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Compound-combination synchronization of chaos in identical and different orders chaotic systems

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This paper proposes a new synchronization scheme called compound-combination synchronization. The scheme is investigated using six chaotic Josephson junctions evolving from different initial conditions based on the drive-response configuration via the active backstepping technique. The technique is applied to achieve compound-combination synchronization of: (i) six identical third order resistive-capacitive-inductive-shunted Josepshon junctions (RCLSJJs) (with three as drive and three as response systems); (ii) three third order RCLSJJs (as drive systems) and three second order resistive-capacitive-shunted Josepshon junctions (RCSJJs (as response systems). In each case, sufficient conditions for global asymptotic stability for compound-combination synchronization to any desired scaling factors are achieved. Numerical simulations are employed to verify the feasibility and effectiveness of the compound-combination synchronization scheme. The result shows that this scheme could be used to vary the junction signal to any desired level and also give a better insight into synchronization in biological systems wherein different organs of different dynamical structures and orders are involved. The scheme could also provide high security in information transmission due to the complexity of its dynamical formulation.

Keywodrs: control and applications of chaos, low- and high-dimensional chaos, numerical simulations of chaotic models, synchronization, coupled oscillators.

1. Introduction

Josephson in 1962 predicted that a Cooper pair of electron can tunnel through the junction of two superconductors separated by a thin layer of nonsuperconducting material in the absence of voltage difference, a phenomenon referred to as Josephson junction effect [1]. The tunneling of the Cooper pairs of electrons of opposite spin and momenta results in quantum-mechanical current, called the superconducting current. Several devices have been developed based on the fundamental idea of Josephson junction effect and as a result Josephson junction has become a subject of intense study of

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considerable physical interest. Josephson junction plays very important role in physics of superconductors and nonlinear physics, and can be used for designing future devices such as emitters, filters, detectors and waveguides working in the sub-terahertz and terahertz frequency ranges, which could be very significant for various applications in different disciplines. Josephson junctions play a major role in the fabrication of low-noise microwave amplifiers, as the only nonlinear non-dissipating element useable at microwave frequencies [2]. Josephson junction is also one of the basic element in the design of superconducting quantum interference devices (SQIUD) [3, 4] which are used for sensing the magnetic fields created by neurological currents.

Josephson junction is a strong nonlinear device that has received considerable attention due to its advantages in devices that require ultra low noise, low power consumption, high frequency. Motivated by the important applications of Josephson junctions, researchers have proposed different models of Josephson junctions as follows: the shunted linear resistive-capacitive Josephson junction (RCSJJ) [5], the shunted nonlinear resistive-capacitive Josephson junction (SNRCJJ) [6], the shunted nonlinear resistive-capacitive-inductive Josephson junction (RCLSJJ) [7] and periodically modulated Josephson junction (PMJJ) [8]. The chaotic nature of Josephson junctions makes them important systems in secure information transmission. Great attention has been given to studies of chaotic dynamical behaviour of Josephson junction in the nonlinear dynamics community due to its extensive applications in many areas like SQUIDs, microwave devices where the high critical-current junctions are preferred [9, 10]. Meanwhile, the present research will utilize the RCLSJJ and the RCSJJ to investigate the proposed synchronization scheme compound-combination synchronization scheme.

Synchronization between coupled chaotic systems [11] is an interesting area of study for understanding the collective behaviour of nonlinear systems [12]. Synchronization of the superconducting junction arrays is important for the purpose of generating reasonably large output power [13]. Also, chaos synchronization in superconducting Josephson junction of parallel array of coupled Josephson junctions linked together by inductors has been used in the fabrication of high sensitive detectors [14, 15, 16]. Synchronization of RCLSJJs could be a suitable superconduting junction that can be used as high frequency transmitter and receiver in chaotic secure communications since it has been found to be appropriate for high frequency applications and there is a good agreement between its experimental and numerical results [7, 17, 18]. Several research papers have reported on the synchronization of Josephson junctions [19, ?, 20, 21, 22, 23] to mention a few. Notable among these research is the paper on generalized control and synchronization of RCL-shunted Josephson junction using backstepping design [19] wherein chaos control, tracking and synchronization were generalized such that the designed control functions for the Josephson junction could be used to tune the output signal of the Josephson junction into desired form and the generalized projective synchronization could be used to amplify the Josephson junction signal.

In the last two decades, there has been a considerable interest in understanding the process of synchronization in chaotic oscillators and their stability criteria due to their real life applications in natural and artificial systems. This interest has led to the discovery of different synchronization types and schemes such as complete synchronization [24], phase synchronization, anti-synchronization [25], projective synchronization [26], time delay synchronization [27], generalized synchronization [28], function projective synchronization [29], increased order synchronization [30], reduced order synchronization [31] and others [23, 32]. Most of the previous discoveries on synchronization focus on synchronization between one drive and one response oscillator only.

Among all these chaos synchronization scheme, hybrid synchronization is very interesting because involves coexistence of synchronization and anti-synchronization in a synchronization scheme that is one part of the system synchronized while the other part of the system anti-synchronized [33]. Then, hybrid projective synchronization involves coexistence of projective synchronization and projective anti-synchronization in a synchronization scheme so that one part of the synchronizes to a positive scaling factor while, other part synchronizes to a negative scaling factor. One of the most significant feature of hybrid projective synchronization is that it can be used to achieve faster and enhanced security in communication and chaotic encryption scheme [34, 33].

Meanwhile, there is increasing interest in the study of chaotic synchronization with different structures and different orders due to its wide existence in biological science and social science [35, 36, 37, 38]. For example, the order of the thalamic neurons can be different from the hippocampal neurons yet they exhibit synchronous behaviour. One more instance is the synchronization that occurs between heart and lungs, where one can observe that circulatory and respiratory systems synchronize with different orders. Hence, the investigation of synchronization of different chaotic systems with different orders is very important from the perspective of practical application and control theory. Synchronization of system with different orders is very interesting and challenging, however, it has received less attention perhaps due to the parameters mismatch and difference in the order of the drive and the response systems. There are only a few results in the literature about the synchronization between chaotic systems whose order are different [35, 38, 39].

In 2011 and 2012, two papers were published on combination synchronization scheme for three chaotic systems [40, 41]. These authors were the first to show the possibility of synchronizing the sum of the state variables of two drive systems with the state variables of a response system. In 2012, Finite-time stochastic combination synchronization of three different chaotic systems and its application in secure communication was presented in [42] where the same authors successfully split the information signal into two and added each of them to each of the drive in the presence of noise and were able to recover the information signal in its original form after synchronization has taken place between the two drive systems and the response system. In 2013 combination-combination synchronization scheme for four chaotic systems based on drive-response configuration which investigates synchronization of the sum of state variables of two drive systems with the sum of the state variables of two slave systems was reported in [43]. The authors stated that the disadvantage of combination synchronization scheme as a result of its one response system has been overcome in combination-combination synchronization scheme.

Furthermore, in 2013 a new synchronization scheme for four chaotic systems called compound synchronization was reported in [44]. The authors carried out compound synchronization of four chaotic memristor oscillator systems, applied it to secure communication and highlighted the advantages of their synchronization scheme in security of information transmission. The compound synchronization scheme is different from combination and combination-combination synchronization schemes since it involves multiplication as well as sum of the master systems state variables and a response system state variables while, the combination and combination-combination synchronization involve only addition of the state variables of the systems. In order to overcome the disadvantage of single response system in compound synchronization we propose **compound-combination** synchronization scheme in this work.

The compound-combination synchronization involves multiplication as well as the sum of the master systems state variables with the sum of the response systems state variables. The major difference between compound-combination synchronization and compound synchronization is that compound-combination scheme involves many response systems while compound synchronization involves only one response system. This difference makes compound-combination synchronization to have a wider application to the real world situations than the compound synchronization scheme. Apart from the fact that compound-combination synchronization enables higher security of information transmission due to complex dynamical formulation of the drive systems, it also enables information signal to be transmitted to the desired receiver or all the receivers either at the same time or different time. So, compound-combination will be highly effective in secure information transmission among network of systems since as many systems as possible can be incorporated in the design. Furthermore, the flexibility of the compound-combination synchronization scheme gives the possibility of designing suitable controllers for achieving a desired synchronization goal or target such as generalized synchronization, generalized anti-synchronization, generalized hybrid synchronization, function projective synchronization and chaos control which has many application in biological systems, chemical systems and physical systems. Moreover, the incorporation of scaling factor in this compound-combination scheme enables the output signal of the Josephson junctions to be tuned to any desired level. Motivated by above discussion, this paper presents compound-combination synchronization among six identical third order resistive-capacitive-inductive-shunted Josepshon junctions (RCLSJJs) via the active backstepping technique.

The rest of this paper is organized as follows. Section 2 gives mathematical background of generalized compound-combination synchronization scheme of five chaotic systems. Section 3 deals with compound-combination synchronization of six third order chaotic JJs (with three as drive and three as response systems). Section 4 investigates reduced order compound-combination synchronization of three third order JJs as drive systems and three second order JJs as the response systems. Section 5 concludes the paper.

2. Compound-combination synchronization scheme

In this section, compound-combination synchronization scheme is designed for five chaotic systems based on the drive-response scheme. In this scheme, we shall consider three drive systems and two response systems. The first drive system is given as

$$\dot{x} \;= f(x) \tag{1}$$

The second drive system is given as

$$\dot{y} \;= f(y) \tag{2}$$

The third drive system is given as

$$\dot{z} \;= f(z) \tag{3}$$

The first response system is given as

$$\dot{w} \;= f(w)+U_1 \tag{4}$$

The second response system is given as

$$\dot{s} \;= f(s)+U_2 \tag{5}$$

where: $x=(x_1,x_2,\ldots x_l)^T$, $y=(y_1,y_2,\ldots y_m)^T$, $z=(z_1,z_2,\ldots z_n)^T$, $w=(w_1,w_2,\ldots w_p)^T$ and $s=(s_1,s_2,\ldots s_q)^T$ are the state variables of systems (1)–(5) respectively; $f(x)\in\Re^l, f(y)\in\Re^m, f(z)\in\Re^n, f(w)\in\Re^p, f(s)\in\Re^q$ are continuous functions of the systems; $U_1=(u_1,u_2,\ldots u_p)^T\in\Re^p$, $U_2=(u_1,u_2,\ldots u_q)^T\in\Re^q$ are controllers to be designed.

Definition 1 *If the order of the drive and the response systems are the same and there exists five scaling matrices* $M_1,M_2,M_3,M_4,M_5\in\Re^l$ *such that* $\lim_{t\to\infty}\|(M_5 s+M_4 w)-M_1 x(M_2 y+M_3 z)\|=0$, *where* $\|.\|$ *represent the matrix norm. Then, the drive systems* (1)-(3) *and the response systems* (4) *and* (5) *achieve compound-combination synchronization.*

Remark 1 The drive system (1) is called the scaling drive system while the drive systems (2) and (3) are called the base drive systems.

Remark 2 M_1, M_2, M_3, M_4 *and* M_5 are constant scaling matrices.

Remark 3 If M_4 or M_5 is zero then, the generalized compound-combination synchronization becomes compound synchronization.

Remark 4 If the scaling matrices $M_1\neq 0, M_2=0$ or $M_3=0$ and either M_4 or M_5 is zero then, the generalized compound-combination synchronization becomes a novel function projective synchronization where the scaling matrix is a chaotic system which is different from the usual function projective synchronization scheme where the scaling matrix is usually a constant or a smooth function of time.

Remark 5 If the scaling matrices $M_1=M_2=M_3=0$ and either M_4 or M_5 *i*s zero then, the compound-combination synchronization reduces to chaos control problem.

Remark 6 From definition 1 we can extend the number of chaotic systems in the drive and response systems to any number *n*.

Definition 2 *The drive systems* (1)-(3) *and the response systems* (4),(5) *are said to achieve generalized increased or reduced order compound-combination synchronization if there exists five constant matrices* $M_1 \in \Re^l$, $M_2 \in \Re^m$, $M_3 \in \Re^n$, $M_4 \in \Re^p$ *and* $M_5 \in \Re^q$ *such that* $\lim_{t\to\infty} \|(M_4 w + M_3 z) - (M_1 x + M_2 y)\| = 0$, *where* $\|.\|$ *represent the matrix norm. Where* $l,m,n < p,q$ *for increased order compound-combination synchronization and* $lm,n > p,q$ *for reduced order compound-combination synchronization case.*

Remark 7 If M_4 or M_5 is zero then, the generalized increased/reduced order compound-combination synchronization becomes incereased/reduced order compound synchronization.

Remark 8 If the scaling matrices $M_1 \neq 0, M_2 = 0$ or $M_3 = 0$ and either M_4 or M_5 is zero then, the generalized increased/reduced order compound-combination synchronization becomes an increased/reduced order novel function projective synchronization where the scaling matrix is a chaotic system which is different from the usual function projective synchronization scheme where the scaling matrix is usually a constant or a smooth function of time.

3. Compound-combination synchronization of six third order Josephson junctions via active backstepping technique

In this section, Josephson junction in (6)–(8) are taken as the drive systems and Josephson junctions in (9)–(11) are taken as the response systems in order to achieve generalized compound-combination synchronization among the six chaotic third order Josephson junctions:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) \\ \dot{x}_3 &= \frac{1}{\beta_L}(x_2 - x_3) \end{aligned} \tag{6}$$

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \frac{1}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) \\ \dot{y}_3 &= \frac{1}{\beta_L}(y_2 - y_3) \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{1}{\beta_C}(i - g(z_2)z_2 - \sin z_1 - z_3) \\ \dot{z}_3 &= \frac{1}{\beta_L}(z_2 - z_3) \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{w}_1 &= w_2 + u_1 \\ \dot{w}_2 &= \frac{1}{\beta_C}(i - g(w_2)w_2 - \sin w_1 - w_3) + u_2 \\ \dot{w}_3 &= \frac{1}{\beta_L}(w_2 - w_3) + u_3 \end{aligned} \tag{9}$$

$$\begin{aligned} \dot{s}_1 &= s_2 + u_4 \\ \dot{s}_2 &= \frac{1}{\beta_C}(i - g(s_2)s_2 - \sin s_1 - s_3) + u_5 \\ \dot{s}_3 &= \frac{1}{\beta_L}(s_2 - s_3) + u_6 \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{v}_1 &= v_2 + u_7 \\ \dot{v}_2 &= \frac{1}{\beta_C}(i - g(v_2)v_2 - \sin v_1 - v_3) + u_8 \\ \dot{v}_3 &= \frac{1}{\beta_L}(v_2 - v_3) + u_9 \end{aligned} \tag{11}$$

where $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ and u_9 are the controllers to be designed. The error systems are defined as follows

$$\begin{aligned} e_1 &= \xi v_1 + \varepsilon s_1 + \delta w_1 - \alpha_1 x_1(\beta_1 y_1 + \gamma_1 z_1) \\ e_2 &= \xi v_2 + \varepsilon s_2 + \delta w_2 - \alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) \\ e_3 &= \xi v_3 + \varepsilon s_3 + \delta w_3 - \alpha_3 x_3(\beta_3 y_3 + \gamma_3 z_3) \end{aligned} \tag{12}$$

From (12), the error dynamics is

$$\begin{aligned} \dot{e}_1 &= \xi \dot{v}_1 + \varepsilon \dot{s}_1 + \delta \dot{w}_1 - \alpha_1 \dot{x}_1(\beta_1 y_1 + \gamma_1 z_1) - \alpha_1 x_1(\beta_1 \dot{y}_1 + \gamma_1 \dot{z}_1) \\ \dot{e}_2 &= \xi \dot{v}_2 + \varepsilon \dot{s}_2 + \delta \dot{w}_2 - \alpha_2 \dot{x}_2(\beta_2 y_2 + \gamma_2 z_2) - \alpha_2 x_2(\beta_2 \dot{y}_2 + \gamma_2 \dot{z}_2) \\ \dot{e}_3 &= \xi \dot{v}_3 + \varepsilon \dot{s}_3 + \delta \dot{w}_3 - \alpha_3 \dot{x}_3(\beta_1 y_3 + \gamma_1 z_3) - \alpha_3 x_3(\beta_3 \dot{y}_3 + \gamma_3 \dot{z}_3) \end{aligned} \tag{13}$$

Substituting (6)–(11) into (13) yields

$$\begin{aligned}\dot{e}_1 &= e_2 + A_1 + U_1\\ \dot{e}_2 &= -\frac{e_3}{\beta_C} + A_2 + U_2\\ \dot{e}_3 &= \frac{1}{\beta_L}(e_2 - e_3) + A_3 + U_3\end{aligned} \tag{14}$$

where

$$\begin{aligned}A_1 &= \alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) - \alpha_1 x_2(\beta_1 y_1 + \gamma_1 z_1) - \alpha_1 x_1(\beta_1 y_2 + \gamma_1 z_2)\\ A_2 &= -\frac{1}{\beta_C}(\alpha_3 x_3(\beta_3 y_3 + \gamma_3 z_3)) + \frac{\xi}{\beta_C}(i - g(v_2)v_2 - \sin v_1)\\ &\quad + \frac{\varepsilon}{\beta_C}(i - g(s_2)s_2 - \sin s_1) + \frac{\delta}{\beta_C}(i - g(w_2)w_2 - \sin w_1)\\ &\quad - \frac{\alpha_2}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)(\beta_2 y_2 + \gamma_2 z_2)\\ &\quad - \alpha_2 x_2(\frac{\beta_2}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) + \frac{\gamma_2}{\beta_C}(i - g(z_2)z_2 - \sin z_1 - z_3))\\ A_3 &= \frac{1}{\beta_L}(\alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) - \alpha_3 x_3(\beta_3 y_3 + \gamma_3 z_3)) - \frac{\alpha_3}{\beta_L}(x_2 - x_3)(\beta_3 y_3 + \gamma_3 z_3)\\ &\quad - \alpha_3 x_3(\frac{\beta_3}{\beta_L}(y_2 - y_3) + \frac{\gamma_3}{\beta_L}(z_2 - z_3))\\ U_1 &= \delta u_1 + \varepsilon u_4 + \xi u_7\\ U_2 &= \delta u_2 + \varepsilon u_5 + \xi u_8\\ U_3 &= \delta u_3 + \varepsilon u_6 + \xi u_9\end{aligned}$$

and then the following theorem is obtained.

Theorem 1 *If the controllers are chosen as*

$$\begin{aligned}U_1 &= \alpha_1 x_2(\beta_1 y_1 + \gamma_1 z_1) + \alpha_1 x_1(\beta_1 y_2 + \gamma_1 z_2) - \alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) - kq_1\\ U_2 &= \frac{1}{\beta_C}\alpha_3 x_3(\beta_3 y_3 + \gamma_3 z_3) - \frac{\xi}{\beta_C}(i - g(v_2)v_2 - \sin v_1)\\ &\quad - \frac{\varepsilon}{\beta_C}(i - g(s_2)s_2 - \sin s_1) - \frac{\delta}{\beta_C}(i - g(w_2)w_2 - \sin w_1)\\ &\quad + \frac{\alpha_2}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)(\beta_2 y_2 + \gamma_2 z_2) - q_1 - kq_2\end{aligned}$$

$$+\alpha_2 x_2(\frac{\beta_2}{\beta_C}(i-g(y_2)y_2-\sin y_1-y_3)+\frac{\gamma_2}{\beta_C}(i-g(z_2)z_2-\sin z_1-z_3))$$

$$U_3 = -\frac{1}{\beta_L}(\alpha_2 x_2(\beta_2 y_2+\gamma_2 z_2)+\alpha_3 x_3(\beta_3 y_3+\gamma_3 z_3))+\frac{\alpha_3}{\beta_L}(x_2-x_3)(\beta_3 y_3+\gamma_3 z_3)$$

$$\frac{1}{\beta_C}q_2-\frac{1}{\beta_L}(q_2-q_3)-kq_3+\alpha_3 x_3(\frac{\beta_3}{\beta_L}(y_2-y_3)+\frac{\gamma_3}{\beta_L}(z_2-z_3)) \tag{15}$$

where $q_1=e_1, q_2=e_2, q_3=e_3$ *and k is the positive feedback gain then, the drive systems* (6)–(8) *and the response systems* (9)–(11) *will achieve compound-combination synchronization.*

Proof The objective of this paper is find control functions via the active backstepping technique that would stabilize the error state dynamics (14) in order for the drive systems (6)–(8) and the response systems (9)–(11) to achieve compound-combination synchronization. The design procedures include the following steps.

Step 1: Let $q_1=e_1$, then we obtain its time derivative as

$$\dot{q}_1 = \dot{e}_1 = e_2+U_1+A_1 \tag{16}$$

Now to stabilize subsystem (16), let $e_2=\alpha_1(q_1)$ be regarded as virtual controller and $V_1=\frac{1}{2}q_1^2$ be a Lyapunov function with time derivative is

$$\dot{V}_1 = q_1\dot{q}_1 = q_1(\alpha_1(q_1)+A_1+U_1) \tag{17}$$

Suppose $\alpha_1(q_1)=0$ and the control function U_1 is chosen as

$$U_1 = -(A_1+kq_1) \tag{18}$$

then, $\dot{V}_1=-kq_1^2<0$ where k is a positive constant. So, $\dot{V}_1$ is negative definite and the subsystem q_1 is asymptotically stable. Since, the virtual controller $\alpha_1(q_1)$ is estimative, the error between e_2 and $\alpha_1(q_1)$ can be denoted by $q_2=e_2-\alpha_1(q_1)$. Thus, we have the following (q_1,q_2)-subsystems

$$\begin{aligned}\dot{q}_1 &= q_2-kq_1\\ \dot{q}_2 &= -\frac{1}{\beta_C}e_3+U_2+A_2\end{aligned} \tag{19}$$

Step 2: In order to stabilize subsystem (19) we regard $e_3=\alpha_2(q_1,q_2)$ as a virtual controller choose a Lyapunov function $V_2=V_1+\frac{1}{2}q_2^2$ and obtain its time derivative as

$$\dot{V}_2 \quad = -kq_1^2 + q_2(q_1 - \frac{1}{\beta_C}\alpha_2(q_1,q_2) + A_2 + U_2) \tag{20}$$

If $\alpha_2(q_1,q_2) = 0$ and the control function U_2 is chosen as

$$U_2 \quad = -A_2 - q_1 - kq_2 \tag{21}$$

then $\dot{V}_2 = -kq_1^2 - kq_2^2 < 0$ where k is a positive constant. Then, $\dot{V}_2$ is negative definite and the subsystem (q_1,q_2) in (19) is asymptotically stable. Thus, we have the following (q_1,q_2,q_3) subsystems

$$\begin{aligned} \dot{q}_1 &= q_2 - kq_1 \\ \dot{q}_2 &= -\frac{1}{\beta_C}q_3 - q_1 - kq_2 \\ \dot{q}_3 &= \frac{1}{\beta_L}(q_2 - q_3) + A_3 + U_3 \end{aligned} \tag{22}$$

Step 3: Finally, we stabilize the subsystem (q_1,q_2,q_3) by choosing an appropriate Lyapunov function $V_3 = V_2 + \frac{1}{2}q_1^2$ and obtain its time derivative as

$$\dot{V}_2 \quad = -kq_1^2 - kq_2^2 + q_3(-\frac{1}{\beta_C}q_2 + \frac{1}{\beta_L}(q_2 - q_3) + A_3 + U_3) \tag{23}$$

If

$$U_3 \quad = \frac{1}{\beta_C}q_2 - \frac{1}{\beta_L}(q_2 - q_3) - A_3 - kq_3 \tag{24}$$

then $\dot{V}_3 = -kq_1^2 - kq_2^2 - kq_3^2 < 0$ where k is a positive constant. Then, $\dot{V}_3$ is negative definite and the subsystem (q_1,q_2,q_3) in (22) is asymptotically stable. This shows that compound-combination synchronization between the drive systems (6)-(8) and the response systems (9)-(11) is achieved. Finally, the full (q_1,q_2,q_3) is

$$\begin{aligned} \dot{q}_1 &= q_2 - kq_1 \\ \dot{q}_2 &= -\frac{1}{\beta_C}q_3 - q_1 - kq_2 \\ \dot{q}_3 &= \frac{1}{\beta_C}q_2 - kq_3 \end{aligned} \tag{25}$$

This complete the prove. □

Several corollaries can be deduced from theorem 1 however, only two corollaries related to our investigation shall be considered. Suppose $u_1 = u_4 = u_7$, $u_2 = u_5 = u_8$ and $u_3 = u_6 = u_9$ in (15) then, we have Corollary 1.

Corollary 1 *If the controllers are chosen as*

$$
\begin{aligned}
u_1 &= (\xi+\varepsilon+\delta)^{-1}(\alpha_1 x_2(\beta_1 y_1+\gamma_1 z_1)+\alpha_1 x_1(\beta_1 y_2+\gamma_1 z_2) \\
&\quad -\alpha_2 x_2(\beta_2 y_2+\gamma_2 z_2)-kq_1) \\
u_2 &= (\xi+\varepsilon+\delta)^{-1}(\frac{1}{\beta_C}\alpha_3 x_3(\beta_3 y_3+\gamma_3 z_3)-\frac{\xi}{\beta_C}(i-g(v_2)v_2-\sin v_1) \\
&\quad -\frac{\varepsilon}{\beta_C}(i-g(s_2)s_2-\sin s_1)-\frac{\delta}{\beta_C}(i-g(w_2)w_2-\sin w_1)-q_1-kq_2 \\
&\quad +\frac{\alpha_2}{\beta_C}(i-g(x_2)x_2-\sin x_1-x_3)(\beta_2 y_2+\gamma_2 z_2)+\alpha_2 x_2(\frac{\beta_2}{\beta_C}(i-g(y_2)y_2 \\
&\quad -\sin y_1-y_3)+\frac{\gamma_2}{\beta_C}(i-g(z_2)z_2-\sin z_1-z_3))) \\
u_3 &= (\xi+\varepsilon+\delta)^{-1}(\frac{1}{\beta_C}q_2-kq_3-\frac{1}{\beta_L}(\alpha_2 x_2(\beta_2 y_2+\gamma_2 z_2)-\alpha_3 x_3(\beta_3 y_3+\gamma_3 z_3)+ \\
&\quad q_2-q_3)+\frac{\alpha_3}{\beta_L}(x_2-x_3)(\beta_3 y_3+\gamma_3 z_3)+\alpha_3 x_3(\frac{\beta_3}{\beta_L}(y_2-y_3)+\frac{\gamma_3}{\beta_L}(z_2-z_3)))
\end{aligned}
\tag{26}
$$

where $e_1 = \xi v_1+\varepsilon s_1+\delta w_1-\alpha_1(\beta_1 y_1+\gamma_1 z_1), e_2 = \xi v_2+\varepsilon s_2+\delta w_2-\alpha_2(\beta_2 y_2+\gamma_2 z_2)$, $e_3 = \xi v_3+\varepsilon s_3+\delta w_3-\alpha_3(\beta_3 y_3-\gamma_3 z_3)$ *and k is the positive feedback gain. Then, the drive systems* (6)–(8) *will achieve compound-combination synchronization with response system* (9)–(11).

Solving the drive system (6)–(8) and the response systems (9)–(11) with the controllers defined in (26) using the following initial conditions $(x_1,x_2,x_3)=(0,0,0),(y_1,y_2,y_3)=(1,1,1,),(z_1,z_2,z_3)=(2,2,2),(w_1,w_2,w_3)=(3,3,3)$, $(s_1,s_2,s_3)=(4,4,4),(v_1,v_2,v_3)=(0.5,0.5,0.5)$ the numerical results are considered under three special cases.

1. Compound-combination projective synchronization: Choosing the scaling parameter values as $\delta=\varepsilon=\xi=\gamma_1=\gamma_2=\gamma_3=\beta_1=\beta_2=\beta_3=1, \alpha_1=\alpha_2=\alpha_3=2$ compound-combination projective synchronization of the drive systems (6)-(8) and response systems (9)–(11) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Fig. 1 .

2. Compound-combination projective anti-synchronization: Choosing the scaling parameter values as $\delta = \varepsilon = \xi = \gamma_1 = \gamma_2 = \gamma_3 = \beta_1 = \beta_2 = \beta_3 = 1, \alpha_1 = \alpha_2 = \alpha_3 = -2$ compound-combination projective anti-synchronization of the drive systems (6)–(8) and response systems (9)–(11) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Fig. 2 .

3. Compound-combination hybrid projective synchronization: Choosing the scaling parameter values as $\delta = \varepsilon = \xi = \gamma_1 = \gamma_2 = \gamma_3 = \beta_1 = \beta_2 = \beta_3 = 1, \alpha_1 = 2, \alpha_2 = -2, \alpha_3 = 2$ compound-combination hybrid projective synchronization of the drive systems (6)–(8) and response systems (9)–(11) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Fig. 3 .

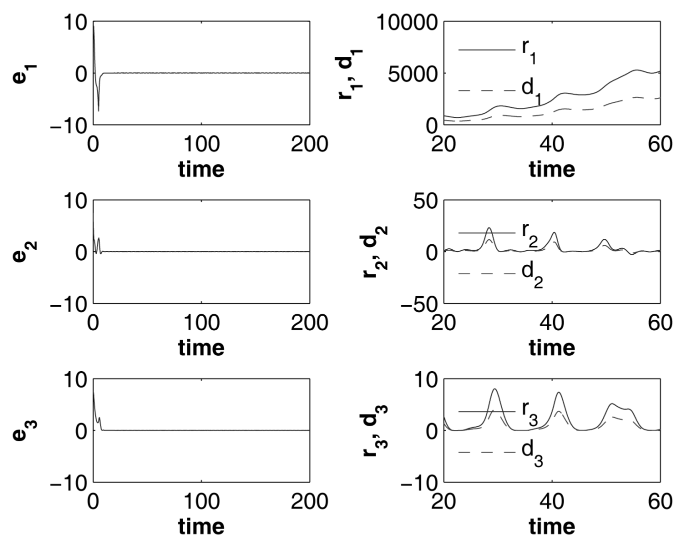

Figure 1: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0 < t < 5$ and activated for $t \geq 5$ where $e_1 = v_1 + s_1 + w_1 - x_1(y_1 + z_1)$, $e_2 = v_2 + s_2 + w_2 - x_2(y_2 + z_2)$, $e_3 = v_3 + s_3 + w_3 - x_3(y_3 + z_3)$, $r_1 = v_1 + s_1 + w_1$, $d_1 = x_1(y_1 + z_1)$, $r_2 = v_2 + s_2 + w_2$, $d_2 = x_2(y_2 + z_2)$ and $r_3 = v_3 + s_3 + w_3$, $d_3 = x_3(y_3 + z_3)$

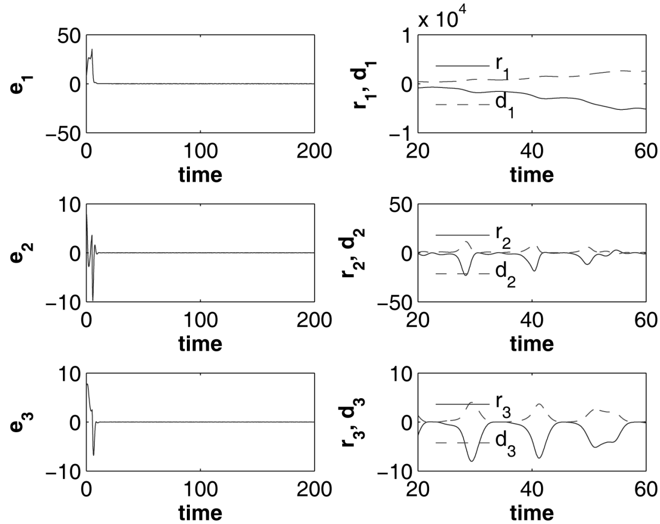

Figure 2: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0<t<5$ and activated for $t \geq 5$ where $e_1 = v_1 + s_1 + w_1 + x_1(y_1 + z_1)$, $e_2 = v_2 + s_2 + w_2 + x_2(y_2 + z_2)$, $e_3 = v_3 + s_3 + w_3 + x_3(y_3 + z_3)$, $r_1 = v_1 + s_1 + w_1$, $d_1 = x_1(y_1 + z_1)$, $r_2 = v_2 + s_2 + w_2$, $d_2 = x_2(y_2 + z_2)$ and $r_3 = v_3 + s_3 + w_3$, $d_3 = x_3(y_3 + z_3)$

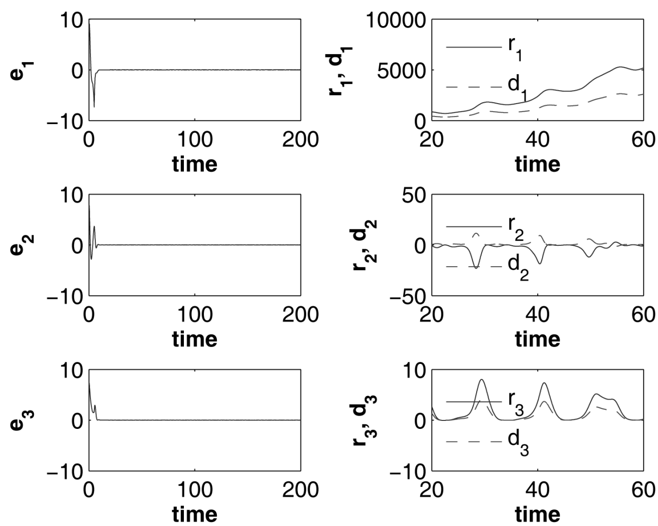

Figure 3: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0<t<5$ and activated for $t \geq 5$ where $e_1 = v_1 + s_1 + w_1 - x_1(y_1 + z_1)$, $e_2 = v_2 + s_2 + w_2 + x_2(y_2 + z_2)$, $e_3 = v_3 + s_3 + w_3 - x_3(y_3 + z_3)$, $r_1 = v_1 + s_1 + w_1$, $d_1 = x_1(y_1 + z_1)$, $r_2 = v_2 + s_2 + w_2$, $d_2 = x_2(y_2 + z_2)$ and $r_3 = v_3 + s_3 + w_3$, $d_3 = x_3(y_3 + z_3)$

Suppose $u_1 = u_4 = u_7$, $u_2 = u_5 = u_8$, $u_3 = u_6 = u_9$, $\xi = \varepsilon = 0$ in (15) then, we have the following corollary.

Corollary 2 *If the controllers are chosen as*

$$\begin{aligned}
u_1 &= (\delta)^{-1}(\alpha_1 x_2(\beta_1 y_1 + \gamma_1 z_1) + \alpha_1 x_1(\beta_1 y_2 + \gamma_1 z_2) \\
&\quad - \alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) - kq_1) \\
u_2 &= (\delta)^{-1}(\frac{1}{\beta_C}\alpha_3 x_3(\beta_3 y_3 + \gamma_3 z_3) - \frac{\delta}{\beta_C}(i - g(w_2)w_2 - \sin w_1) - q_1 - kq_2 \\
&\quad + \frac{\alpha_2}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)(\beta_2 y_2 + \gamma_2 z_2) + \alpha_2 x_2(\frac{\beta_2}{\beta_C}(i - g(y_2)y_2 \\
&\quad - \sin y_1 - y_3) + \frac{\gamma_2}{\beta_C}(i - g(z_2)z_2 - \sin z_1 - z_3))) \\
u_3 &= (\delta)^{-1}(\frac{1}{\beta_C}q_2 - kq_3 - \frac{1}{\beta_L}(\alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) - \alpha_3 x_3(\beta_3 y_3 + \gamma_3 z_3) + \\
&\quad q_2 - q_3) + \frac{\alpha_3}{\beta_L}(x_2 - x_3)(\beta_3 y_3 + \gamma_3 z_3) + \alpha_3 x_3(\frac{\beta_3}{\beta_L}(y_2 - y_3) + \frac{\gamma_3}{\beta_L}(z_2 - z_3)))
\end{aligned} \tag{27}$$

where $e_1 = \delta w_1 - \alpha_1(\beta_1 y_1 + \gamma_1 z_1), e_2 = \delta w_2 - \alpha_2(\beta_2 y_2 + \gamma_2 z_2), e_3 = \delta w_3 - \alpha_3(\beta_3 y_3 - \gamma_3 z_3)$ *and* k *is the positive feedback gain. Then, the drive systems* (6)-(8) *will achieve compound synchronization with response system* (9).

Solving the drive system (6)–(8) and the response system (9) with the controllers defined in (27) using the following initial conditions using the initial conditions of the drive systems and response systems as
$(x_1, x_2, x_3) = (0,0,0), (y_1, y_2, y_3) = (1,1,1,), (z_1, z_2, z_3) = (2,2,2), (w_1, w_2, w_3) = (3,3,3)$,
the numerical results are considered under three special cases.

1. Compound projective synchronization: Choosing the scaling parameter values as $\delta = \gamma_1 = \gamma_2 = \gamma_3 = \beta_1 = \beta_3 = \beta_3 = 1, \alpha_1 = \alpha_2 = \alpha_3 = 2$ compound projective synchronization of the drive systems (6)–(8) and response system (9) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Fig. 4 .

2. Compound projective anti-synchronization: Choosing the scaling parameter values as $\delta = \gamma_1 = \gamma_2 = \gamma_3 = \beta_1 = \beta_3 = \beta_3 = 1, \alpha_1 = \alpha_2 = \alpha_3 = -2$ compound projective anti-synchronization of the drive systems (6)–(8) and response system (9) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the re-

sponse Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Fig. 5 .

3. Compound hybrid projective synchronization: Choosing the scaling parameter values as $\delta = \gamma_1 = \gamma_2 = \gamma_3 = \beta_1 = \beta_3 = \beta_3 = 1, \alpha_1 = 2, \alpha_2 = -2, \alpha_3 = 2$ compound hybrid projective synchronization of the drive systems (6)-(8) and response system (9) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Fig. 4.

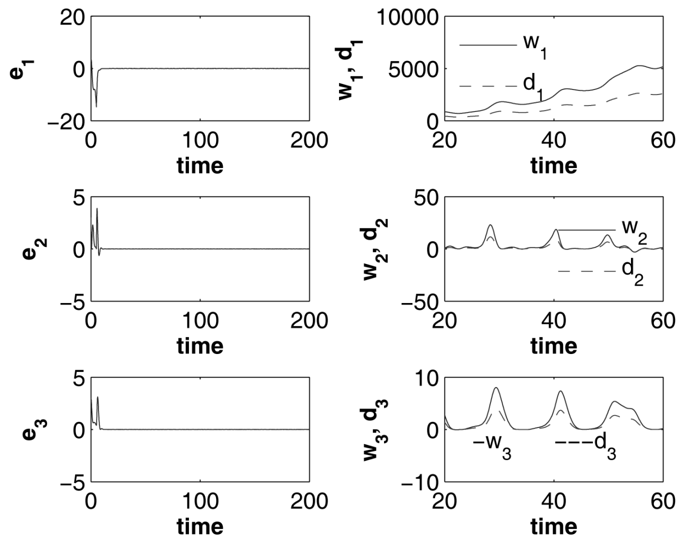

Figure 4: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0 < t < 5$ and activated for $t \geq 5$ where $e_1 = w_1 - x_1(y_1 + z_1)$, $e_2 = w_2 - x_2(y_2 + z_2)$, $e_3 = w_3 - x_3(y_3 + z_3)$, $r_1 = w_1$, $d_1 = x_1(y_1 + z_1)$, $r_2 = w_2$, $d_2 = x_2(y_2 + z_2)$ and $r_3 = w_3$, $d_3 = x_3(y_3 + z_3)$

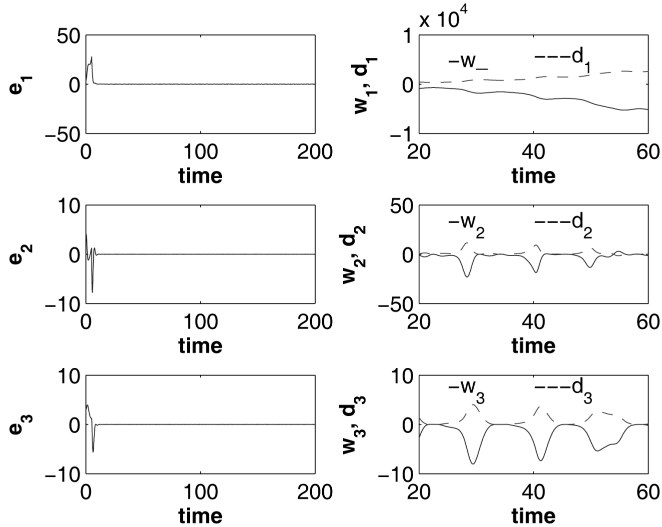

Figure 5: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0 < t < 5$ and activated for $t \geq 5$ where $e_1 = w_1 + x_1(y_1 + z_1)$, $e_2 = w_2 + x_2(y_2 + z_2)$, $e_3 = w_3 + x_3(y_3 + z_3)$, $r_1 = w_1$, $d_1 = x_1(y_1 + z_1)$, $r_2 = w_2$, $d_2 = x_2(y_2 + z_2)$ and $r_3 = w_3$, $d_3 = x_3(y_3 + z_3)$

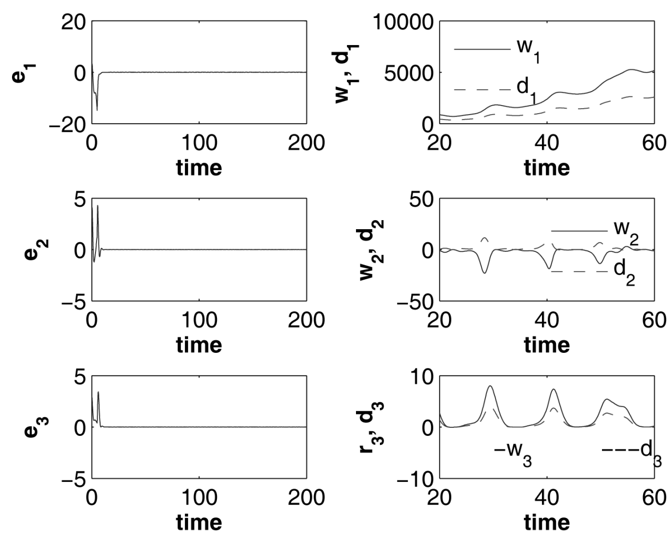

Figure 6: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0 < t < 5$ and activated for $t \geq 5$ where $e_1 = w_1 - x_1(y_1 + z_1)$, $e_2 = w_2 + x_2(y_2 + z_2)$, $e_3 = w_3 - x_3(y_3 + z_3)$, $r_1 = w_1$, $d_1 = x_1(y_1 + z_1)$, $r_2 = w_2$, $d_2 = x_2(y_2 + z_2)$ and $r_3 = w_3$, $d_3 = x_3(y_3 + z_3)$

4. Reduced order compound-combination synchronization of three third and three second order chaotic Josephson junctions

In this section, three third order Josephson junctions in (6)–(8) in section 3 are taken as the drive systems and three second order Josephson junctions (28)–(30) below are taken as the response systems in order to achieve generalized reduced order compound-combination synchronization among three third order and three second order chaotic Josephson junctions:

$$\begin{aligned} \dot{w}_1 &= w_2 + u_1 \\ \dot{w}_2 &= -\alpha w_2 - \sin w_1 + a + b\sin\omega t + u_2 \end{aligned} \tag{28}$$

$$\begin{aligned} \dot{s}_1 &= s_2 + u_3 \\ \dot{s}_2 &= -\alpha s_2 - \sin s_1 + a + b\sin\omega t + u_4 \end{aligned} \tag{29}$$

$$\begin{aligned} \dot{v}_1 &= v_2 + u_5 \\ \dot{v}_2 &= -\alpha v_2 - \sin v_1 + a + b\sin\omega t + u_6 \end{aligned} \tag{30}$$

where u_1, u_2, u_3, u_4, u_5 and u_6 are the controllers to be designed. The error variables are defined as follows

$$\begin{aligned} e_1 &= \xi v_1 + \varepsilon s_1 + \delta w_1 - \alpha_1 x_1(\alpha_3 x_3 + \beta_1 y_1 + \beta_3 y_3 + \gamma_1 z_1 + \gamma_3 z_3) \\ e_2 &= \xi v_2 + \varepsilon s_2 + \delta w_2 - \alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) \end{aligned} \tag{31}$$

From error variables in (31), the error dynamical systems can be obtained as follows

$$\begin{aligned} \dot{e}_1 &= \xi \dot{v}_1 + \varepsilon \dot{s}_1 + \delta \dot{w}_1 - \alpha_1 \dot{x}_1(\alpha_3 x_3 + \beta_1 y_1 + \beta_3 y_3 + \gamma_1 z_1 + \gamma_3 z_3) \\ &\quad - \alpha_1 x_1(\alpha_3 \dot{x}_3 + \beta_1 \dot{y}_1 + \beta_3 \dot{y}_3 + \gamma_1 \dot{z}_1 + \gamma_3 \dot{z}_3) \\ \dot{e}_2 &= \xi \dot{v}_2 + \varepsilon \dot{s}_2 + \delta \dot{w}_2 - \alpha_2 \dot{x}_2(\beta_2 y_2 + \gamma_2 z_2) - \alpha_2 x_2(\beta_2 \dot{y}_2 + \gamma_2 \dot{z}_2) \end{aligned} \tag{32}$$

Substituting (6)–(8) and (28)–(30) into (32) yields the error dynamics

$$\begin{aligned} \dot{e}_1 &= e_2 + B_1 + U_1 \\ \dot{e}_2 &= -\alpha e_2 + B_2 + U_2 \end{aligned} \tag{33}$$

where

$$\begin{aligned} B_1 &= \alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) - \alpha_1 x_2(\alpha_3 x_3 + \beta_1 y_1 + \beta_3 y_3 + \gamma_1 z_1 + \gamma_3 z_3) \\ &\quad - \alpha_1 x_1\Big(\frac{\alpha_3}{\beta_L}(x_2 - x_3) + \frac{\beta_3}{\beta_L}(y_2 - y_3) + \frac{\gamma_3}{\beta_L}(z_2 - z_3) + \beta_1 y_2 + \gamma_1 z_2\Big) \\ &\quad + \xi u_5 + \varepsilon u_3 + \delta u_1 \end{aligned}$$

$$
\begin{aligned}
B_2 &= -a(\alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2)) + \xi(-\sin v_1 + a + b\sin\omega t) \\
&\quad + \varepsilon(-\sin s_1 + a + b\sin\omega t) + \delta(-\sin w_1 + a + b\sin\omega t) + \xi u_6 + \varepsilon u_4 + \delta u_2 \\
&\quad - \frac{\alpha_2}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)(\beta_2 y_2 + \gamma_2 z_2) \\
&\quad - \alpha_2 x_2\left(\frac{\beta_2}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) + \frac{\gamma_2}{\beta_C}(i - g(z_2)z_2 - \sin z_1 - z_3)\right) \\
U_1 &= \xi u_5 + \varepsilon u_3 + \delta u_1 \\
U_2 &= \xi u_6 + \varepsilon u_4 + \delta u_2
\end{aligned}
$$

Then, the following theorem is obtained.

Theorem 2 *If the controllers are chosen as*

$$
\begin{aligned}
U_1 &= \alpha_1 x_2(\alpha_3 x_3 + \beta_1 y_1 + \beta_3 y_3 + \gamma_1 z_1 + \gamma_3 z_3) - \alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) - kq_1 \\
&\quad + \alpha_1 x_1\left(\frac{\alpha_3}{\beta_L}(x_2 - x_3) + \frac{\beta_3}{\beta_L}(y_2 - y_3) + \frac{\gamma_3}{\beta_L}(z_2 - z_3) + \beta_1 y_2 + \gamma_1 z_2\right) \\
U_2 &= \alpha\alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) - \xi(-\sin v_1 + a + b\sin\omega t) + (\alpha - k)q_2 \\
&\quad - \varepsilon(-\sin s_1 + a + b\sin\omega t) - \delta(-\sin w_1 + a + b\sin\omega t) \\
&\quad + \frac{\alpha_2}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)(\beta_2 y_2 + \gamma_2 z_2) - q_1 \\
&\quad + \alpha_2 x_2\left(\frac{\beta_2}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) + \frac{\gamma_2}{\beta_C}(i - g(z_2)z_2 - \sin z_1 - z_3)\right)
\end{aligned}
\tag{34}
$$

where $q_1 = e_1, q_2 = e_2, q_3 = e_3$ *and* k *is the positive feedback gain then, the drive systems* (6)–(8) *and the response systems* (28)–(30) *will achieve reduced order compound-combination synchronization.*

Proof The objective of this section is to find control functions via the active backstepping technique that would stabilize the error state dynamics (33) in order for the drive systems (6)–(8) and the response systems (28)–(30) to achieve generalized combination-combination synchronization. The design procedure includes the following steps.

Step 1

Let $q_1 = e_1$, its time derivative is

$$
\dot{q}_1 = \dot{e}_1 = \frac{\delta_1}{\delta_2} e_2 + U_1 + B_1 \tag{35}
$$

Where $e_2 = \alpha_1(q_1)$ can be regarded as virtual controller. In order to stabilize q_1-subsystem, we choose the following Lyapunov function $V_1 = \frac{1}{2} q_1^2$. The time derivative of v_1 is

$$\dot{V}_1 = q_1\dot{q}_1 = q_1\left(\frac{\delta_1}{\delta_2}\alpha_1(q_1) + B_1 + U_1\right) \tag{36}$$

Suppose $\alpha_1(q_1) = 0$ and the control function U_1 is chosen as

$$U_1 = -(B_1 + kq_1) \tag{37}$$

then $\dot{V}_1 = -kq_1^2 < 0$ where k is positive constant which represents the feedback gain. Hence, $\dot{v}_1$ is negative definite and the subsystem q_1 is asymptotically stable. Since, the virtual controller $\alpha_1(q_1)$ is estimative, the error between e_2 and $\alpha_1(q_1)$ can be denoted by $q_2 = e_2 - \alpha_1(q_1)$. Thus, the following (q_1, q_2)-subsystems

$$\begin{aligned} \dot{q}_1 &= \frac{\delta_1}{\delta_2}q_2 - kq_1 \\ \dot{q}_2 &= -\alpha q_2 + U_2 + B_2 \end{aligned} \tag{38}$$

Step 2

In order to stabilize subsystem (38), the following Lyapunov function can be chosen as $V_2 = V_1 + \frac{1}{2}q_2^2$. Its time derivative is

$$\dot{V}_2 = -kq_1^2 + q_2\left(\frac{\delta_1}{\delta_2}q_1 - \alpha q_2 + U_2 + B_2\right) \tag{39}$$

If the control function U_2 is chosen as

$$U_2 = -B_2 - kq_2 + \alpha q_2 - \frac{\delta_1}{\delta_2}q_1 \tag{40}$$

then $\dot{V}_2 = -kq_1^2 - kq_2^2 < 0$ where k is a positive constant. Hence, $\dot{V}_2$ is negative definite and the subsystem (q_1, q_2) in (38) is asymptotically stable. This implies that generalized compound-combination synchronization of the drive systems (6)–(8) with the response system (28)–(30) is achieved. Finally, the subsystem (38) becomes

$$\begin{aligned} \dot{q}_1 &= \frac{\delta_1}{\delta_2}q_2 - kq_1 \\ \dot{q}_2 &= -\frac{\delta_1}{\delta_2}q_1 - kq_2 \end{aligned} \tag{41}$$

□

This completes the prove. Several Corollaries can be deduced from theorem 9 however, only two Corollaries related to our investigation shall be considered.

Suppose $u_1 = u_3 = u_5$, $u_2 = u_4 = u_6$ in (34) then, we have Corollary 3.

Corollary 3 *If the controllers are chosen as*

$$
\begin{aligned}
u_1 &= \frac{1}{(\delta+\varepsilon+\xi)}(\alpha_1 x_2(\alpha_3 x_3+\beta_1 y_1+\beta_3 y_3+\gamma_1 z_1+\gamma_3 z_3)-\alpha_2 x_2(\beta_2 y_2+\gamma_2 z_2)-kq_1 \\
&\quad +\alpha_1 x_1(\frac{\alpha_3}{\beta_L}(x_2-x_3)+\frac{\beta_3}{\beta_L}(y_2-y_3)+\frac{\gamma_3}{\beta_L}(z_2-z_3)+\beta_1 y_2+\gamma_1 z_2)) \\
u_2 &= \frac{1}{(\delta+\varepsilon+\xi)}(\alpha\alpha_2 x_2(\beta_2 y_2+\gamma_2 z_2)-\delta(-\sin w_1+a+b\sin\omega t)+(\beta_2 y_2+\gamma_2 z_2) \\
&\quad \frac{\alpha_2}{\beta_C}(i-g(x_2)x_2-\sin x_1-x_3)+\alpha_2 x_2(\frac{\beta_2}{\beta_C}(i-g(y_2)y_2-\sin y_1-y_3) \\
&\quad +\frac{\gamma_2}{\beta_C}(i-g(z_2)z_2-\sin z_1-z_3))-\varepsilon(-\sin s_1+a+b\sin\omega t)-kq_2-q_1 \\
&\quad -\xi(-\sin v_1+a+b\sin\omega t))
\end{aligned}
\tag{42}
$$

where $e_1=\xi v_1+\varepsilon s_1+\delta w_1-\alpha_1(\beta_1 y_1+\gamma_1 z_1), e_2=\xi v_2+\varepsilon s_2+\delta w_2-\alpha_2(\beta_2 y_2+\gamma_2 z_2)$ *and* k *is the positive feedback gain. Then, the drive systems* (6)–(9) *will achieve reduced order compound-combination synchronization with response system* (28)–(30).

Solving the drive system (6)-(8) and the response systems (28)–(30) with the controllers defined in (42) using the initial conditions of the drive systems and response systems as $(x_1,x_2,x_3)=(0,0,0),(y_1,y_2,y_3)=(1,1,1),(z_1,z_2,z_3)=(2,2,2),(w_1,w_2)=(3,3),$ $(s_1,s_2)=(-1,-2),(v_1,v_2)=(0.4,0.1)$, the numerical results are considered under three special cases.

1. Reduced order compound-combination projective synchronization: Choosing the scaling parameter values as $\delta=\xi=\varepsilon=\gamma_1=\beta_1=\beta_2=1.0,\alpha_1=2,\alpha_2=2.0$ reduced order compound-combination projective synchronization of the drive systems (6)–(8) and response systems (28)–(30) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Figure 7 .

2. Reduced order compound-combination projective anti-synchronization: Choosing the scaling parameter values as $\delta=\xi=\varepsilon=\gamma_1=\beta_1=\beta_2=1.0,\alpha_1=-2,\alpha_2=-2.0$ reduced order compound-combination projective anti-synchronization of the drive systems (6)–(8) and response systems (28)–(30) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Figure 8.

3. Reduced order compound-combination hybrid projective synchronization: Choosing the scaling parameter values as $\delta = \xi = \varepsilon = \gamma_1 = \beta_1 = \beta_2 = 1.0, \alpha_1 = -2, \alpha_2 = 2.0$ reduced order compound-combination hybrid projective synchronization of the drive systems (4)–(8) and response systems (28)–(30) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Fig. 9.

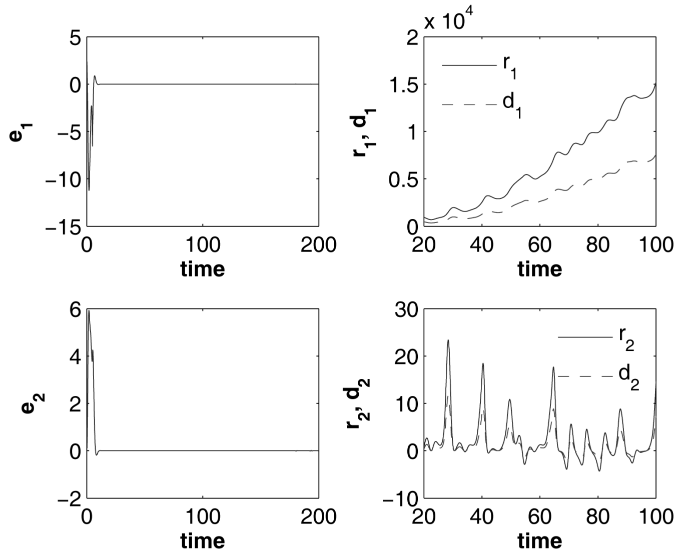

Figure 7: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0 < t < 5$ and activated for $t \geq 5$ where $e_1 = v_1 + s_1 + w_1 - 2x_1(x_3 + y_1 + y_3 + z_1 + z_3)$, $e_2 = v_2 + s_2 + w_2 - 2x_2(y_2 + z_2)$, $r_1 = v_1 + s_1 + w_1$, $d_1 = x_1(y_1 + z_1 + x_3 + y_3 + z_3)$, $r_2 = v_2 + s_2 + w_2$ and $d_2 = x_2(y_2 + z_2)$

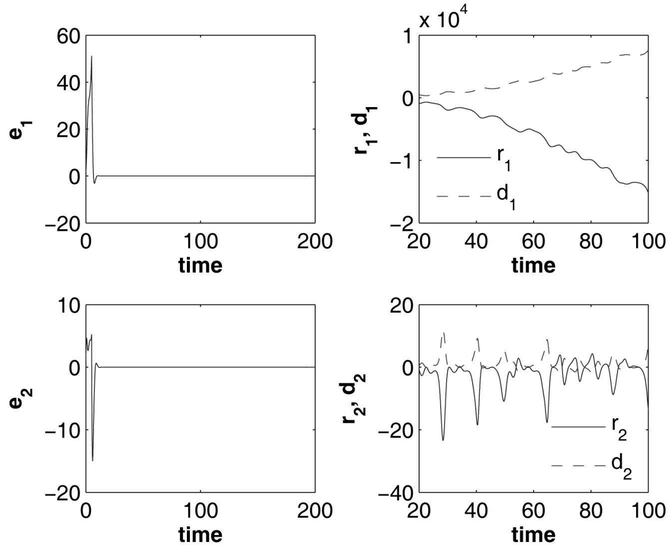

Figure 8: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0<t<5$ and activated for $t \geq 5$ where $e_1 = v_1 + s_1 + w_1 + 2x_1(x_3 + y_1 + y_3 + z_1 + z_3)$, $e_2 = v_2 + s_2 + w_2 + 2x_2(y_2 + z_2)$, $r_1 = v_1 + s_1 + w_1$, $d_1 = x_1(y_1 + z_1 + x_3 + y_3 + z_3)$, $r_2 = v_2 + s_2 + w_2$ and $d_2 = x_2(y_2 + z_2)$

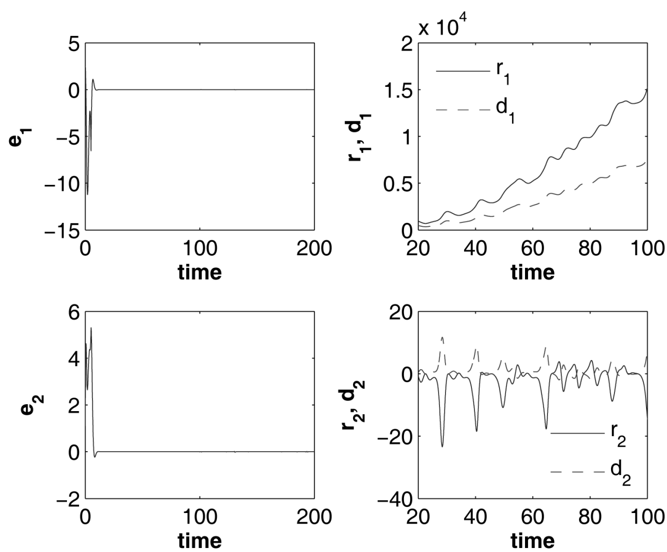

Figure 9: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0<t<5$ and activated for $t \geq 5$ where $e_1 = v_1 + s_1 + w_1 - 2x_1(x_3 + y_1 + y_3 + z_1 + z_3)$, $e_2 = v_2 + s_2 + w_2 + 2x_2(y_2 + z_2)$, $r_1 = v_1 + s_1 + w_1$, $d_1 = x_1(y_1 + z_1 + x_3 + y_3 + z_3)$, $r_2 = v_2 + s_2 + w_2$ and $d_2 = x_2(y_2 + z_2)$

Suppose $u_1 = u_3 = u_5$, $u_2 = u_4 = u_6$, $\xi = \varepsilon = 0$ in (34) then, we have Corollary 18.

Corollary 4 *If the controllers are chosen as*

$$
\begin{aligned}
u_1 &= \frac{1}{(\delta)}(-\alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) + \alpha_1 x_2(\beta_1 y_1 + \gamma_1 z_1) + \alpha_1 x_1(\beta_1 y_2 + \gamma_1 z_2) - kq_1) \\
u_2 &= \frac{1}{(\delta)}(\alpha\alpha_2 x_2(\beta_2 y_2 + \gamma_2 z_2) - \delta(-\sin w_1 + a + b\sin\omega t) \\
&\quad + \alpha_2(-\alpha x_2 - \sin x_1 + a + b\sin\omega t)(\beta_2 y_2 + \gamma_2 z_2) + (\alpha - k)q_2 - q_1 \\
&\quad + \alpha_2 x_2(\beta_2(-\alpha y_2 - \sin y_1 + a + b\sin\omega t) + \gamma_2(-\alpha z_2 - \sin z_1 + a + b\sin\omega t)))
\end{aligned}
\tag{43}
$$

where $e_1 = \delta w_1 - \alpha_1(\beta_1 y_1 + \gamma_1 z_1), e_2 = \delta w_2 - \alpha_2(\beta_2 y_2 + \gamma_2 z_2), e_3 = \delta w_3 - \alpha_3(\beta_3 y_3 - \gamma_3 z_3)$ *and* k *is the positive feedback gain. Then, the drive systems* 6)-(8) *will achieve compound synchronization with response system* (28).

Solving the drive system (6)–(8) and the response systems (28) with the controllers defined in (43) using the initial conditions of the drive systems and response systems as $(x_1, x_2, x_3) = (0,0,0), (y_1, y_2, y_3) = (1,1,1), (z_1, z_2, z_3) = (0,0,0), (w_1, w_2) = (0.4, 0.1)$, the numerical results are considered under three special cases.

1. Reduced order compound synchronization: Choosing the scaling parameter values as $\gamma_1 = \beta_1 = \beta_2 = 1.0, \alpha_1 = \alpha_2 = 1$ reduced order compound projective synchronization of the drive systems (6)–(8) and response system (28) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Fig. 10 .

2. Reduced order compound anti-synchronization: Chosen the scaling parameter values as $\gamma_1 = \beta_1 = \beta_2 = 1.0, \alpha_1 = \alpha_2 = -1$ reduced order compound projective anti-synchronization of the drive systems (6)–(8) and response system (28) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Fig. 11 .

3. Reduced order compound hybrid synchronization: Chosen the scaling parameter values as $\gamma_1 = \beta_1 = \beta_2 = 1.0, \alpha_1 = 1, \alpha_2 = -1$ reduced order compound hybrid projective synchronization of the drive systems (6)–(8) and response system (28) is achieved as indicated by the convergence of the error state variables to zero and the projection of the state variables of the drive Josephson junctions on the response Josephson junctions when the controllers are activated for $t \geq 5$ as shown in Fig. 12.

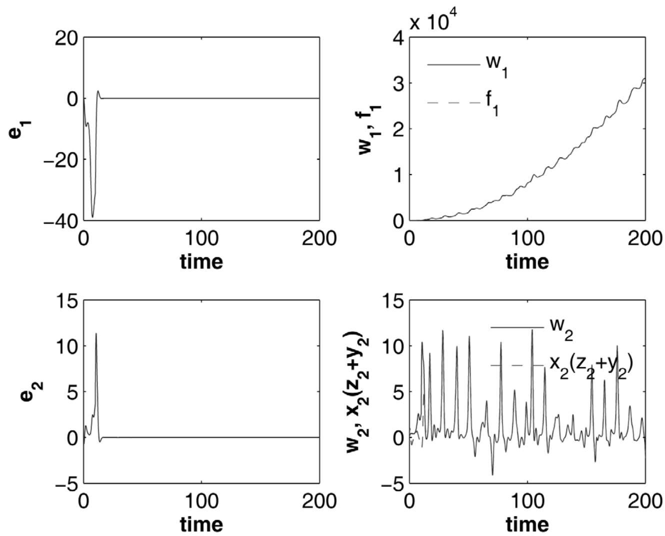

Figure 10: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0 < t < 5$ and activated for $t \geq 5$ where $e_1 = w_1 - x_1(x_3 + y_1 + y_3 + z_1 + z_3)$, $e_2 = w_2 - x_2(y_2 + z_2)$, $r_1 = w_1$, $d_1 = x_1(y_1 + z_1 + x_3 + y_3 + z_3)$, $r_2 = w_2$ and $d_2 = x_2(y_2 + z_2)$

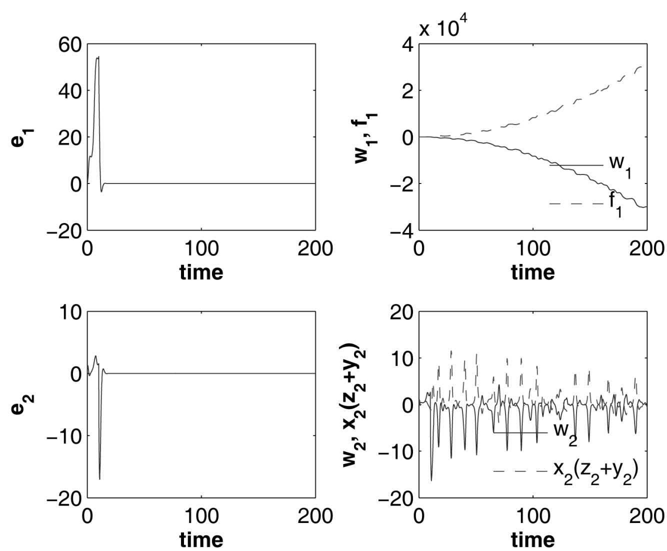

Figure 11: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0 < t < 5$ and activated for $t \geq 5$ where $e_1 = w_1 + x_1(x_3 + y_1 + y_3 + z_1 + z_3)$, $e_2 = w_2 + x_2(y_2 + z_2)$, $r_1 = w_1$, $d_1 = x_1(y_1 + z_1 + x_3 + y_3 + z_3)$, $r_2 = w_2$ and $d_2 = x_2(y_2 + z_2)$

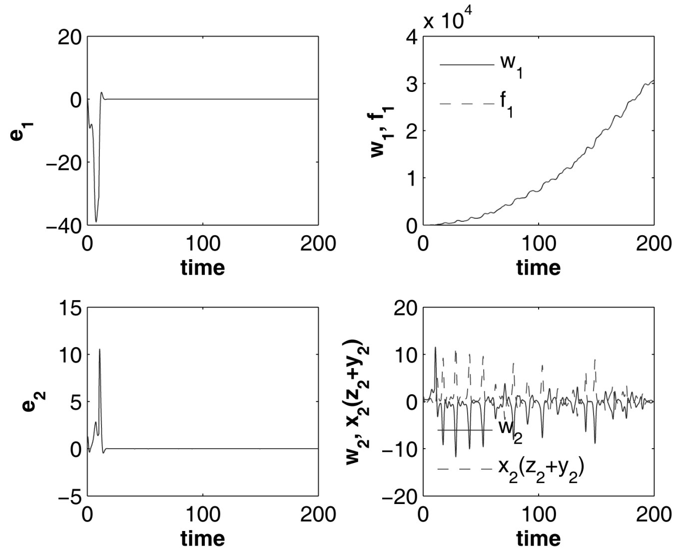

Figure 12: Error dynamics among the drive and the response systems (column one) and the corresponding dynamics (time series) of the state variable of the drive (dashed line) and the response (solid line) variables (column two) with controllers deactivated for $0<t<5$ and activated for $t\geq 5$ where $e_1=w_1-x_1(x_3+y_1+y_3+z_1+z_3)$, $e_2=w_2+x_2(y_2+z_2)$, $r_1=w_1$, $d_1=x_1(y_1+z_1+x_3+y_3+z_3)$, $r_2=w_2$ and $d_2=x_2(y_2+z_2)$

5. Conclusion

A new synchronization scheme called compound-combination synchronization has been proposed and investigated using six chaotic Josephson junctions evolving from different initial conditions based on the drive-response configuration (with three as drive and three as response systems) via the active backstepping technique. The technique has been used to achieve identical and reduced order compound-combination synchronization of RCLSJ and RCSJ. The scheme will no doubt improve security of information transmission due the complex dynamical structure of the drive systems and also enable secure transmission of information to any of the response systems or all the response systems at a desired time. The result shows that this scheme could be used to vary the junction signal to any desired level and also give a better insight into synchronization in biological systems wherein different organs of different dynamical structures and orders are involved.

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