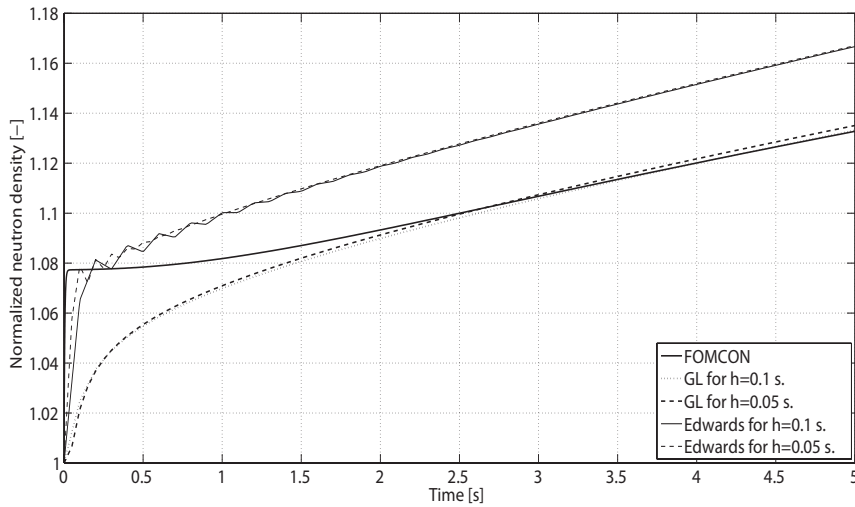Figure 4. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-4} s^k$ and $k = 0.50$.

Figure 5. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-4} s^k$ and $k = 0.50$.
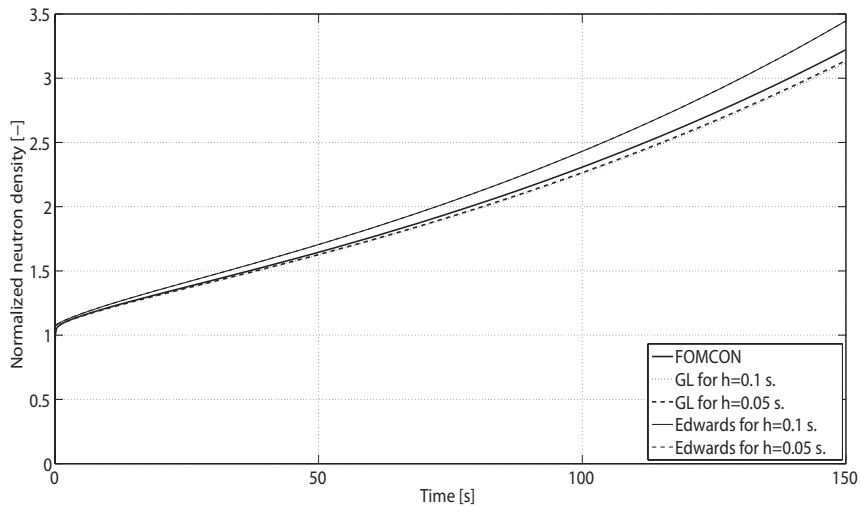

Figure 6. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-4} s^k$ and $k = 0.75$.
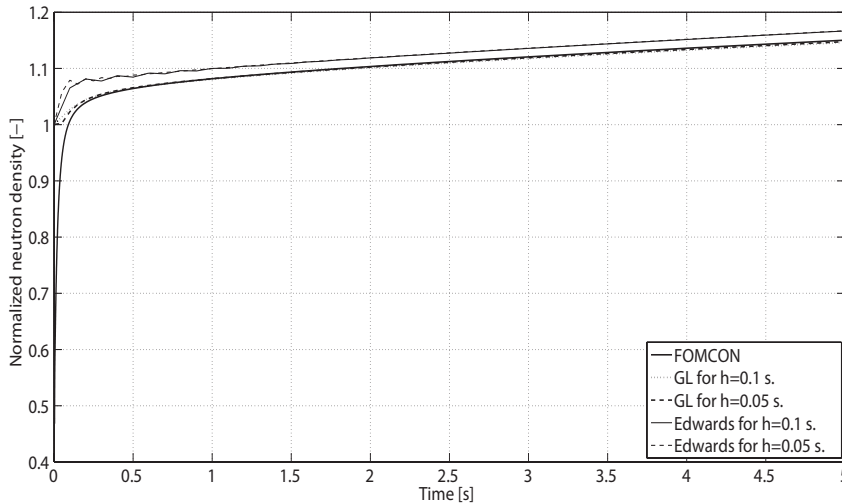
Figure 7. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-4}s^k$ and $k = 0.75$.

For positive step reactivity the neutron density increases. The neutron density is supposed to be greater than its initial value (before tests) for the whole simulation. This was not satisfied only for the second method where $k = 0.75$. In this case neutron density firstly decreases and is lower than its initial value. The neutron density for the first method after 3 s. of simulation is the highest for the third method and the lowest for the first method.

The neutron density progress was different for tested values of fractional derivative. The higher value of k the maximum neutron density is higher. This relationship was seen for three methods. For higher k the difference between methods for neutron density is lower.

In Figs 8-13 neutron density for $\tau^k = 10^{-5}s^k$ is presented.

There is a difference between results obtained for $\tau^k = 10^{-5}s^k$ and $\tau^k = 10^{-4}s^k$. The smaller τ^k, the smaller the difference between the methods. It can be explained by the fact that $\tau^k \to 0$ causes change in Eq. (5). The parts of equation containing fractional derivative are becoming no longer relevant. The difference between used methods is in calculating the values of fractional derivative. Due to this fact, the difference between results for tested value of fractional derivative is smaller for $\tau^k = 10^{-5}s^k$ than for $\tau^k = 10^{-4}s^k$.

Based on the results three methods methods may be used to approximate neutron kinetics in a nuclear reactor. The obtained results were different for different values of relaxation time, order of fractional derivative and step size - the model was characterized by different dynamics. It can be noticed that for the second method order of fractional

derivative, relaxation time and frequency range have to be chosen wisely because the model's response may contradict the law of physics.
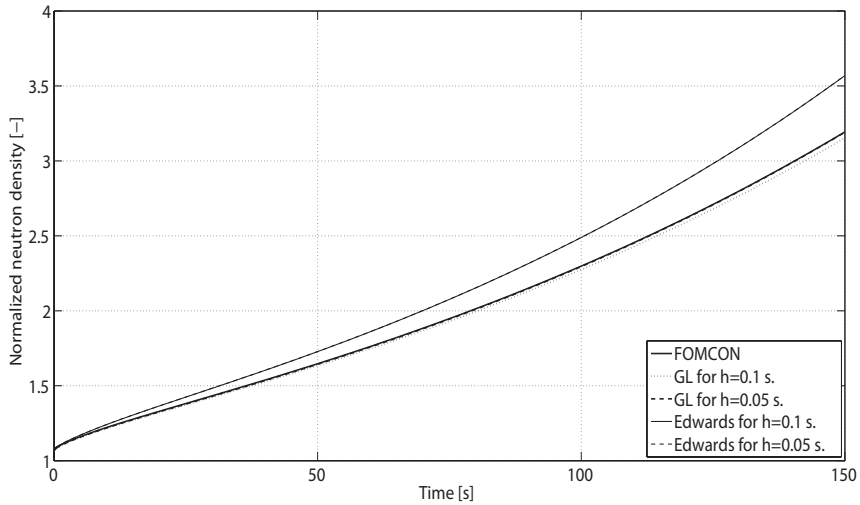

Figure 8. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-5} s^k$ and $k = 0.25$.
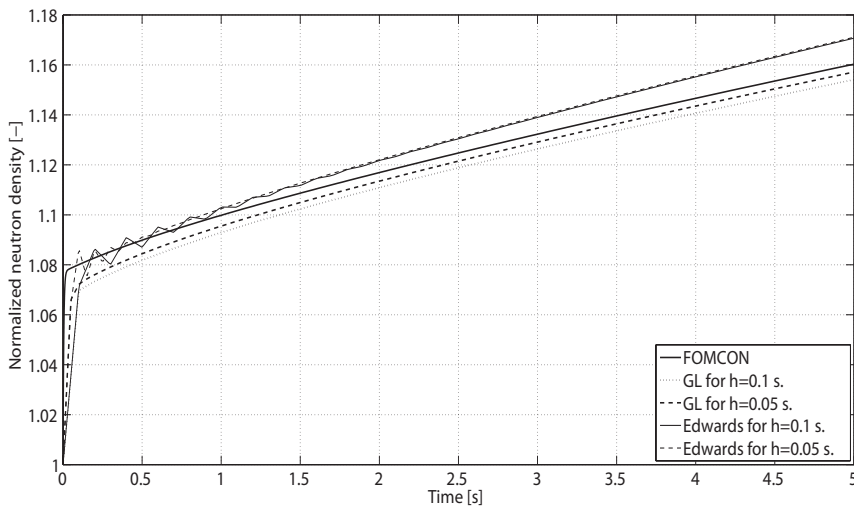

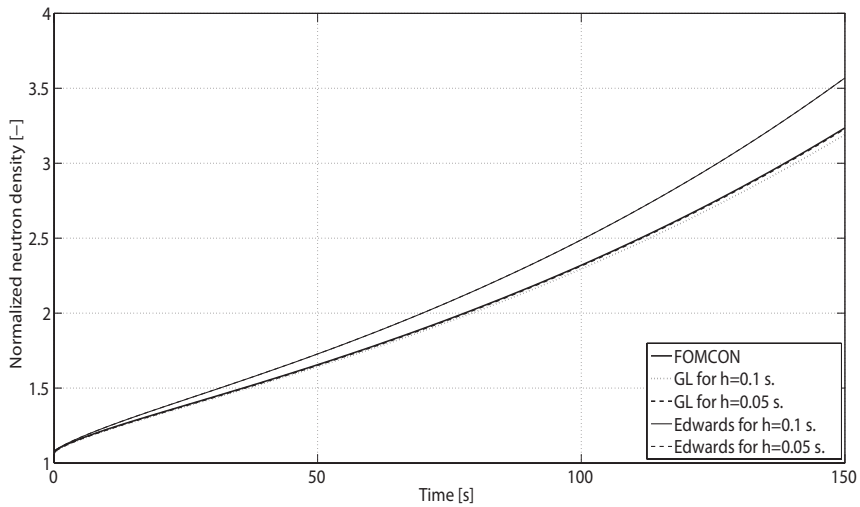Figure 9. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-5} s^k$ and $k = 0.25$.

Figure 10. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-5} s^k$ and $k = 0.50$.
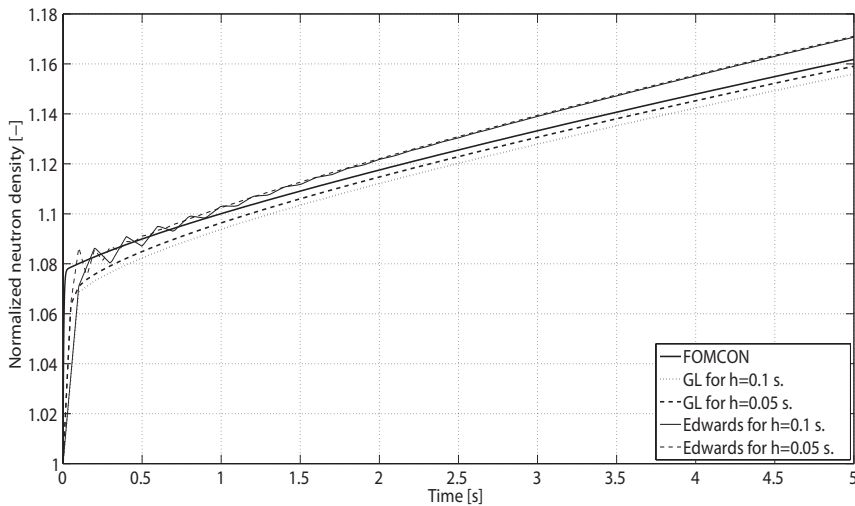

Figure 11. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-5} s^k$ and $k = 0.50$.
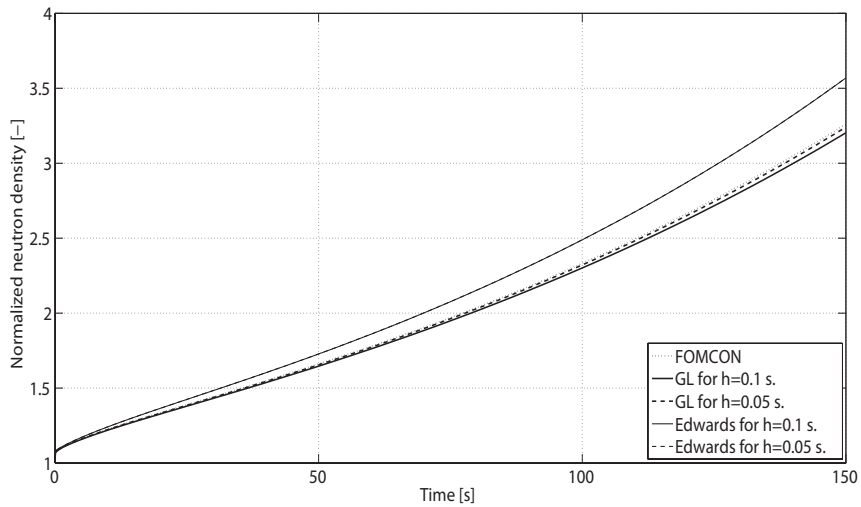
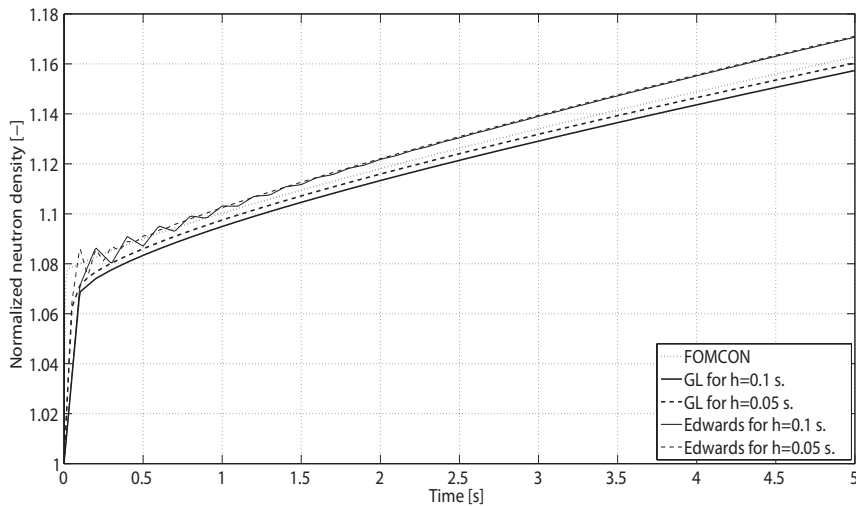Figure 12. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-5}s^k$ and $k = 0.75$.

Figure 13. Normalized neutron density for $\rho = 0.0005$, $\tau^k = 10^{-5}s^k$ and $k = 0.75$.

6.   Comparison of the methods

In order to decide which one of three methods is more accurate for FNPK model, a comparison to a special case was drawn. The special case is critical state of nuclear reactor where $\rho = 0$. For the zero value of the reactivity the nuclear power output and so neutron density is constant n(t)=1. The difference between this value and obtained for the three methods will be compared.

The results are shown in Figs 14-19.

The third method seems to be the most appropriate for the FNPK model. It can be seen that lower value of τ^k the lower difference between $n(t) = 1$ and results obtained by application of methods.

In order to eliminate any doubts the comparison consisted of 2 indicators. They were: the maximum value of the difference between $n(t) = 1$ and the first, second or third method (MAX) and integral of the absolute value of this difference (IAD). The results are shown in Tabs 3-8.

The third method returned the lowest absolute value of three indicators. The lower step-size the better results for the first method. The third method turned out to be the best from all used methods in the paper. This was confirmed by the indicators' values used for comparison. The lower value of τ^k the lower the difference for three methods. The same conclusion can be made for k.
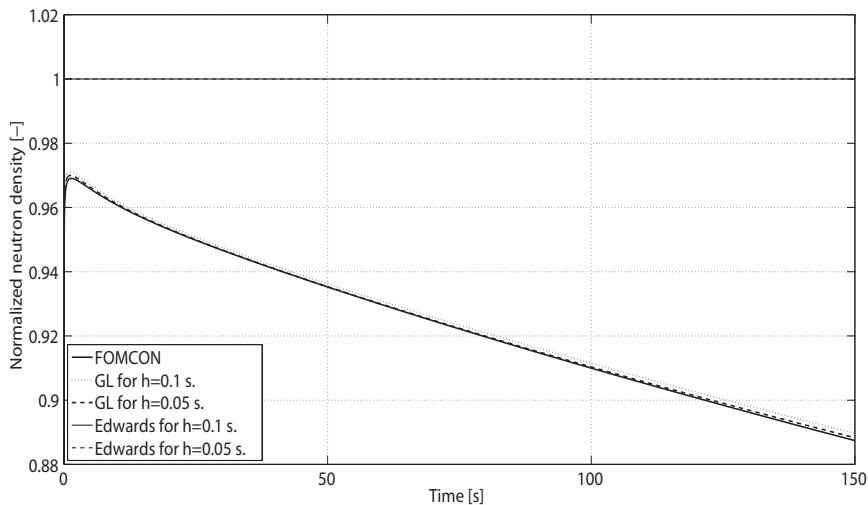

Figure 14. Difference between $n(t) = 1$ and results obtained by considered methods for $\rho = 0$, $\tau^k = 10^{-4}s^k$, $k = 0.25$.
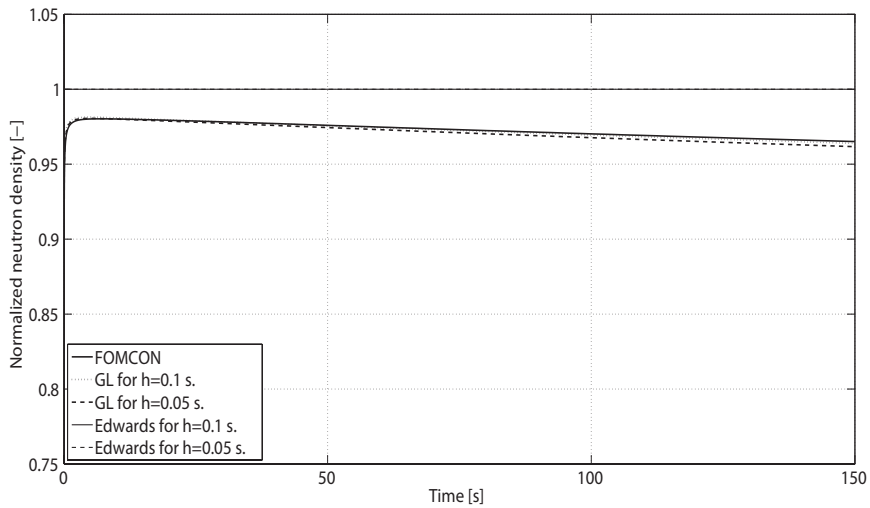
Figure 15. Difference between $n(t) = 1$ and results obtained by considered methods for $\rho = 0$, $\tau^k = 10^{-4} s^k$, $k = 0.50$.

Figure 16. Difference between $n(t) = 1$ and results obtained by considered methods for $\rho = 0$, $\tau^k = 10^{-4} s^k$, $k = 0.75$.

Figure 17. Difference between $n(t) = 1$ and results obtained by considered methods for $\rho = 0$, $\tau^k = 10^{-5} s^k$, $k = 0.25$.

Figure 18. Difference between $n(t) = 1$ and results obtained by considered methods for $\rho = 0$, $\tau^k = 10^{-5} s^k$, $k = 0.50$.
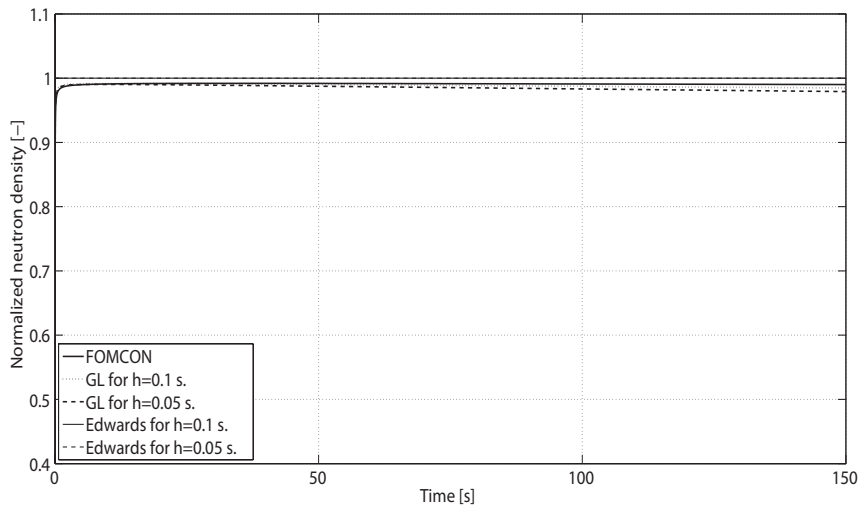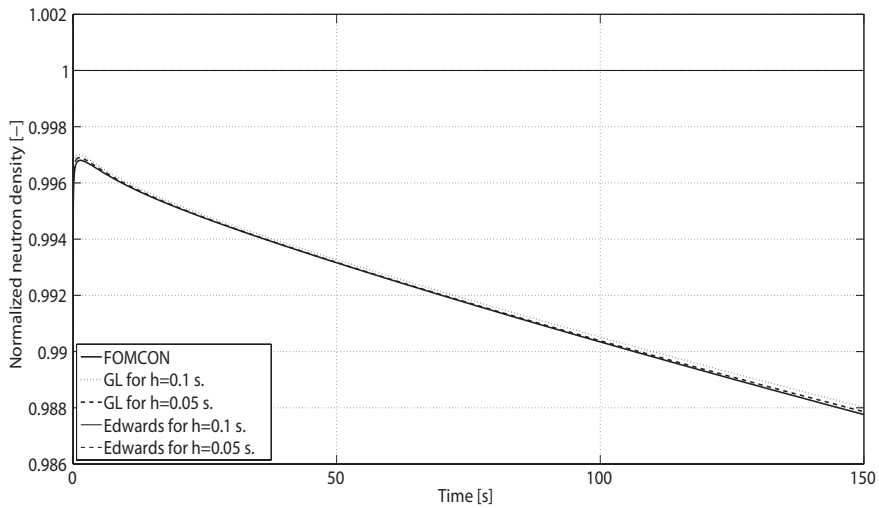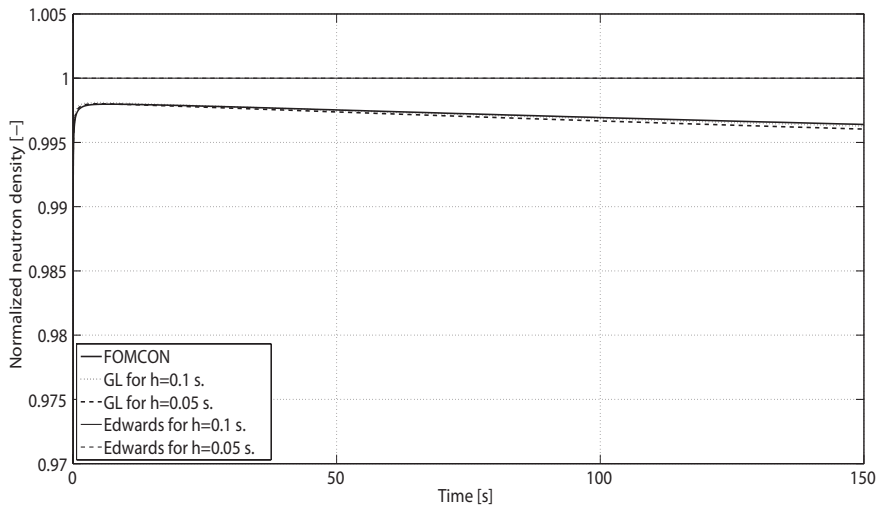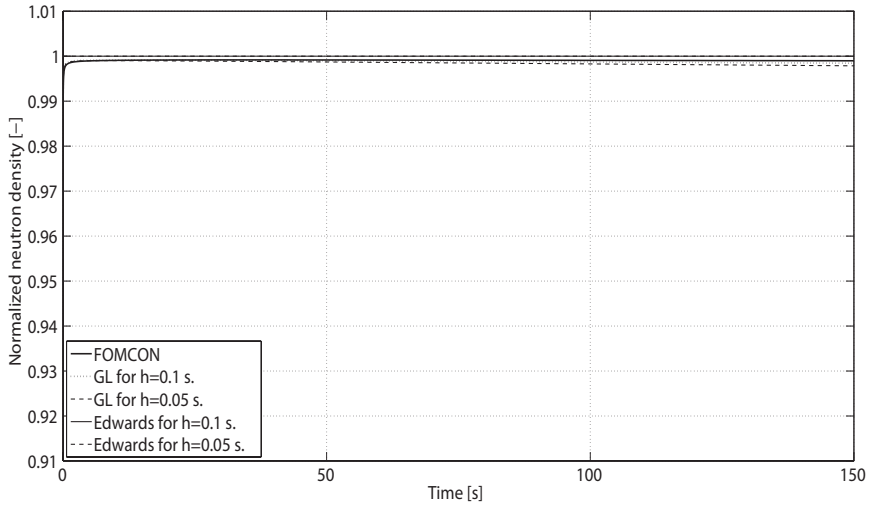
Figure 19. Difference between $n(t) = 1$ and results obtained by considered methods for $\rho = 0$, $\tau^k = 10^{-5}s^k$, $k = 0.75$.

Table 3. Indicators to compare the methods for $k = 0.25$ and $\tau^k = 10^{-4}s^k$.

Indicator	$GL^a_{h=0.1}$	$GL^a_{h=0.05}$	FOMCON[b]	Edwards$^c_{h=0.1}$	Edwards$^c_{h=0.05}$
MAX	0.11257	0.11033	0.11168	7.3275e-11	1.2812e-13
IAD	11.422	11.213	11.367	7.303e-12	1.2633e-11

[a] The difference between results obtained from $n(t) = 1$ and the first method

[b] The difference between results obtained from $n(t) = 1$ and the second method

[c] The difference between results obtained from $n(t) = 1$ and the third method

Table 4. Indicators to compare the methods for $k = 0.50$ and $\tau^k = 10^{-4}s^k$.

Indicator	$GL^a_{h=0.1}$	$GL^a_{h=0.05}$	FOMCON[b]	Edwards$^c_{h=0.1}$	Edwards$^c_{h=0.05}$
MAX	0.2104	0.040079	0.052878	7.3275e-14	1.2879e-13
IAD	4.0588	4.1118	4.3349	7.3e-12	1.2629e-11

Table 5. Indicators to compare the methods for $k = 0.75$ and $\tau^k = 10^{-4}s^k$.

Indicator	$GL^a_{h=0.1}$	$GL^a_{h=0.05}$	FOMCON[b]	Edwards$^c_{h=0.1}$	Edwards$^c_{h=0.05}$
MAX	0.54392	0.033726	0.050363	7.3608e-14	1.2812e-12
IAD	1.3651	1.7183	2.2261	7.302e-12	1.2627e-11

Table 6. Indicators to compare the methods for $k = 0.25$ and $\tau^k = 10^{-5}s^k$.

MAX	0.012241	0.011981	0.01214	3.7748e-11	3.7381e-13
IAD	1.2199	1.1964	1.2137	3.2732e-13	2.7976e-11

Table 7. Indicators to compare the methods for $k = 0.50$ and $\tau^k = 10^{-5}s^k$.

Indicator	$GL^a_{h=0.1}$	$GL^a_{h=0.05}$	FOMCON[b]	Edwards$^c_{h=0.1}$	Edwards$^c_{h=0.05}$
MAX	0.02512	0.0043232	0.00058491	3.7748e-15	3.7381e-13
IAD	0.41761	0.42327	0.44636	3.2941e-13	2.7975e-11

Table 8. Indicators to compare the methods for $k = 0.75$ and $\tau^k = 10^{-5}s^k$.

Indicator	$GL^a_{h=0.1}$	$GL^a_{h=0.05}$	FOMCON[b]	Edwards$^c_{h=0.1}$	Edwards$^c_{h=0.05}$
MAX	0.088156	0.0038377	0.0061347	3.8858e-15	3.7381e-13
IAD	0.13953	0.17559	0.22743	3.3014e-13	2.7974e-11

7. Conclusions

Properties of the numerical solution of fractional neutron point kinetics model were analyzed in this paper. Results were obtained by using three methods. The first was application of discrete Grünwald-Letnikov definition of the fractional derivative, the second was application of the FOMCON Toolbox, whose blocks use the Oustaloup recursive filter and the third was the method proposed by Edwards. Quantitative differences between the methods were observed. The effect of relaxation time, order of fractional derivative and step-size on the obtained results were examined. It was proven that three methods can be used to approximate neutron kinetics. A comparison between the three methods was drawn. Method proposed by Edwards turned out to be the best from proposed method for considered model.

The task for researchers is to check what values should be assumed by τ^k, k (and h for first method) in order to obtain a satisfactory approximation for the nuclear reactor's 1D [3] and 3D model.

References

[1] G. ACKERMANN: Operation of Nuclear Reactors. WNT, 1987, (in Polish).

[2] H. ANGLART: Nuclear Reactor Dynamics and stability. Institute of Heat Engineering, Warsaw University of Technology, 2013, (in Polish).

[3] G. BAUM and K. DUZINKIEWICZ: Simulation model of 1D dynamic reactor WWER-440. Technical report. Electric Power and Control Engineering Institute, Gdansk University of Technology, 1989, (in Polish)

[4] R. CAPONETTO, G. DONGOLA, L. FORTUNA and I. PETRÁŠ: Fractional order systems. Modeling and Control Applications. World Scientific Series of Nonlinear Science, **A**(72), 2010.

[5] J.J. DUDERSTADT and L.J. HAMILTON: Nuclear Reactor Analysis. John Wiley & Sons, New York, 1976.

[6] K. DUZINKIEWICZ, G. BAUM and A. MICHALAK: Simulation model of basic dynamic processes in nuclear power plant WWER based on model with lumped parameters. Technical report. Electric Power and Control Engineering Institute, Gdansk University of Technology, 1989, (in Polish).

[7] J.T. EDWARDS, N.J. FORD and A.CH. SIMPSON: The numerical solution of linear multi-term fractional differential equations. *J. of Computational and Applied Mathematics*, **148**(2), (2002), 401-418.

[8] G. ESPINOSA-PAREDES, M.-A. POLO-LABARRIOS, E.-G ESPINOSA-MARTINEZ and E. DEL VALLE-GALLEGOS: Fractional neutron point kinetics equations for nuclear reactor dynamics. *Annals of Nuclear Energy*, **38**(2-3), (2011), 307-330.

[9] H. JAFARI and V. DAFTARDAR-GEJJI: Solving system of nonlinear fractional differential equations using Adomian decomposition. *J. of Computational and Applied Mathematics*, **196**(2), (2006), 644-651.

[10] M. KINARD and K.E.J. ALLEN: Efficient numerical solution of the point kinetics equations in nuclear reactor dynamics. *Annals of Nuclear Energy*, **31** (2004), 1039-1051.

[11] T.K. NOWAK, K. DUZINKIEWICZ and R. PIOTROWSKI: Fractional neutron point kinetics equations for nuclear reactor dynamics – numerical investigations. *Annals of Nuclear Energy*, in review.

[12] I. PETRÁŠ: Engineering Education and Research Using MATLAB. InTech, Rijeka, 2011.

[13] I. PETRÁŠ: Fractional-Order Nonlinear Systems. Modelling, Analysis and Simulation. Springer, New York, 2011.

[14] I. PODLUBNY: Fractional Differential Equations. Academic Press, San Diego, 1999.

[15] M.A. SCHULTZ: Control of Nuclear Reactors and Power Plants. McGew-Hill, New York, 1961.

[16] A. TEPJAKOV, E. PETLENKOV and J. BELIKOV: FOMCON: a MATLAB toolbox for fractional-order system identification and control. *J. of Microelectronics and Computer Science*, **2**(2), (2011), 51-62.