Inverse systems of linear systems

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Abstract: The concept of inverse systems for standard and positive linear systems is introduced. Necessary and sufficient conditions for the existence of the positive inverse system for continuous-time and discrete-time linear systems are established. It is shown that: 1) The inverse system of continuous-time linear system is asymptotically stable if and only if the standard system is asymptotically stable. 2) The inverse system of discrete-time linear system is asymptotically stable if and only if the standard system is unstable. 3) The inverse system of continuous-time and discrete-time linear systems are reachable if and only if the standard systems are reachable. The considerations are illustrated by numerical examples.

Key words: inverse, linear, positive system, reachability, stability

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [2, 4].

The linear systems and control systems have been the classical field of research and they have been considered in many books [1, 5-8].

In this paper a new concept of inverse systems for standard and positive linear systems will be proposed. The positivity, stability and reachability of the linear inverse systems will be investigated.

The paper is organized as follows. In Section 2 two lemmas concerning the function of matrices are recalled. The concept of inverse systems for standard and positive continuous-time and discrete-time linear systems are introduced in Section 3. Necessary and sufficient

conditions for the existence of positive inverse systems are established. The asymptotic stability of inverse linear systems is addressed in Section 4. Necessary and sufficient conditions for the asymptotic stability of the inverse continuous-time and discrete-time linear systems are given. In Section 5 it is shown that the inverse systems are reachable if and only if the standard systems are reachable. Concluding remarks are given in Section 6.

The following notation will be used: $\Re$ – the set of real numbers, Z_+ – the set of non-negative integers, $\Re^{n \times m}$ – the set of $n \times m$ real matrices, $\Re_+^{n \times m}$ – the set of $n \times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n \times 1}$, the $n \times n$ identity matrix will be denoted by I_n.

2. Preliminaries

The following well-known [3, 6] lemmas will be used in proofs of the main results of the paper.

Lemma 2.1. If $\lambda_k = \alpha_k + i\beta_k, k = 1, \ldots, n$ are the eigenvalues (not necessary distinct) of the matrix $A \in \Re^{n \times n}$ then $\overline{\lambda}_k = \lambda_k^{-1}, k = 1, \ldots, n$ are the eigenvalues of the inverse matrix $\overline{A} = A^{-1}$.

Lemma 2.2. Let the scalar function $f(\lambda)$ be well-defined on the spectrum of the matrix $A \in \Re^{n \times n}$ i.e. the values

$$f^{(i)}(\lambda_k) = \frac{d^i f(\lambda)}{d\lambda^i}\bigg|_{\lambda = \lambda_k}, \quad i = 0, 1, \ldots, m_k - 1; \quad k = 1, \ldots, r \left(\sum_{k=1}^{r} m_k = n \right) \quad (2.1)$$

are finite. Then the function $f(A)$ of the matrix A is given by

$$f(A) = \sum_{k=1}^{r} [f(\lambda_k)Z_{k1} + f^{(1)}(\lambda_k)Z_{k2} + \ldots + f^{(m_k-1)}(\lambda_k)Z_{km_k}], \quad (2.2)$$

where

$$Z_{kj} = \sum_{i=j-1}^{m_k-1} \frac{\Psi_k(A)}{(i-j+1)!(j-1)!} \frac{d^{(i-j+1)}}{d\lambda^{(i-j+1)}} \left[\frac{1}{\Psi_k(\lambda)} \right]_{\lambda = \lambda_k} (A - I_n \lambda_k)^i \quad (2.3)$$

and the minimal polynomial of the matrix A has the form

$$\Psi_k(\lambda) = \frac{\Psi(\lambda)}{(\lambda - \lambda_k)^{m_k}}, \quad \Psi(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_r)^{m_r}. \quad (2.4)$$

In particular case when $m_1 = m_2 = \ldots = m_r = 1$ $(r = n)$ then we have

$$\Psi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n) \quad (2.5)$$

and

$$f(A) = \sum_{k=1}^{n} f(\lambda_k) Z_k, \tag{2.6}$$

where

$$Z_k = \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{(A - I_n \lambda_i)}{\lambda_k - \lambda_i}. \tag{2.7}$$

Remark 2.1. [3, 6] The matrices Z_{kj} $k = 1, ..., r$; $j = 1, ..., m_k$ defined by (2.3) are linearly independent and independent of the function $f(\lambda)$.

3. Standard and positive inverse systems

3.1. Continuous-time systems

Consider the standard autonomous continuous-time linear system

$$\dot{x}(t) = Ax(t), \quad t \geq 0, \tag{3.1}$$

where $x(t) \in \mathfrak{R}^n$ is the state vector and $A \in \mathfrak{R}^{n \times n}$. It is assumed that the matrix A is nonsingular, i.e. $\det A \neq 0$.

Definition 3.1. The system

$$\dot{\overline{x}}(t) = \overline{A}\,\overline{x}(t), \quad \overline{A} = A^{-1} \tag{3.2}$$

is called the inverse system of the system (3.1).

The inverse system (3.2) exists if and only if the matrix A of the system (3.1) is nonsingular.

Definition 3.2. The system (3.1) is called positive if $x(t) \in \mathfrak{R}_+^n$, $t \geq 0$ for any initial conditions $x_0 = x(0) \in \mathfrak{R}_+^n$.

Theorem 3.1. [4] The system (3.1) is positive if and only if its matrix A is the Metzler matrix (off-diagonal entries are nonnegative).

Theorem 3.2. The inverse system (3.2) of the positive asymptotically stable system (3.1) is positive only if the matrix A is diagonal.

Proof. If the positive system (3.1) is asymptotically stable then its matrix is a Hurwitz Metzler matrix and it satisfied the condition [4]

$$-A^{-1} \in \mathfrak{R}_+^{n \times n}. \tag{3.3}$$

Therefore, the inverse system (3.2) is positive only if the matrix A is diagonal, otherwise it is not positive.

From Theorem 3.2 we have the following corollary.

Corollary 3.1. The inverse system (3.2) is positive only if the positive system (3.1) is unstable.

For example the positive system (3.1) with the matrix:

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}.$$

is unstable since $\det A = -4 < 0$. In this case the inverse system (3.2) is positive since

$$\overline{A} = A^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.75 & 0.25 \end{bmatrix} \in M_2.$$

Example 3.1. Consider the electrical R, L and R, C circuits shown on Fig. 1 which are positive linear systems.
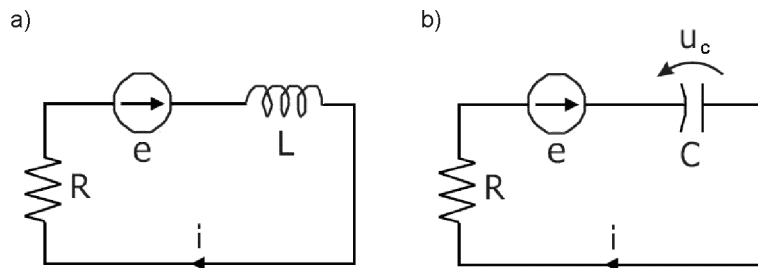

Fig. 1. R, L and R, C circuits

The R, L circuit is described by the equation

$$e = Ri + L\frac{di}{dt} \qquad (3.4)$$

and the R, C circuit by the equation

$$e = RC\frac{du_c}{dt} + u_c. \qquad (3.5)$$

From (3.4) and (3.5) we have

$$\frac{di}{dt} = -\frac{R}{L}i + \frac{1}{L}e, \quad \frac{du_c}{dt} = -\frac{1}{RC}u_c + \frac{1}{RC}e, \qquad (3.6)$$

For $x = i$, $\overline{x} = u_c$, $A = -R/L$, $\overline{A} = -1/RC$ and $C = 1/L$ the R, C circuit is the inverse circuit of the R, L circuit and vice versa.

Theorem 3.2. Let the minimal polynomial of the matrix A has the form (2.4). Then the solutions of the equations (3.1) and (3.2) are given by the formulas

$$x(t) = e^{At}x_0 = \sum_{k=1}^{r}[Z_{k1} + Z_{k2}t + ... + Z_{km_k}t^{m_k-1}]e^{\lambda_k t}x_0 \qquad (3.7)$$

and

$$\bar{x}(t) = e^{\bar{A}t}\bar{x}_0 = \sum_{k=1}^{r}\left[Z_{k1}e^{\frac{t}{\lambda_k}} + Z_{k2}\frac{de^{\frac{t}{\lambda_k}}}{d\lambda_k} + ... + Z_{km_k}\frac{d^{m_k-1}e^{\frac{t}{\lambda_k}}}{d\lambda_k^{m_k-1}}\right]\bar{x}_0 \qquad (3.8)$$

respectively and the matrices Z_{kj} $k = 1, ..., r;$ $j = 1, ..., m_k$ are defined by (2.3).

Proof. By Lemma 2.1 if λ_k, $k = 1,..., n$ are the eigenvalues of the matrix $A \in \Re^{n \times n}$ then $\bar{\lambda}_k = 1/\lambda_k$ $k = 1,...,n$ are the eigenvalues of the matrix $\bar{A} = A^{-1} \in \Re^{n \times n}$. The matrices Z_{kj} $k = 1, ..., r;$ $j = 1, ..., m_k$ are the same for the matrix A and A^{-1} (see Remark 2.1). Substituting $f(\lambda) = e^{\lambda t}$ and $f(\lambda) = e^{t/\lambda}$ in (2.2) respectively we obtain the formula (3.7) and (3.8) respectively.

   In particular case when $m_1 = m_2 = ... = m_r = 1$ $(r = n)$ and the minimal polynomial has the form (2.5) then the formulas (3.7) and (3.8) takes the forms

$$x(t) = e^{At}x_0 = \sum_{k=1}^{r}Z_k e^{\lambda_k t}x_0 \qquad (3.7')$$

and

$$\bar{x}(t) = e^{\bar{A}t}\bar{x}_0 = \sum_{k=1}^{r}Z_k e^{\frac{t}{\lambda_k}}\bar{x}_0, \qquad (3.8')$$

where

$$Z_k = \prod_{\substack{i=1 \\ i \neq k}}^{n}\frac{(A - I_n\lambda_i)}{\lambda_k - \lambda_i}. \qquad (2.3')$$

Example 3.2. Consider the linear system (3.1) with the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}. \qquad (3.9)$$

The characteristic (minimal) polynomial of the matrix (3.9) has the form

$$\Psi(\lambda) = \det[I_n\lambda - A] = \begin{vmatrix} \lambda & -1 \\ 4 & \lambda+4 \end{vmatrix} = (\lambda + 2)^2. \qquad (3.10)$$

Taking into account that in this case $k = 1$, $m_k = 2$, $\lambda_1 = -2$ and using (3.7) we obtain:

$$Z_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Z_{12} = A - I_2\lambda_1 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

and

$$x(t) = e^{At}x_0 = (Z_{11} + Z_{12}t)x_0 = \begin{bmatrix} (1+2t)e^{-2t} & te^{-2t} \\ -4te^{-2t} & (1-2t)e^{-2t} \end{bmatrix} x_0. \tag{3.11}$$

The inverse matrix $\overline{A}$ of (3.9) has the form

$$\overline{A} = A^{-1} = \begin{bmatrix} -1 & -0.25 \\ 1 & 0 \end{bmatrix} \tag{3.12}$$

and its characteristic polynomial is

$$\det[I_n\overline{\lambda} - \overline{A}] = \begin{vmatrix} \overline{\lambda}+1 & 0.25 \\ -1 & \overline{\lambda} \end{vmatrix} = \overline{\lambda}^2 + \overline{\lambda} + 0.25 = (\overline{\lambda} + 0.5)^2. \tag{3.13}$$

Using (3.8) for $k = 1$, $m_1 = 2$ and $\overline{\lambda} = -0.5$ we obtain

$$\overline{x}(t) = e^{\overline{A}t}\overline{x}_0 = (Z_{11} - Z_{12}\overline{\lambda}^2 t)e^{\overline{\lambda}t}\overline{x}_0 = \begin{bmatrix} 1-0.5t & -0.25t \\ t & 1+0.5t \end{bmatrix} e^{-0.5t}\overline{x}_0. \tag{3.14}$$

3.2. Discrete-time systems

Consider the standard discrete-time autonomous linear system

$$x_{i+1} = Ax_i \quad i \in Z_+ = \{0, 1, ...\}, \tag{3.15}$$

where $x_i \in \mathfrak{R}^n$ is the state vector and $A \in \mathfrak{R}^{n \times n}$. It is assumed that the matrix A is nonsingular $\det A \neq 0$.

Definition 3.3. The system

$$\overline{x}_{i+1} = \overline{A}\overline{x}_i, \quad \overline{A} = A^{-1} \tag{3.16}$$

is called the inverse system of the system (3.15).

The inverse system (3.16) exists if and only if the matrix A of the system (3.15) is nonsingular.

Definition 3.4. The system (3.15) is called positive if $x_i \in \mathfrak{R}^n_+$, $i \in Z_+$ for any initial conditions $x_0 \in \mathfrak{R}^n_+$.

Theorem 3.3. [4] The system (3.15) is positive if and only if its matrix A has nonnegative entries, $A \in \mathfrak{R}^{n \times n}_+$.

A matrix $A \in \Re^{n \times n}$ is called monomial if in each row and in each column only one entry is positive and the remaining entries are zero.

Theorem 3.4. The inverse system (3.16) is positive if and only if the matrix A of the system (3.15) is monomial.

Proof. It is well-known [4] that $A^{-1} \in \Re_+^{n \times n}$ if and only if the matrix A is monomial. By Theorem 3.3 the system (3.15) is positive if and only if $A^{-1} \in \Re_+^{n \times n}$. Therefore, the system (3.16) is positive if and only if the matrix A is monomial.

Theorem 3.5. Let the minimal polynomial of the matrix A has the form (2.4). Then the solutions of the equations (3.15) and (3.16) are given by the formulas

$$x_i = A^i x_0 = \sum_{k=1}^{r} [Z_{k1}\lambda_k^i + Z_{k2}i\,\lambda_k^{i-1} + ... + Z_{km_k}i(i-1)...(i-m_k+2)\lambda_k^{i-m_k+1}]x_0 \qquad (3.17)$$

and

$$\bar{x}_i = \overline{A}^i \bar{x}_0 = \sum_{k=1}^{r} \left[Z_{k1}\left(\frac{1}{\lambda_k}\right)^i + Z_{k2}\frac{d}{d\lambda_k}\left(\frac{1}{\lambda_k}\right)^i + ... + Z_{km_k}\frac{d^{m_k-1}}{d\lambda_k^{m_k-1}}\left(\frac{1}{\lambda_k}\right)^i \right] \bar{x}_0 \qquad (3.18)$$

respectively and the matrices Z_{kj} $k=1,...,r$; $j=1,...,m_k$ are defined by (2.3).

Proof is similar to the proof of Theorem 3.2.

In particular case when the minimal polynomial of A has the form (2.5) then the formulas (3.17) and (3.18) takes the forms

$$x_i = \sum_{k=1}^{r} Z_k \lambda_k^i x_0 \qquad (3.17')$$

and

$$\bar{x}_i = \sum_{k=1}^{r} Z_k \left(\frac{1}{\lambda_k}\right)^i \bar{x}_0, \qquad (3.18')$$

where Z_k is defined by (2.7').

Example 3.3. Consider the linear system (3.15) with the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}. \qquad (3.19)$$

The characteristic (minimal) polynomial of the matrix (3.19) has the form

$$\Psi(\lambda) = \det[I_2\lambda - A] = \begin{vmatrix} \lambda & -1 \\ 6 & \lambda+5 \end{vmatrix} = (\lambda+2)(\lambda+3).$$

Taking into account that in this case $\lambda_1 = -2$, $\lambda_1 = -3$ and using (2.6) and (3.17'), (2.7') we obtain

$$Z_1 = \frac{A - I_2\lambda_2}{\lambda_1 - \lambda_2} = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}, \quad Z_2 = \frac{A - I_2\lambda_1}{\lambda_2 - \lambda_1} = \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} \qquad (3.20)$$

and

$$x_i = (Z_1\lambda_1^i + Z_2\lambda_2^i)x_0 = \begin{bmatrix} 3(-2)^i - 2(-3)^i & (-2)^i - (-3)^i \\ 6(-3)^i - 6(-2)^i & 3(-3)^i - 2(-2)^i \end{bmatrix} x_0. \qquad (3.21)$$

The inverse matrix $\overline{A}$ of (3.19) has the form

$$\overline{A} = A^{-1} = \begin{bmatrix} -\dfrac{5}{6} & -\dfrac{1}{6} \\ 1 & 0 \end{bmatrix} \qquad (3.22)$$

and its characteristic polynomial is

$$\det[I_2\overline{\lambda} - \overline{A}] = \begin{vmatrix} \overline{\lambda} + \dfrac{5}{6} & \dfrac{1}{6} \\ -1 & \overline{\lambda} \end{vmatrix} = (\overline{\lambda} + \dfrac{1}{2})(\overline{\lambda} + \dfrac{1}{3}). \qquad (3.23)$$

Using (3.18') for $\overline{\lambda}_1 = 1/\lambda_1 = -1/2$, $\overline{\lambda}_2 = 1/\lambda_2 = -1/3$ and (3.20) we obtain

$$\overline{x}_i = (Z_1\overline{\lambda}_1^i + Z_2\overline{\lambda}_2^i)\overline{x}_0 = \begin{bmatrix} 3\left(-\dfrac{1}{2}\right)^i - 2\left(-\dfrac{1}{3}\right)^i & \left(-\dfrac{1}{2}\right)^i - \left(-\dfrac{1}{3}\right)^i \\ 6\left(-\dfrac{1}{3}\right)^i - 6\left(-\dfrac{1}{2}\right)^i & 3\left(-\dfrac{1}{3}\right)^i - 2\left(-\dfrac{1}{2}\right)^i \end{bmatrix} \overline{x}_0. \qquad (3.14)$$

4. Stability of inverse systems

4.1. Continuous-time systems

The continuous-time linear system (3.1) is called asymptotically stable if

$$\lim_{t \to \infty} x(t) = 0 \text{ for any initial conditions } x_0 \in \Re^n. \qquad (4.1)$$

In a similar way we define the asymptotic stability of the inverse system (3.2).

Theorem 4.1. The inverse system (3.2) is asymptotically stable if and only if the system (3.1) is asymptotically stable.

Proof. It is well-known that the system (3.1) (and (3.2)) is asymptotically stable if and only if $\mathrm{Re}\,\lambda_i = -\alpha_i < 0$ for all eigenvalues $\lambda_i = -\alpha_i + j\beta_i$, $i = 1, \ldots, n$ of the matrix A $(\overline{A} = A^{-1})$. By Lemma 2.1 the eigenvalues $\overline{\lambda}_i$, $i = 1, \ldots, n$ of the matrix $\overline{A}$ are related with the eigenvalues λ_i of the matrix A by the equality

$$\overline{\lambda}_i = \frac{1}{\lambda_i} = \frac{1}{-\alpha_i + j\beta_i} = \frac{-\alpha_i - j\beta_i}{\alpha_i^2 + \beta_i^2} = -\overline{\alpha}_i - j\overline{\beta}_i, \quad i = 1, \ldots, n, \tag{4.2}$$

where

$$\overline{\alpha}_i = \frac{\alpha_i}{\alpha_i^2 + \beta_i^2}, \quad \overline{\beta}_i = \frac{\beta_i}{\alpha_i^2 + \beta_i^2}. \tag{4.3}$$

From (4.3) it follows that $\mathrm{Re}\,\overline{\lambda}_i = -\overline{\alpha}_i < 0$ if and only if $\mathrm{Re}\,\lambda_i = -\alpha_i < 0$. Therefore, the inverse system (3.2) is asymptotically stable if and only if the system (3.1) is asymptotically stable.

The considerations can be easily extended for positive continuous-time linear systems.

4.2. Discrete-time systems

The discrete-time linear system (3.15) is called asymptotically stable if

$$\lim_{i \to \infty} x_i = 0 \quad \text{for any initial conditions } x_0 \in \mathfrak{R}^n. \tag{4.4}$$

In a similar way we define the asymptotic stability of the inverse system (3.16).

Theorem 4.2. The inverse system (3.16) is asymptotically stable if and only if the system (3.15) is unstable.

Proof. It is well-known that the system (3.15) (and (3.16)) is asymptotically stable if and only if $|\lambda_i| < 1$ for all eigenvalues λ_i $(\overline{\lambda}_i)$, $i = 1, \ldots, n$ of the matrix $A\,(\overline{A} = A^{-1})$. By Lemma 2.1 the module of eigenvalues $\overline{\lambda}_i$, $i = 1, \ldots, n$ of the matrix $\overline{A}$ with module of eigenvalues λ_i of the matrix A are related by the equality

$$\left| \overline{\lambda}_i \right| = \frac{1}{|\lambda_i|}, \quad i = 1, \ldots, n. \tag{4.5}$$

From (4.5) it follows that $\left| \overline{\lambda}_i \right| < 1$, if and only if $|\lambda_i| > 1$. Therefore, the inverse system (3.16) is asymptotically stable if and only if the system (3.15) is unstable.

The considerations can be easily extended for positive discrete-time linear systems.

Example 4.1. Consider the positive linear system (3.15) with the matrix

$$A = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.55 \end{bmatrix}. \tag{4.6}$$

The system with (4.6) is asymptotically stable since the coefficients of the polynomial

$$\det[I_2\lambda - A + I_2] = \begin{vmatrix} \lambda + 0.6 & -0.1 \\ -0.2 & \lambda + 0.45 \end{vmatrix} = \lambda^2 + 1.05\lambda + 0.25 \qquad (4.7)$$

are positive. The eigenvalues of the matrix (4.6) are $\lambda_1 = 0.635$, $\lambda_2 = 0.315$.
The inverse matrix $\overline{A}$ of (4.6) has the form

$$\overline{A} = A^{-1} = \frac{1}{0.2}\begin{bmatrix} 0.55 & -0.1 \\ -0.2 & 0.4 \end{bmatrix} \qquad (4.8)$$

and its eigenvalues are $\overline{\lambda}_1 = 1/\lambda_1 = 1/0.635 = 1.575$, $\overline{\lambda}_2 = 1/\lambda_2 = 1/0.315 = 3.175$. There-
fore, the inverse system is not positive and it is unstable.

5. Reachability of inverse systems

5.1. Continuous-time systems

Consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \qquad (5.1)$$

where $x(t) \in \mathfrak{R}^n$, $u(t) \in \mathfrak{R}^m$ are the state and input vectors and $A \in \mathfrak{R}^{n\times n}$, $B \in \mathfrak{R}^{n\times m}$.

Definition 5.1. [1, 5] The system (5.1) is called reachable if for any given final state $x_f \in \mathfrak{R}^n$
there exists time t_f and input $u(t)$ $0 \leq t \leq t_f$ such that $x(t_f) = x_f$ for zero initial state
$x(0) = 0$.

To simplify the notation we shall assume that the matrix A has distinct eigenvalues
$\lambda_1, ..., \lambda_n$.

Lemma 5.1. The system (5.1) is reachable if and only if

$$rank[Z_1B \quad Z_2B \quad ... \quad Z_nB] = n. \qquad (5.2)$$

Proof. It is well-known [5] that the system (5.1) is reachable if and only if the rows of the
matrix $e^{At}B$ are linearly independent. Using (2.6) for $f(\lambda) = e^{\lambda t}$ we obtain

$$e^{At}B = \sum_{k=1}^{n} Z_k e^{\lambda_k t} B = [Z_1B \quad Z_2B \quad ... \quad Z_nB]\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}, \qquad (5.3)$$

where Z_k are defined by (2.7). From (5.3) it follows that the rows of the matrix $e^{At}B$ are
linearly independent if and only if the condition (5.2) is satisfied.

The considerations can be easily extended for multiple eigenvalues when the minimal
polynomial of the matrix A has the form (2.4). It is assumed that the matrix A of the system
(5.1) is nonsingular.

Definition 5.2. The system

$$\dot{\bar{x}}(t) = \bar{A}\,\bar{x}(t) + \bar{B}u(t), \quad \bar{A} = A^{-1}, \ \bar{B} = B \tag{5.4}$$

is called the inverse system of the system (5.1).

Theorem 5.1. The inverse system (5.4) is reachable if and only if the system (5.1) is reachable.

Proof. If the eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix A are distinct then by Lemma 2.1 the eigenvalues of $\bar{A} = A^{-1}$ are also distinct and $\bar{\lambda}_k = 1/\lambda_k$, $k = 1, \dots, n$. Using (2.6) for matrix $\bar{A}$ we obtain

$$e^{\bar{A}t}B = \sum_{k=1}^{n} Z_k e^{\frac{t}{\lambda_k}} B = [Z_1 B \quad Z_2 B \quad \dots \quad Z_n B] \begin{bmatrix} e^{\frac{t}{\lambda_1}} \\ e^{\frac{t}{\lambda_2}} \\ \vdots \\ e^{\frac{t}{\lambda_n}} \end{bmatrix}. \tag{5.5}$$

From comparison of (5.3) and (5.5) and Lemma 5.1 it follows that the inverse system (5.4) is reachable if and only if the system (5.1) is reachable.

Example 5.1. Consider the system (5.1) with the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{5.6}$$

The pair (5.6) is reachable since

$$rank\,[B \quad AB] = rank \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = 2. \tag{5.7}$$

The same result we obtain using the condition (5.2).

The matrix A has the eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$ and

$$Z_1 = \frac{A - I_2 \lambda_2}{\lambda_1 - \lambda_2} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}, \ Z_2 = \frac{A - I_2 \lambda_1}{\lambda_2 - \lambda_1} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}. \tag{5.8}$$

Using (5.2) we obtain

$$rank\,[Z_1 B \quad Z_2 B] = rank \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} = 2. \tag{5.9}$$

Taking into account that

$$\bar{A} = A^{-1} = \frac{1}{2} \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \text{ and } \bar{B} = B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{5.10}$$

for the inverse system (5.4) we obtain

$$rank[\overline{B} \quad \overline{A}\overline{B}] = rank\begin{bmatrix} 1 & -1.5 \\ 0 & 1 \end{bmatrix} = 2. \tag{5.11}$$

Therefore, the inverse system is also reachable.

5.2. Discrete-time systems

Consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+, \tag{5.12}$$

where $x_i \in \mathfrak{R}^n$, $u_i \in \mathfrak{R}^n$ are the state and input vectors and $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$. It is assumed that the matrix A of (5.12) is nonsingular.

Definition 5.3.
The system

$$\overline{x}_{i+1} = \overline{A}\overline{x}_i + \overline{B}u_i, \quad \overline{A} = A^{-1}, \quad \overline{B} = B \tag{5.13}$$

is called the inverse system of the system (5.12).

Definition 5.4. The system (5.12) (and also the system (5.13)) is reachable if for any given final state $x_f \in \mathfrak{R}^n$ ($\overline{x}_f \in \mathfrak{R}^n$) there exists a number of step q and the sequence of input $u_0, u_1, ..., u_{q-1}$ such that $x_q = x_f$ ($\overline{x}_q = \overline{x}_f$).

Theorem 5.2. The inverse system (5.13) is reachable if and only if the system (5.12) is reachable.

Proof. It is well-known [1,5, 9] that the system (5.13) is reachable if and only if

$$rank[I_n\lambda - A \quad B] = n \text{ for all } \lambda \in \delta_A, \tag{5.14}$$

where δ_A is the spectrum of A.
By assumption the matrix A is nonsingular and its eigenvalues (all elements of δ_A) are nonzero. Using (5.14) for the inverse system (5.13) we obtain (for all $\lambda \in \delta_A$)

$$rank[I_n\lambda^{-1} - A^{-1} \quad B] = rank\left\{[I_n\lambda^{-1} - A^{-1} \quad B]\begin{bmatrix} \lambda A & 0 \\ 0 & I_n \end{bmatrix}\right\} \tag{5.15}$$

$$= rank[A - I_n\lambda \quad B] = rank[I_n\lambda - A \quad B].$$

From (5.15) it follows that the inverse system is reachable if and only if the system (5.12) is reachable.
Note that this way of proving can be also applied to prove the Theorem 5.1.

Example 5.2. Consider the system (5.12) with the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{5.16}$$

The eigenvalues of the matrix A are $\lambda_1 = -2, \ \lambda_2 = -3$. The pair (5.16) is reachable since

$$rank\,[I_2\lambda - A \quad B] = rank \begin{bmatrix} \lambda & -1 & 0 \\ 6 & \lambda+5 & 1 \end{bmatrix} = 2 \tag{5.17}$$

for all $\lambda \in \delta_A = \{-2, -3\}$.

The inverse matrix

$$\overline{A} = A^{-1} = \frac{1}{6}\begin{bmatrix} -5 & -1 \\ 6 & 0 \end{bmatrix} \text{ and } \overline{B} = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.18}$$

has the eigenvalues $\overline{\lambda}_1 = 1/\lambda_1 = -1/2, \ \overline{\lambda}_2 = 1/\lambda_2 = -1/3$ and

$$rank\,[I_2\overline{\lambda} - \overline{A} \quad \overline{B}] = rank \begin{bmatrix} \overline{\lambda} + \dfrac{5}{6} & \dfrac{1}{6} & 0 \\ -1 & \overline{\lambda} & 1 \end{bmatrix} = 2 \tag{5.19}$$

for all $\overline{\lambda} \in \delta_A = \{-1/2, -1/3\}$. Therefore, the inverse system is also reachable.

Those considerations can be easily extended for: 1) the observability of the inverse systems 2) the reachability and observability of the positive linear systems.

6. Concluding remarks

The concept of inverse systems for standard and positive linear systems has been investigated. Necessary sufficient conditions for the existence of the positive inverse systems for continuous-time and discrete-time linear systems have been established (Theorem 3.1 and 3.4). The solutions to inverse systems have been derived (Theorem 3.5). It has been shown that: 1) the inverse system (3.2) of continuous-time linear system is asymptotically stable if and only if the system (3.1) is asymptotically stable (Theorem 4.1), 2) the inverse system (3.16) of discrete-time system (3.15) is asymptotically stable if and only if the system (3.15) is unstable (Theorem 4.2), 3) the inverse system (5.4) of continuous-time system (5.1) and inverse system (5.13) of discrete-time system (3.12) are reachable if and only if the systems (5.1) and (5.12) are reachable (Theorem 5.1 and 5.2). The considerations concerning reachability can be easily extended for the observability and for positive continuous-time and discrete-time linear systems. An extension of these considerations for 2D linear discrete-time systems and 2D linear hybrid systems are open problems.

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