Approximation method for a fractional order transfer function with zero and pole

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The paper presents an approximation method for elementary fractional order transfer function containing both pole and zero. This class of transfer functions can be applied for example to build model - based special control algorithms. The proposed method bases on Charef approximation. The problem of cancelation pole by zero with useful conditions was considered, the accuracy discussion with the use of interval approach was done also. Results were depicted by examples.

Key words: fractional order systems, fractional order transfer function, Charef approximation

1. Introduction

Fractional order transfer differential equations and consequently, fractional-order transfer functions describe a number of real physical phenomena (see for example [1], [10], [4], [11]). The possibility of effective modeling fractional order systems at the MATLAB/SIMULINK platform is determined by possibility of their integer order approximation. The approximation of elementary operator s^α was proposed by Oustaloup (see for example [1], [16]), the approximation of elementary inertial transfer function $\frac{k}{(Ts+1)^\alpha}$ was proposed by Charef [2]. The idea of both approximations is very similar and consists in fitting Bode magnitude plots of exact and approximated transfer functions. However, author of this paper does not know the simple approximation method dedicated to elementary fractional order transfer function containing both one pole and one zero $\frac{(T_\beta s+1)^\beta}{(T_\alpha s+1)^\alpha}$. This elementary function can describe a part of special, model - based controller (for example - cancelation controller). In this paper is presented a proposition of generalization the Charef approximation to describe an elementary fractional order transfer function with zero and pole.

Particularly, in the paper the following problems will be discussed:

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- A plant under consideration.

- The Charef approximation.

- The proposed approximation method for the considered transfer function.

    - Problem of cancelation poles by zeros.
    - The accuracy of the proposed approximation.

- Examples.

2. A plant under consideration

Let us consider the plant described by the following, elementary transfer function:

$$G(s) = \frac{(T_\beta s + 1)^\beta}{(T_\alpha s + 1)^\alpha} \tag{1}$$

where: $0 < \alpha, \beta < 1$, $\alpha \neq \beta$ denote fractional orders of numerator and denominator, $T_\alpha, T_\beta > 0$, $T_\alpha \neq T_\beta$ denote time constants of numerator and denominator respectively. We do not make any assumptions about conmeasurability of these coefficients (they can be conmeasurable or not).

2.1. The exact Bode magnitude plot for the considered plant

The exact magnitude Bode plot for the plant we deal with can be obtained with the use of MATLAB. The spectrum transfer function of the plant described by (1) as a function of pulsation ω can be expressed as follows:

$$G(j\omega) = \frac{(j\omega T_\beta + 1)^\beta}{(j\omega T_\alpha + 1)^\alpha} \tag{2}$$

The module $M(\omega)$ of the transfer function (2) as a function of ω is expressed as underneath:

$$M(\omega) = \sqrt{\frac{((\omega T_\beta)^2 + 1)^\beta}{((\omega T_\alpha)^2 + 1)^\alpha}} \tag{3}$$

The module (3) can be expressed in $[dB]$ as a following sum:

$$20\lg M(\omega) = 20\lg M_\beta(\omega) + 20\lg M_\alpha(\omega) \tag{4}$$

where:

$$20\lg M_\beta(\omega) = 10\beta lg((\omega T_\beta)^2 + 1) \tag{5}$$

$$20\lg M_\alpha(\omega) = -10\alpha lg((\omega T_\alpha)^2 + 1) \tag{6}$$

Equations (3) to (6) describe the exact Bode module plot for the transfer function we deal with. It is easy to numerical calculation with the use of MATLAB, it can be also easily drawn with the use of Bode approximation. The shape of a diagram is determined by values of orders α and β and time constants T_α and T_β. The factor (6) can be directly approximated with the use of Charef approximation, the factor (5) can also approximated with the use of Charef approximation, but it requires additional assumptions.

3. The Charef approximation

The possibility of application of the fractional order transfer function in MATLAB platform is determined by possibility of its finite-dimensionality and integer order approximation. The approximation of fractional-order transfer functions has been presented by many authors. Fundamental results were given by Oustaloup (see for example: [3], [8], [16], and Charef [2]). Interesting results were also presented by Djouambi, Charef and Besancon in 2007 [6]. The approximation proposed by Charef (see [2]) approximation allows us to approximate the fractional order transfer function of inertial plant, described as underneath (see [2])

$$G(s) = \frac{1}{(Ts+1)^\gamma} \tag{7}$$

where T denotes the time constant of the plant, $0 < \gamma < 1$ denotes the fractional order of the plant.

The finite-dimensional approximation of the transfer function (7) is expressed as underneath:

$$G_{ch}(s) = \frac{\prod_{n=0}^{N-1}\left(1+\frac{s}{z_i}\right)}{\prod_{n=0}^{N}\left(1+\frac{s}{p_i}\right)} \tag{8}$$

where z_i and p_i denote zeros and poles of approximation, N denotes order of the approximation. An idea of this approximation is to best fit the Bode magnitude plot of approximation to Bode magnitude plot of plant in given frequency band. Zeros and poles are calculated with the use of following recursive dependencies (see [2]):

$$p = \frac{1}{T} \tag{9}$$

$$p_0 = p\sqrt{b} \tag{10}$$

$$z_0 = ap_0 \tag{11}$$

$$\dots \tag{12}$$

$$p_i = p_0(ab)^i \quad i = 1...N \tag{13}$$

$$z_i = ap_0(ab)^i \quad i = 1...N \tag{14}$$

where:

$$a = 10^{\frac{\Delta}{10(1-\alpha)}}$$
$$b = 10^{\frac{\Delta}{10\alpha}}$$
$$ab = 10^{\frac{\Delta}{10\alpha(1-\alpha)}}$$

(15)

In (15) $\Delta > 0$ denotes maximal permissible error of Charef approximation, defined as the difference between Bode magnitude plot for model and plant, expressed in [dB]. The order of approximation N can be estimated as follows (see [2]):

$$N = Int\left(\frac{log(\omega_{max}T)}{log(ab)}\right) + 1 =$$
$$Int\left(\frac{10\alpha(1-\alpha)log(\omega_{max}T)}{\Delta}\right) + 1$$

(16)

where ω_{max} denotes the maximal frequency band, for which the approximation will be applied.

4. Proposition of approximation for the considered transfer function

The approach presented above can also be applied to approximate multimodal fractional order transfer functions (see [2]). The Bode magnitude plot of the function we deal with is a sum of plots described by (5) and (6), because the transfer function (1) we deal with, can be presented as the following product:

$$G(s) = G_\alpha(s)G_\beta(s)$$

(17)

where:

$$G_\alpha(s) = \frac{1}{(T_\alpha s + 1)^\alpha}$$

(18)

$$G_\beta(s) = (T_\beta s + 1)^\beta = \frac{1}{\frac{1}{(T_\beta s + 1)^\beta}}$$

(19)

The transfer function (18) can be directly approximated with the use of approximation (8) to (16). Denote its approximation by $G_{ch\alpha}(s)$:

$$G_{ch\alpha}(s) = \frac{\prod\limits_{n_\alpha=0}^{N_\alpha-1}\left(1+\frac{s}{z_{n_\alpha}}\right)}{\prod\limits_{n_\alpha=0}^{N_\alpha}\left(1+\frac{s}{p_{n_\alpha}}\right)} = \frac{L_{ch\alpha}(s)}{D_{ch\alpha}(s)}$$

(20)

In (20) notation is the same, as in (8). The order of approximation is equal N_α. More problematic is the approximation of the second factor of transfer function (1), expressed by (19). Notice, that this transfer function is the inverse of inertial transfer function,

expressed by (1). This implies, that its approximation can be proposed as an inverse of approximation for inertial plant described by (20):

$$G_{ch\beta}(s) = \frac{\prod\limits_{n_\beta=0}^{N_\beta-1}\left(1+\frac{s}{p_{n_\beta}}\right)}{\prod\limits_{n_\beta=0}^{N_\beta}\left(1+\frac{s}{z_{n_\beta}}\right)} = \frac{D_{ch\beta}(s)}{L_{ch\beta}(s)} \qquad (21)$$

Consequently, the approximation $G_{ch}(s)$ of the whole transfer function we deal with can be expressed as follows:

$$G_{ch}(s) = G_{ch\alpha}(s)G_{ch\beta}(s) = \frac{L_{ch\alpha}(s)D_{ch\beta}(s)}{D_{ch\alpha}(s)L_{ch\beta}(s)} \qquad (22)$$

The summarized order of the numerator and denominator of the transfer function (22) is the same and it is equal $N_\alpha + N_\beta - 1$, the both orders N_α and N_β can be estimated with the use of 16):

$$N_\alpha = Int\left(\frac{10\alpha(1-\alpha)log(\omega_{max}T_\alpha)}{\Delta}\right) + 1$$
$$N_\beta = Int\left(\frac{10\beta(1-\beta)log(\omega_{max}T_\beta)}{\Delta}\right) + 1 \qquad (23)$$

where ω_{max} describes the maximal frequency, for which the approximation is going to be applied.

4.1. Problem of cancelation poles by zeros

The above approximation has the form of quotient of two pairs polynomials (see (22)). Set of zeros transfer function (22) contains both zeros of polynomial $L_{ch\alpha}(s)$ and poles $D_{ch\beta}(s)$ and analogically: set of poles transfer function (22) contains both poles of polynomial $D_{ch\alpha}(s)$ and zeros $L_{ch\beta}(s)$. This implies, that for particular values of plant parameters T_α, T_β, α or β and approximation parameters: error Δ and orders: N_α and N_β a cancelation poles by zeros for approximating transfer function can occur. This situation can be written as underneath:

$$\exists n_\alpha = 0...N_\alpha, \exists n_\beta = 0...N_\beta : \quad z_{n\alpha} = z_{n\beta} \vee p_{n\alpha} = p_{n\beta} \qquad (24)$$

where $n_\alpha = 0...N_\alpha$, $n_\beta = 0...N_\beta$. The situation described by (24) can cause the lost of an approximation accuracy and it should be avoided during application of the proposed approximation. Conditions eliminating the poles by zeros cancelation are given underneath.

Proposition 1 Sufficient and necessary condition of cancelation absence. *Let us assume that:*

- *We consider the fractional order plant described with the use of transfer function (1) with parameters: T_α, T_β, α and β,*

- *the both parts of the fractional order transfer function (1) are approximated with the use of Charef approximation, described by (8) - (16) and (20) (21),*

- *the maximal approximation error is equal Δ for both parts of approximation,*

- *orders of both parts of approximation are equal N_α and N_β respectively,*

Thesis:
The cancelation poles and zeros of the both parts of approximation will not occur if and only if: $\forall n_\alpha = 0...N_\alpha$, $\forall n_\beta = 0...N_\beta$:

$$F_z(T_\alpha, T_\beta, \alpha, \beta, \Delta, n_\alpha, n_\beta) \neq 0 \wedge F_z(T_\alpha, T_\beta, \alpha, \beta, \Delta, n_\alpha, n_\beta) \neq 0$$

where $F_z(...)$ and $F_p(...)$ are described as follows:

$$F_z(T_\alpha, T_\beta, \alpha, \beta, \Delta, n_\alpha, n_\beta) = \frac{20}{\Delta}(\log\frac{T_\beta}{T_\alpha}) - \frac{2n_\beta + 1 + \beta}{\beta(1-\beta)} + \frac{2n_\alpha + 1 + \alpha}{\alpha(1-\alpha)} \tag{25}$$

$$F_p(T_\alpha, T_\beta, \alpha, \beta, \Delta, n_\alpha, n_\beta) = \frac{20}{\Delta}(\log\frac{T_\beta}{T_\alpha}) - \frac{2n_\beta + 1 - \beta}{\beta(1-\beta)} + \frac{2n_\alpha + 1 - \alpha}{\alpha(1-\alpha)} \tag{26}$$

Proof At the beginning let us remember that the maximal permissible error for the both approximations is the same and equal Δ. This implies, that the direct dependency between poles and zeros of transfer function (22) and parameters of the plant can be expressed as underneath:

$$z_{n\alpha} = \frac{1}{T_\alpha} 10^{\frac{\Delta}{20} \frac{2n_\alpha + 1 + \alpha}{(1-\alpha)\alpha}} \tag{27}$$

$$z_{n\beta} = \frac{1}{T_\beta} 10^{\frac{\Delta}{20} \frac{2n_\beta + 1 + \beta}{(1-\beta)\beta}} \tag{28}$$

$$p_{n\alpha} = \frac{1}{T_\alpha} 10^{\frac{\Delta}{20} \frac{2n_\alpha + 1 - \alpha}{(1-\alpha)\alpha}} \tag{29}$$

$$p_{n\beta} = \frac{1}{T_\beta} 10^{\frac{\Delta}{20} \frac{2n_\beta + 1 - \beta}{(1-\beta)\beta}} \tag{30}$$

Remember that condition of zeros cancelation is formulated as follows: $z_{n_\alpha} = z_{n_\beta}$ (see (24))After inserting (27) - (30) into this equation and any elementary transformations we obtain directly condition (25) describing the zeros cancelation:

$$\underbrace{\frac{20}{\Delta}(\log\frac{T_\beta}{T_\alpha}) - \frac{2n_\beta + 1 + \beta}{\beta(1-\beta)} + \frac{2n_\alpha + 1 + \alpha}{\alpha(1-\alpha)}}_{F_z(T_\alpha, T_\beta, \alpha, \beta, \Delta, n_\alpha, n_\beta)} = 0 \tag{31}$$

The condition of poles cancelation $p_{n_\alpha} = p_{n_\beta}$ we obtain analogically with the use of (27) - (30):

$$\underbrace{\frac{20}{\Delta}(\log\frac{T_\beta}{T_\alpha}) - \frac{2n_\beta + 1 - \beta}{\beta(1-\beta)} + \frac{2n_\alpha + 1 - \alpha}{\alpha(1-\alpha)} = 0}_{F_p(T_\alpha, T_\beta, \alpha, \beta, \Delta, n_\alpha, n_\beta)} \tag{32}$$

Finally, if the neither of conditions (31) nor (32) is met, the cancelation zeros and poles will not occur. This describes the thesis of Proposition 1 and finishes the proof. □

Direct use of the Proposition 1 to test the cancelation requires us to calculate values of functions $F_z(...)$ and $F_p(...)$ for each node of grid built by sets: $n_\alpha = 0...N_\alpha$ and $n_\beta = 0...N_\beta$. It will be shown in examples.

However the Proposition 1 allows us to test the "ready" approximation only. It does not give us any guidelines, which value of the parameter Δ, for given: T_α, T_β, α and β and estimated N_α and N_β should be selected to avoid the cancelation? The response at this question can be directly obtained from Proposition 1 and it is presented as Proposition 2.

Proposition 2 Sufficient and necessary condition of cancelation absence associated to error Δ

Let us assume that:

- *we consider the fractional order plant described with the use of transfer function (1),*

- *The both factors of the fractional order transfer function (1) are approximated with the use of Charef approximation, described by (8) - (16) and (20) (21),*

- *Maximal approximation error is equal Δ for both parts of approximation*

Thesis:
The cancelation poles and zeros of the both parts of approximation will not occur if and only if:

$$\Delta \notin \{\Delta_{cp}\} \cup \{\Delta_{cz}\}$$

where $\{\Delta_{cp}\}$ and $\{\Delta_{cz}\}$ are defined as underneath:

$$\begin{aligned}
\{\Delta_{cp}\} &= \{\Delta_{cp}(n_\alpha, n_\beta) = \frac{20log(\frac{T_\beta}{T_\alpha})}{\frac{2n_\beta+1-\beta}{\beta(1-\beta)} - \frac{2n_\alpha+1-\alpha}{\alpha(1-\alpha)}} > 0 : n_\alpha = 0...N_\alpha, \ n_\beta = 0...N_\beta\} \\
\{\Delta_{cz}\} &= \{\Delta_{cz}(n_\alpha, n_\beta) = \frac{20log(\frac{T_\beta}{T_\alpha})}{\frac{2n_\beta+1+\beta}{\beta(1-\beta)} - \frac{2n_\alpha+1+\alpha}{\alpha(1-\alpha)}} > 0 : n_\alpha = 0...N_\alpha, \ n_\beta = 0...N_\beta\}
\end{aligned} \tag{33}$$

Proof The positive solutions of (31) and (32) directly describe the "forbidden" values of error Δ. □

Use of the both presented propositions will be shown in Examples.

4.2. The accuracy of the proposed approximation

The next problem, which should be analyzed during the using of the proposed approximation is an estimation of the approximation error and associating it to maximal permissible error Δ of each partial approximation. Denote this summarized approximation error by $\Delta_s(\omega)$. The estimation of this error is given in the Proposition 3:

Proposition 3 *Let us assume that:*

- *we consider the approximation of fractional order transfer function* (1)*, described by* (20)-(24)*,*

- *maximal errors of both partial approximations are the same and equal* Δ*,*

- *The pole-zero compensation does not occur (Proposition* 1 *is not met).*

The maximal error of the proposed approximation can be estimated as follows:

$$max(\Delta_s(\omega)) = 2\Delta \tag{34}$$

Proof The error $\Delta_s(\omega)$ can be defined as the following function of the pulsation ω:

$$\Delta_s(\omega) = M(\omega) - M_{ch}(\omega) \tag{35}$$

where $M(\omega)$ and $M_{ch}(\omega)$ denote modules of considered transfer function (1) and their approximation proposed here (22) respectively. To estimate the above maximal error an interval approach will be used.

At the beginning notice that an idea of Charef approximation assumes that the module $20logM(\omega)$ of approximation of each partial transfer function: $G_{ch\alpha}(s)$ and $G_{ch\beta}(s)$ is "from definition" inaccurate and the maximal error of this approximation is equal Δ. This error is applied to calculate poles and zeros of approximation (see (15),(16)). Δ is expressed in $[dB]$ and it can be applied to express both partial modules as following intervals:

$$M_{ch\alpha} = \left[\underline{M_{ch\alpha}}; \overline{M_{ch\alpha}}\right] \tag{36}$$

$$M_{ch\beta} = \left[\underline{M_{ch\beta}}; \overline{M_{ch\beta}}\right] \tag{37}$$

where:

$$\underline{M_{ch\alpha}} = M_{ch\alpha} - \Delta \tag{38}$$

$$\overline{M_{ch\alpha}} = M_{ch\alpha} + \Delta \tag{39}$$

$$\underline{M_{ch\beta}} = M_{ch\beta} - \Delta \tag{40}$$

$$\overline{M_{ch\beta}} = M_{ch\beta} + \Delta \tag{41}$$

The width of each interval (36) and (37) is equal 2Δ. Notice that the module M_{ch} of the whole approximation (22) expressed in $[dB]$ can be calculated as a difference between modules of the both transfer functions $G_{ch\alpha}(s)$ and $G_{ch\beta}(s)$ expressed in $[dB]$:

$$M_{ch} = M_{ch\alpha} - M_{ch\beta} \tag{42}$$

The module (42) can be presented as the following interval:

$$M_{ch} = \left[\underline{M_{ch}}; \overline{M_{ch}}\right] \tag{43}$$

where:

$$\underline{M_{ch}} = M_{ch\alpha} - M_{ch\beta} - 2\Delta \tag{44}$$

$$\overline{M_{ch}} = M_{ch\alpha} - M_{ch\beta} + 2\Delta \tag{45}$$

In (44) and (45) Δ denotes the maximal error of each partial approximation, applied to calculate coefficients (9) to (16). The maximal error of the the proposed approximation can be estimated as the width of interval (43)) and it is easy to see that it is equal 2Δ, where Δ is the maximal error of each partial approximation. This finishes the proof. $\square$

The estimation presented above is very "cautious" and it describes upper limits of inaccuracy. Real error is much smaller (it will be shown in examples), but it can significantly increase when the cancelation poles or zeros occurs.

5. Examples

5.1. Example 1

As the first example let us consider the following transfer function :

$$G(s) = \frac{(5s+1)^{0.5}}{(47s+1)^{0.7}} \tag{46}$$

The approximation of the above transfer function will be built for pulsation range from 10^{-3} to 10 $[sec^{-1}]$, the maximal error $\Delta = 1$ [dB]. With the use of (24) we calculate orders of both parts of approximation. They are equal:

$$N_\alpha = 5, N_\beta = 6$$

To test the cancelation zeros and poles the Proposition 1 will be used. Results of tests are shown in Tab. 1 and 2

From Tab. 1 and 2 we can conclude that cancelation poles and zeros will not occur in the considered example, because each table contains no zeros. The exact Bode magnitude plot for plant described by (46), plotted with the use of MATLAB/SIMULINK is shown
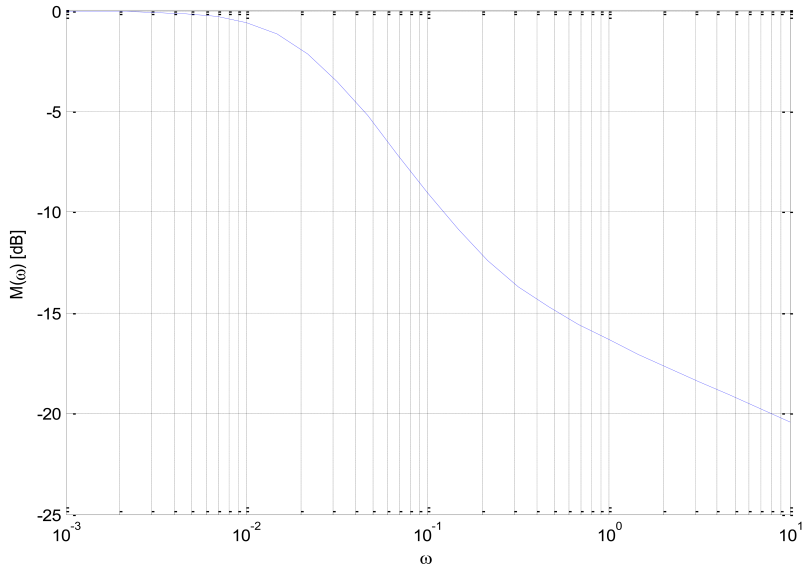
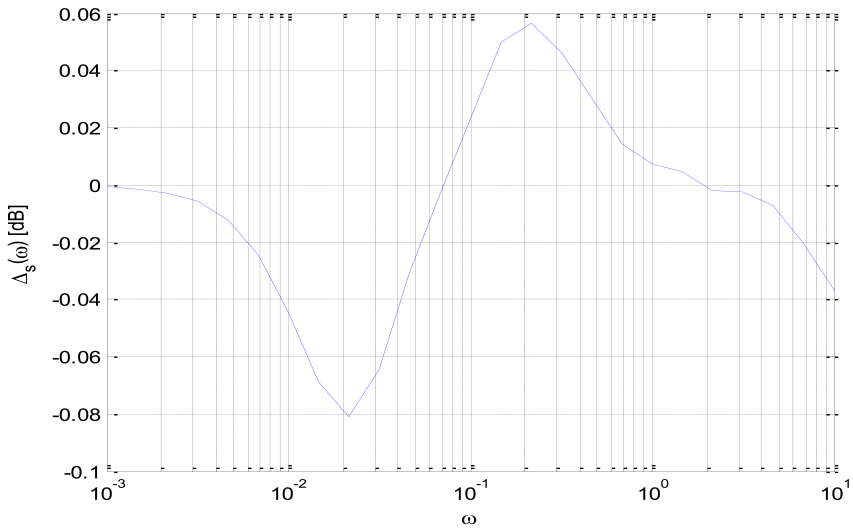Figure 1: The exact Bode magnitude plot for transfer function (46)

Figure 2: The error of approximation (35) as a function of pulsation for transfer function (46)

Table 1: values of $F_z(...)$ for whole grid in Example 2

N_α N_β	0	1	2	3	4	5	6
0	-17.3673	-7.8435	1.6803	11.2041	20.7279	30.2517	39.7755
1	-25.3673	-15.8435	-6.3197	3.2041	12.7279	22.2517	31.7755
2	-33.3673	-23.8435	-14.3197	-4.7959	4.7279	14.2517	23.7755
3	-41.3673	-31.8435	-22.3197	-12.7959	-3.2721	6.2517	15.7755
4	-49.3673	-39.8435	-30.3197	-20.7959	-11.2721	-1.7483	7.7755
5	-57.3673	-47.8435	-38.3197	-28.7959	-19.2721	-9.7483	-0.2245

Table 2: values of $F_p(...)$ for whole grid in Example 2

N_α N_β	0	1	2	3	4	5	6
0	-20.0340	-10.5102	-0.9864	8.5374	18.0613	27.5851	37.1089
1	-28.0340	-18.5102	-8.9864	0.5374	10.0613	19.5851	29.1089
2	-36.0340	-26.5102	-16.9864	-7.4626	2.0613	11.5851	21.1089
3	-44.0340	-34.5102	-24.9864	-15.4626	-5.9387	3.5851	13.1089
4	-52.0340	-42.5102	-32.9864	-23.4626	-13.9387	-4.4149	5.1089
5	-60.0340	-50.5102	-40.9864	-31.4626	-21.9387	-12.4149	-2.8911

in Fig. 1, the error of approximation (35) is shown in Fig. 2, Bode magnitude plots exact and approximated are shown in Fig. 3.

The maximal value of the approximation error $\Delta_s(\omega)$ for this case is smaller than 0.06 [dB]. From analysis of plots presented in figures 3 and 2 we can conclude that the approximation is correct and its maximal error is not greater than 0.06 [dB], what is much smaller, than maximal estimated error, equal 2[dB].

5.2.   Example 2

As the next example let us consider the transfer function (46) from Example 1 in the pulsation range from 10^{-3} to 10 $[sec^{-1}]$, the orders of approximation we set from previous example also: $N_\alpha = 5, N_\beta = 6$. But now we will deal with the "forbidden" values of error Δ. We calculate both sets $\{\Delta_{cz}\}$ and $\{\Delta_{cp}\}$ with respect to (33). They are shown in Tab. 3 and 4.

Next let us consider one of "forbidden" error values causing poles cancelation: $\Delta_{cp}(4,0) = 0.5187$ (see Tab. 4). The Bode diagrams of approximation with canceled zeros and approximation error $\Delta_s(\omega)$ are shown in Fig. 4 and 5. The maximal value of the approximation error $\Delta_s(\omega)$ for this case is equal 1.7 [dB]. Next we consider the exemplary value of error causing canceling zeros (see Tab. 3): $\Delta_{cz}(3,3) = 2.9194$. The
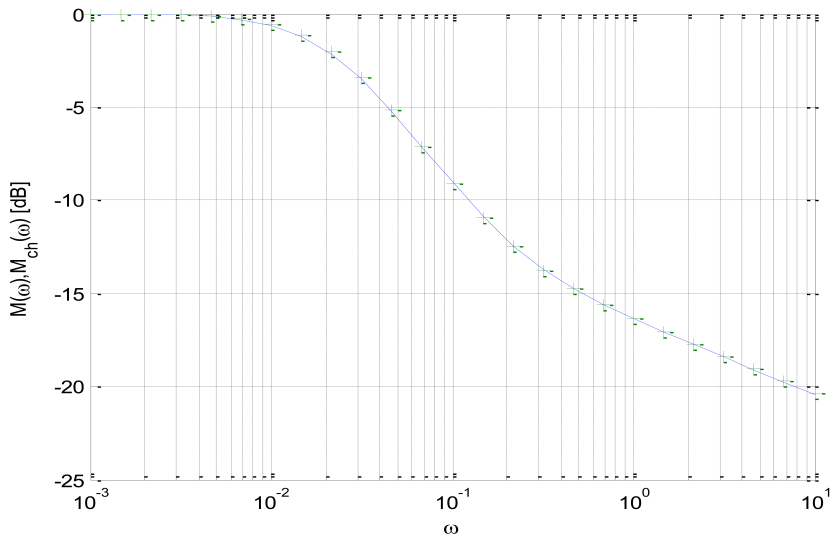
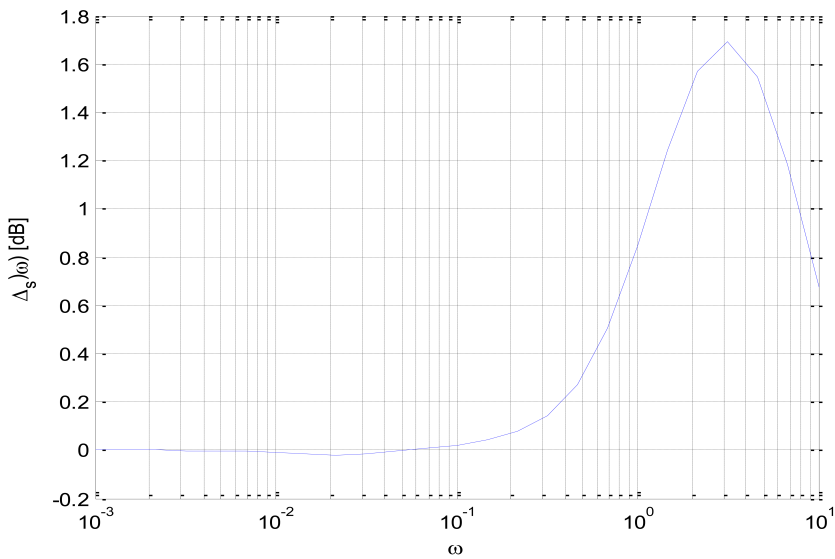Figure 3: Bode magnitude plots: exact (solid line) and approximated (cross) for transfer function (46)

Figure 4: The error of approximation (35) as a function of pulsation for transfer function (46) and canceled poles of approximation

Table 3: Set $\{\Delta_{cz}\}$ [dB] for the whole grid in Example 2, "-" denotes negative values of error

N_α / N_β	0	1	2	3	4	5	6
0	9.2889	1.6751	0.9205	0.6346	0.4843	0.3915	0.3285
1	-	5.3778	1.4808	0.8586	0.6046	0.4666	0.3798
2	-	-	3.7844	1.3270	0.8046	0.5773	0.4501
3	-	-	-	2.9194	1.2021	0.7569	0.5523
4	-	-	-	-	2.3762	1.0987	0.7145
5	-	-	-	-	1.0217	2.0035	1.0117

Table 4: Set $\{\Delta_{cp}\}$ [dB] for the whole grid in Example 2 "-" denotes negative values of error

N_α / N_β	0	1	2	3	4	5	6
0	-	2.1740	1.0534	0.6951	0.5187	0.4137	0.3440
1	-	20.4357	1.8578	0.9731	0.6592	0.4984	0.4007
2	-	-	7.8599	1.6219	0.9042	0.6269	0.4797
3	-	-	-	4.8656	1.4391	0.8444	0.5975
4	-	-	-	-	3.5234	1.2934	0.7921
5	-	-	-	-	-	2.7616	1.1745

Bode diagrams of approximation with canceled zeros and approximation error $\Delta_s(\omega)$ are shown in figures 7 and 6.

The maximal value of the approximation error $\Delta_s(\omega)$ for this case is equal 1.3 [dB]. After comparing the above results with example 1 (the same plant, but different parameters of approximation) we can see that the cancelation poles or zeros decreases the performance of proposed approximation.

6. Final conclusions

Final conclusions from the above paper can be formulated as follows:

- The proposed approximation method is a generalization of well known Charef approximation for fractional order transfer function with one zero and one pole,
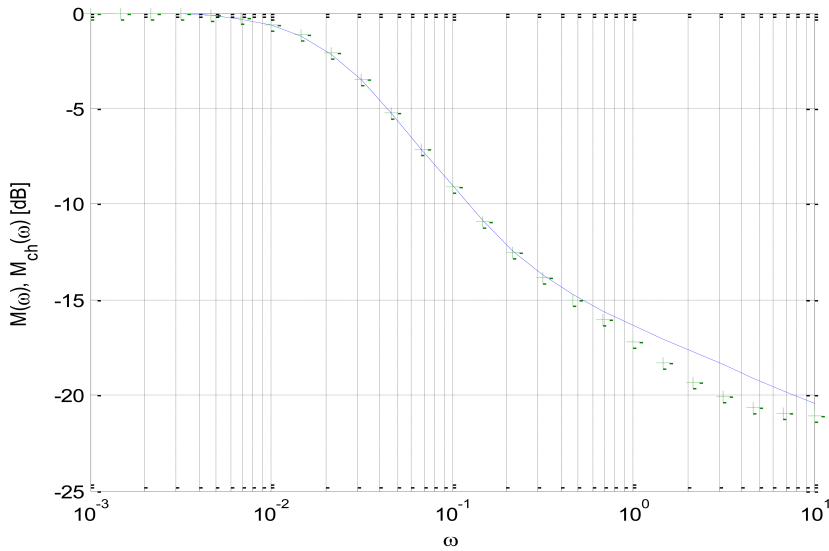
Figure 5: The exact Bode diagram (solid) and approximation (35)(cross) functions of pulsation for transfer function (46) and canceled poles of approximation
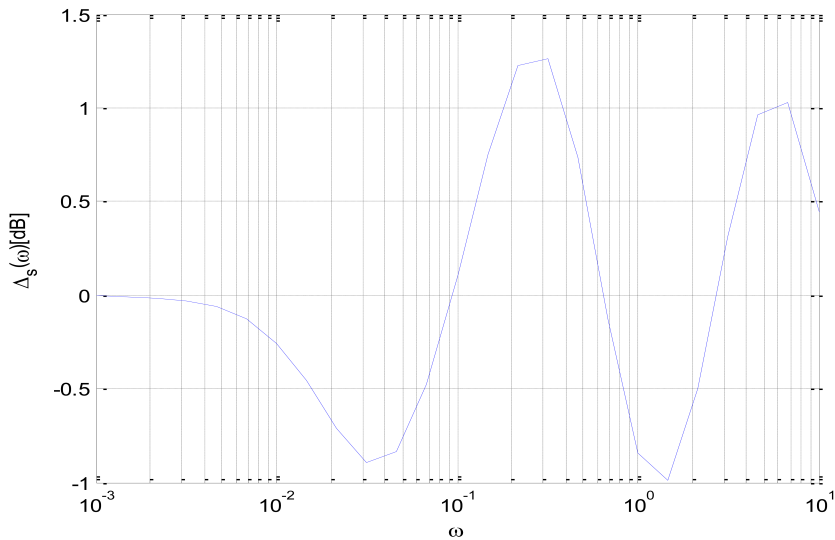

Figure 6: The error of approximation (35) as a function of pulsation for transfer function (46) and canceled zeros of approximation
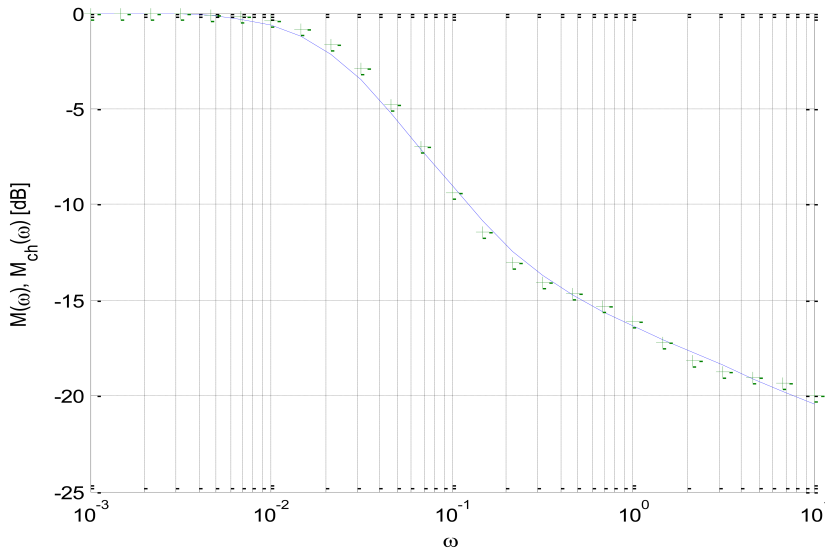
Figure 7: The exact Bode diagram (solid) and approximation (35)(cross) functions of pulsation for transfer function (46) and canceled zeros of approximation

- Estimations and results of simulations show that the proposed method is accurate and their maximal error is localized in expected places (corners of Bode magnitude plot),

- The cancelation poles or zeros can significantly decrease the approximation performance, but suitable conditions to avoid it were proposed,

- Further investigations of the presented problem are going to cover:

    – more exact accuracy analysis,

    – more precise pole-zero compensation conditions,

    – the case of multimodal pole-zero fractional order transfer function.

- The proposed approximation will be applied to simulations of a fractional order cancelation controller.

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