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IMPLEMENTATION OF A GEOMETRIC CONSTRAINT REGULARIZATION FOR MULTIBODY SYSTEM MODELS

Redundant constraints in MBS models severely deteriorate the computational performance and accuracy of any numerical MBS dynamics simulation method. Classically this problem has been addressed by means of numerical decompositions of the constraint Jacobian within numerical integration steps. Such decompositions are computationally expensive. In this paper an elimination method is discussed that only requires a single numerical decomposition within the model preprocessing step rather than during the time integration. It is based on the determination of motion spaces making use of Lie group concepts. The method is able to reduce the set of loop constraints for a large class of technical systems. In any case it always retains a sufficient number of constraints. It is derived for single kinematic loops.

1. Introduction

Redundant constraints in MBS models lead to singular coefficient matrices in the dynamic motion equations, and thus to increased computational effort and stability problems of the numerical solution. This problem has been addressed in number of publications [1, 2, 5, 10, 13, 16, 17, 19, 24, 25, 26].

There are two types of redundancy: *topological* and *geometrical*. While topological redundancy is a generic property inherent to all mechanisms with the same topology, geometric redundancy is due to special geometric conditions. The first has historically been a central topic in rigidity theory and structural analysis, whereas the latter has been on the agenda in mechanism theory and multibody dynamics since its very beginning. Fig. 1a) shows a topologically redundant planar mechanism. For any (compatible) length of links 5 and 6 the upper substructure is redundantly constrained.

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Geometrically overconstrained mechanisms are shown in Fig. 1b) and in Fig. 2, which are fundamentally different in nature, however. The spherical 4-bar mechanism in Fig. 1b) owes its mobility to the fact that its joint axes intersect in one point. The crucial observation is that this is an inherent invariant characteristic of the kinematic chain rather than the closed loop, and is thus preserved if the loop is cut open (e.g. removing joint J_3). In other words, the motion space of the closed kinematic loop, and thus the existence of dependent constraints, can be inferred from the kinematics of the unconstrained system. This is a representative of so-called trivial mechanisms [6, 7]. This is slightly different for the example in Fig. 2a). The motion of this mechanism is determined by the intersection of the motion spaces of the three planar kinematic chains with mutually perpendicular motion planes. Again these motion spaces are intrinsic invariant properties of the respective partial chain. Consequently the redundancy of the constraints can be determined solely from the intersection of motion spaces of the open kinematic chains. Such mechanisms are termed exceptional. Apparently trivial mechanisms are just special cases of exceptional mechanisms. A fundamentally different situation is observed for the Bricard 6-bar mechanism in Fig. 2b). The redundant constraints arise from the intersection of all joint axes with a common line causing one dependent loop constraint. In contrast to the preceding two examples this intersection, and thus the redundancy, is due to a special geometric arrangement of the loop, and is lost when the loop is opened. The double 4-bar in Fig. 3 is another example where the redundancy is this not inherent to the kinematic chain but the mobility is owed to a special assembly. This mechanism is mobile with 1 DOF only since links 1, 2, and 3 have the same length giving rise to redundant constraints. This redundancy cannot be inferred from the kinematics of the opened loop.

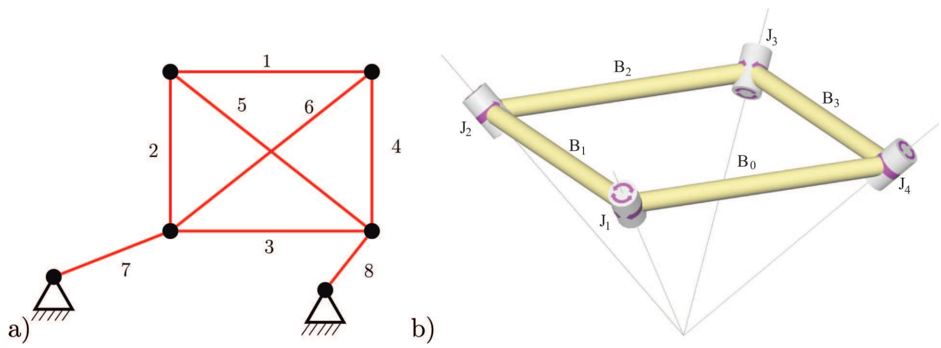

Fig. 1. a) A topologically overconstrained planar mechanism, b) A geometrically overconstrained spherical mechanism

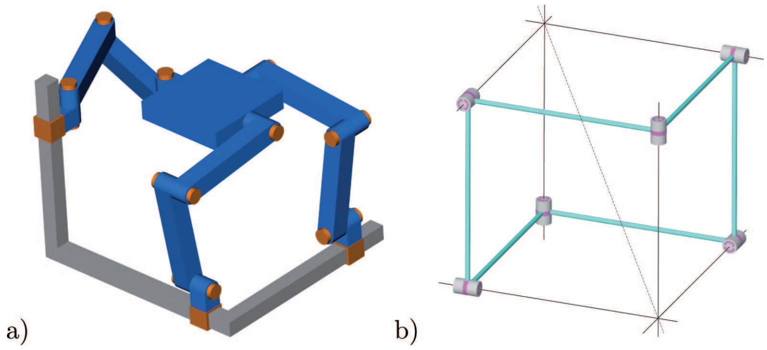

Fig. 2. a) An overconstrained parallel manipulator, b) The geometrically overconstrained Bricard mechanism

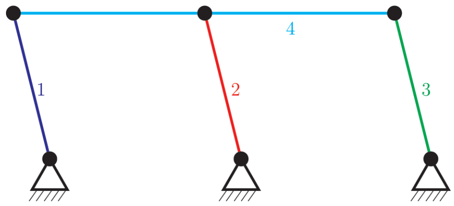

Fig. 3. A geometrically overconstrained double 4-bar mechanism

From these examples it shall be clear that redundant geometric loop constraints can be distinguished as those that can be explained by the motion spaces of partial kinematic chains of a mechanism without reference to the actual assembly and as those that only occur because of special arrangements of the closed loop. The crucial implication is that for the first class of mechanisms the smallest constraint subspace can be deduced algebraically/geometrically merely from the joint arrangement and link geometries, allowing for a permanent elimination of redundant loop constraints, and thus a regularization of the motion equations [16, 17]. The basic idea of this approach is to determine an involutive closure covering the motion space of the partial kinematic chains when a kinematic loop is cut open, and to restrict the constraint to this. Since this is always a conservative approximation the elimination always yields a sufficient set of constraints. MBS codes for spatial mechanisms treat any kinematic loop as a system of 6 independent constraints. The proposed method aids to reduce this to a set of independent constraints by means of a *preprocessing procedure independent from the actual MBS code* (using relative or absolute coordinates).

The method rests on the theory of screw systems and Lie groups, and accordingly its implementation requires numerical treatment of such objects. In this paper the algorithmic steps are discussed and related to the standard vector operations in spatial multibody dynamics. The principle is introduced for a single kinematic loop for simplicity, the extension to multi-loop MBS is the topic of a forthcoming paper. For background on the relevant Lie group $SE(3)$ and its algebra $se(3)$ the reader is referred to [18, 23].

2. MBS Motion Equations and the Problem of Redundant Constraints

Denoting with $\mathbf{q} \in \mathbb{V}^n$ the vector of generalized coordinates and with $\mathbf{V} \in \mathbb{R}^n$ the vector of generalized (usually non-holonomic) velocities, the EOM of a general (for simplicity holonomically and scleronomically) constrained MBS can be written in the form of the constrained Boltzmann-Hamel equations

$$\begin{aligned} \mathbf{M}(\mathbf{q})\dot{\mathbf{V}} + \mathbf{J}^T\lambda &= \mathbf{Q}(\mathbf{q},\mathbf{V},t) && (1)\\ \mathbf{A}(\mathbf{q})\dot{\mathbf{q}} &= \mathbf{V} && (2)\\ h(\mathbf{q}) &= \mathbf{0} && (3) \end{aligned}$$

where $\mathbf{J}$ is the $m \times n$ Jacobian of the system of m geometric constraints (3). The latter define the configuration space $V = \{\mathbf{q} \in \mathbb{V}^n, h(\mathbf{q}) = \mathbf{0}\}$, which is a variety in $\mathbb{V}^n$. Assume in the following that the number of independent constraint equations is r, so that the dimension of V is $n - r$. That is, the rank of $\mathbf{J}$ is always $n - r \geq n - m$ except in singular configurations. Hence only $n - r$ generalized coordinates and velocities are independent.

This is a system of differential-algebraic equations (DAE). In order to apply time integration schemes for ordinary differential equations (ODE) the equations are commonly transformed to an ODE system. To this end the velocity and acceleration constraints, $\mathbf{J}(\mathbf{q})\mathbf{V} = \mathbf{0}$ and $\mathbf{J}(\mathbf{q})\dot{\mathbf{V}} + \dot{\mathbf{J}}(\mathbf{q},\dot{\mathbf{q}})\mathbf{V} = \mathbf{0}$ are introduced. A widely used approach is to use the formulation in terms of dependent velocities

$$\begin{pmatrix} \mathbf{M} & \mathbf{J}^T \\ \mathbf{J} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{V}} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{Q} \\ -\dot{\mathbf{J}}\mathbf{V} \end{pmatrix} \quad (4)$$

which is solved for the accelerations in every integration step. The actual constraints and constraint Jacobian depend on the introduced coordinates. As such commonly either absolute coordinates (position vector and rotation parameters for each rigid body describing the configuration w.r.t. to a spatial frame) or relative coordinates (joint displacements and angles describing the relative configurations of adjacent bodies) are used. For instance in the spherical 4-bar mechanism in Fig. 1b) (treating B_0 as fixed ground) the absolute coordinate model comprises 5 constraints for each revolute joint (i.e. 20 constraints) formulated in terms of the 6 configuration parameters for each one of the 3 bodies (18 coordinates). An estimate for the DOF would be 18 – 20 = −2. Modeling this mechanism in terms of relative coordinates requires to open loop by replacing one joint, e.g. J_3, by a set of 5 constraints so to obtain a tree-structure MBS. This would give a model in terms of the 3 angles of joints J_1, J_2, J_4 subject to 5 constraints. Again an estimated for the DOF would then be $3-5 = -2$. It is known, however, that this mechanism has 1 DOF implying that 3 constraints are redundant for either model. Moreover, since both models describe the same mechanisms the problem of redundant constraints exists for both formulations.

A unique solution of (4) only exists if the coefficient matrix is regular. This matrix is full rank $n+m$ if and only if the m constraints are independent. Hence the existence of redundant constraints is problematic for any formulation, regardless of whether relative or absolute coordinates are used, leading to a singular constraint Jacobian. The rank deficiency can occur at specific configurations forming a lower-dimensional subvariety of V, or be permanent. The first situation refers to singular configurations that always require special treatment, whereas in the second situation the drop of rank persists for a submanifold of V, i.e. there exist permanently redundant constraints.

A straightforward method to tackle this problem is to resort to a pseudoinverse solution using SVD or other numerical decomposition algorithms. In fact the various approaches [10, 13, 19, 24] to address the problem of singular constraint systems rely in the end on such a decomposition. From a computational point of view such methods are extremely expensive. In practice the pseudoinverse is only invoked when the Jacobian is rank deficient. This is acceptable for systems where $\mathbf{J}$ is full rank m except at singular points. But for systems with permanently singular Jacobian the computational costs are significant, and any means to reduce the application of numerical decomposition is beneficial.

It may by intuitively clear from the introductory discussion in section 1 that inspecting the particular geometry reveals information about the redundant constraints. The crucial point is to identify certain *invariant* properties inherent to the specific kinematics and geometry. This was pursued for 'trivial' mechanisms in [16] and later extended to 'exceptional' mechanisms in [17]. Despite the names such mechanisms prevail in technical applications.

In this paper the method is revised and its basic computational steps are presented with emphasize on the implementation aspect. The method relies on the use of relative coordinates as the geometry of kinematic chains is the basis for the approach. The actual elimination is applicable to relative and absolute coordinate models, however.

3. Kinematics of an Open Chain

For sake of simplicity a kinematic chain connected to the ground and comprising n 1-DOF joints is considered. The bodies of the chain are numbered from 1 to n toward the terminal body n. To the ground is assigned the index 0. Accordingly the joint connecting body i to body $i-1$ is indexed with i.

Configurations Each body is kinematically represented by a body-fixed reference frame (RFR), usually at the COM. A space-fixed inertial frame (IFR) is defined at the ground used to measure the (absolute) configuration of the bodies. Configurations are represented by elements of the Lie group $SE(3)$ [18, 23]. Joint i determines the relative motion of a joint frame (JFR) on body i w.r.t. to body $i-1$. The relative configuration of the connected bodies is given by the variable configuration of the JFR on either body, and the constant configuration of the JFR w.r.t. to the RFR on the respective body (Fig. 4). The latter is modeled by a constant transformation $\mathbf{S}_{i,i-1}$ from JFR of joint i to the RFR on body $i-1$, and the constant transformation $\mathbf{S}_{i,i}$ from JFR of joint i to the RFR on body i. To any 1-DOF joint can be associated a unique screw axis. Denote this screw coordinates expressed in the JFR on body $i-1$ with $\mathbf{Z}_i \in \mathbb{R}^6$, then the variable part, i.e. actual joint motion, is given in terms of the joint parameter q^i (angle or displacement) as $\exp(q^i\widehat{\mathbf{Z}}_i)$. Hence the motion of body i relative to body $i-1$ is given as

$$\mathbf{S}_{i,i-1}\exp(q^i\widehat{\mathbf{Z}}_i)\mathbf{S}_{i,i}^{-1} = \mathbf{S}_{i,i-1}\mathbf{S}_{i,i}^{-1}\mathbf{S}_{i,i}\exp(q^i\widehat{\mathbf{Z}}_i)\mathbf{S}_{i,i}^{-1} = \mathbf{M}_i\exp(\widehat{\mathbf{X}}_i q^i). \quad (5)$$

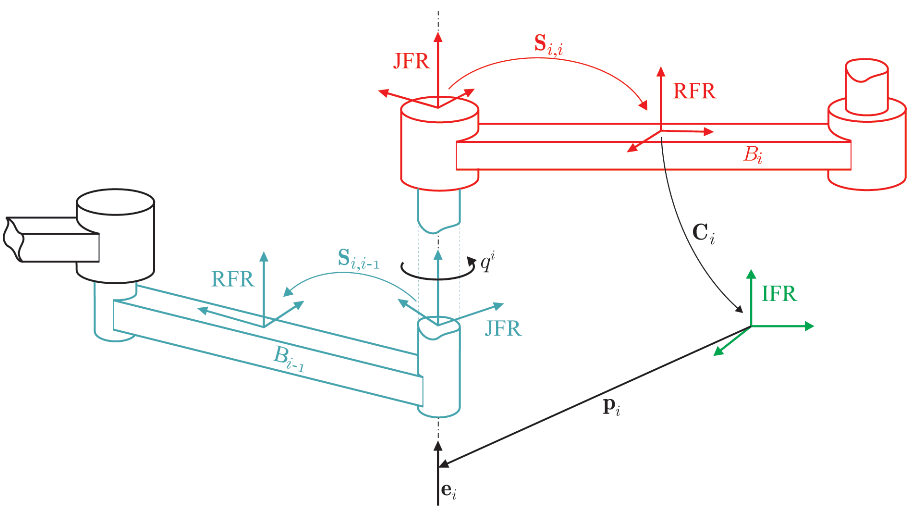

Fig. 4. Relative kinematics of adjacent bodies

Therein $\mathbf{M}_i := \mathbf{S}_{i,i-1}\mathbf{S}_{i,i}^{-1}$ is the constant reference configuration of body i w.r.t. to body $i-1$ (for $q^i = 0$), and

$$\mathbf{X}_i = \mathbf{Ad}_{\mathbf{S}_{i,i}}\mathbf{Z}_i \tag{6}$$

is the screw coordinate vector of joint i expressed in the RFR on body i. Recursive application of these transformations for all joints from ground to body i yields its configuration w.r.t. to the IFR

$$\begin{aligned} \mathbf{C}_i(\mathbf{q}) &= \mathbf{M}_1 \exp(\widehat{\mathbf{X}}_1 q^1) \cdot \mathbf{M}_2 \exp(\widehat{\mathbf{X}}_2 q^2) \cdot \ldots \cdot \mathbf{M}_i \exp(\widehat{\mathbf{X}}_i q^i) && (7) \\ &= \exp(\widehat{\mathbf{Y}}_1 q^1) \cdot \exp(\widehat{\mathbf{Y}}_2 q^2) \cdot \ldots \cdot \exp(\widehat{\mathbf{Y}}_i q^i)\mathbf{m}_i && (8) \end{aligned}$$

with $\mathbf{m}_i := \mathbf{M}_1 \cdots \mathbf{M}_i$ is the reference configuration of body i and

$$\mathbf{Y}_j := \mathbf{Ad}_{\mathbf{m}_j}\mathbf{X}_j = \begin{pmatrix} \mathbf{e}_j \\ \mathbf{p}_j \times \mathbf{e}_j + h_j\mathbf{e}_j \end{pmatrix} \tag{9}$$

is the screw coordinate vector of joint i expressed in the IFR for the reference configuration $\mathbf{q} = \mathbf{0}$. These are readily determined from a reference assembly in terms of the unit vector $\mathbf{e}_j$ along the joint axis and $\mathbf{p}_j$ the position vector to any point on the axis. This 'zero reference' formulation has been first reported by Gupta [9] in terms of screw transformations and latter by Ploen and Park [20, 21, 22] using the formulation (8). It is important to notice that it allows to formulate the kinematics without introduction of body-fixed JFR since only the screw coordinates are w.r.t. the IFR are needed. The body-fixed description, i.e. using body-fixed RFR has been originally be introduced in [3]. Both is are occasionally referred to as *product of exponentials* (POE) formulae.

Velocities The *body-fixed velocity* (or twist) of a body moving according to $\mathbf{C}(t)$ is defined as

$$\widehat{\mathbf{V}}^{\mathrm{b}} := \mathbf{C}^{-1}\dot{\mathbf{C}} = \begin{pmatrix} \mathbf{R}^T\dot{\mathbf{R}} & \mathbf{R}^T\dot{\mathbf{r}} \\ \mathbf{0} & 0 \end{pmatrix} = \begin{pmatrix} \widehat{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{pmatrix} \tag{10}$$

which is the $se(3)$ matrix corresponding to the twist vector $\mathbf{V}^{\mathrm{b}} = \left(\boldsymbol{\omega}^{\mathrm{b}}, \mathbf{v}^{\mathrm{b}}\right)^T$, where $\widehat{\boldsymbol{\omega}}^{\mathrm{b}} = \mathbf{R}^T\dot{\mathbf{R}}$ is the body-fixed definition of angular velocity and $\mathbf{v}^{\mathrm{b}} = \mathbf{R}^T\dot{\mathbf{r}}$ is the translation velocity of the origin of the body-fixed RFR expressed in this RFR. Here $\mathbf{R}$ is the rotation matrix and $\mathbf{r}$ the position vector of the body. The body Jacobian can be derived analytically from the product of exponentials (7) or (8). The columns of the body Jacobian $\mathbf{J}_i^{\mathrm{b}}$ for body i, so that

$$\mathbf{V}_i^{\mathrm{b}} = \sum_{j \leq i} \mathbf{J}_{ij}^{\mathrm{b}} \dot{q}^j, \tag{11}$$

are explicitly given by [15, 16, 18, 23]

$$\mathbf{J}_{ij}^{\mathrm{b}} = \mathbf{Ad}_{\mathbf{C}_i^{-1}\mathbf{C}_j} \mathbf{X}_j^{\mathrm{b}} \tag{12}$$

$$= \mathbf{Ad}_{\mathbf{C}_i^{-1}\mathbf{C}_j\mathbf{m}_j^{-1}} \mathbf{Y}_j^{\mathrm{s}}, \quad j < i. \tag{13}$$

An alternative definition of velocities is the *spatial velocity* $\mathbf{V}^{\mathrm{s}} = (\boldsymbol{\omega}^{\mathrm{s}}, \mathbf{v}^{\mathrm{s}})^T$ defined as

$$\widehat{\mathbf{V}}^{\mathrm{s}} := \dot{\mathbf{C}}\mathbf{C}^{-1} = \begin{pmatrix} \dot{\mathbf{R}}\mathbf{R}^T & \dot{\mathbf{r}} - \dot{\mathbf{R}}\mathbf{R}^T\mathbf{r} \\ \mathbf{0} & 0 \end{pmatrix} = \begin{pmatrix} \widehat{\boldsymbol{\omega}}^{\mathrm{s}} & \mathbf{v}^{\mathrm{s}} \\ \mathbf{0} & 0 \end{pmatrix}. \tag{14}$$

ω^{s} is the angular velocity expressed in the spatial frame, and $\mathbf{v}^{\mathrm{s}} = \dot{\mathbf{r}} - \boldsymbol{\omega}^{\mathrm{s}} \times \mathbf{r}$ is the velocity of a (possibly imaginary) point on the body momentarily traveling through the origin of the spatial frame. The columns of the spatial Jacobian for body i in the kinematic chain, so that

$$\mathbf{V}_i^{\mathrm{s}} = \sum_{j \leq i} \mathbf{J}_j^{\mathrm{s}} \dot{q}^j, \tag{15}$$

are with (7) and (8) found as [15]

$$\mathbf{J}_j^{\mathrm{s}} = \mathbf{Ad}_{\mathbf{C}_j} \mathbf{X}_j, \quad j \leq i \tag{16}$$

$$= \mathbf{Ad}_{\mathbf{C}_j\mathbf{m}_j^{-1}} \mathbf{Y}_j, \quad j \leq i. \tag{17}$$

4. Closed Loop Kinematics and Constraints

Geometric loop closure constraints A single loop is considered that is connected to the ground. This is transformed to an MBS with kinematic tree structure by eliminating one joint (the cut-joint) indexed with α. For sake of clarity the two kinematic chains are numbered sequentially starting with 1, and the index sets are distinguished by primes, so that $\mathcal{K}_{k'} = \{1', \ldots, k'\}$ and $\mathcal{K}_{l''} = \{1'', \ldots, l''\}$ denote the two chains (Fig. 5). The configurations of the terminal bodies k' and l'' are known from (8)

$$\mathbf{C}_{k'}(\mathbf{q}) = \exp(\widehat{\mathbf{Y}}_{1'} q^{1'}) \cdot \ldots \cdot \exp(\widehat{\mathbf{Y}}_{k'} q^{k'}) \mathbf{m}_{k'}$$

$$\mathbf{C}_{l''}(\mathbf{q}) = \exp(\widehat{\mathbf{Y}}_{1''} q^{1''}) \cdot \ldots \cdot \exp(\widehat{\mathbf{Y}}_{l''} q^{l''}) \mathbf{m}_{l''}. \tag{18}$$

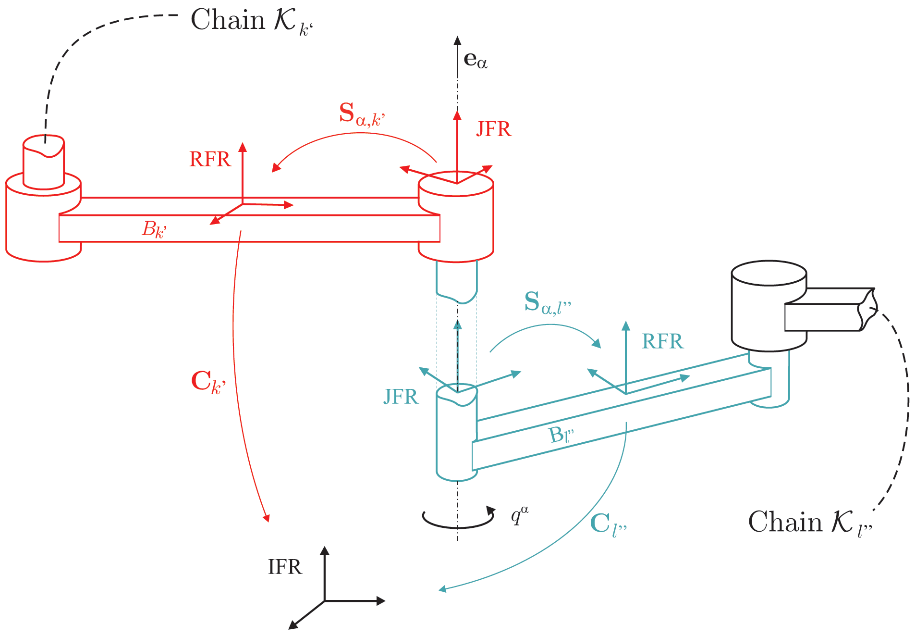

Fig. 5. Cut-joint kinematics

The relative motion of body l'' w.r.t. body k' is $\mathbf{C}_{k'}^{-1}\mathbf{C}_{l''}$. Joint α is not part of the MBS but rather delivers the closure constraints. Its motion can be represented as $\exp(\mathbf{Z}_\alpha q^\alpha)$ in terms of screw coordinate vectors $\mathbf{Z}_{\alpha_i}^{\mathrm{b}}$ expressed in cut-joint frame on body k' and joint variables q^α. Denote with $\mathbf{S}_{\alpha,k'}$ and $\mathbf{S}_{\alpha,l''}$ the transformation from cut-joint frame to RFR on body k' and l'', respectively. The loop closure condition is then

$$\mathbf{S}_{\alpha,k'}^{-1}\mathbf{C}_{r,k'}^{-1}\mathbf{C}_{r,l''}\mathbf{S}_{\alpha,l''} = \exp(\mathbf{Z}_\alpha q^\alpha). \tag{19}$$

The $n = k + l$ joint coordinates of the two chains are the generalized coordinates of the MBS model. The loop closure requires certain elements of the $SE(3)$ matrix on the left hand side of (19) to remain constant, corresponding to the motions disabled by the cut-joint.

Velocity constraints The loop closure requires the relative twist of the terminal bodies k' and l'' be compatible with the cut-joint twist, expressed in the cut-joint frame on body k'. For an admissible configuration $\mathbf{q} \in V$ (i.e. the geometric closure is presumed) there is a $\dot{q}^\alpha$ such that

$$\mathbf{Ad}_{\mathbf{S}_{\alpha,k'}}^{-1}\mathbf{Ad}_{\mathbf{C}_{k'}}^{-1}(\mathbf{V}_{l''}^{\mathrm{s}} - \mathbf{V}_{k'}^{\mathrm{s}}) = \mathbf{Z}_\alpha\dot{q}^\alpha. \tag{20}$$

The closure condition is that certain components of the twist vector on the left hand side of (20) vanish. These can be identified systematically with help of a $(6-\nu)\times 6$ matrix denoted $\mathbf{W}_\alpha$ [14, 16]. It serves as a selection matrix. Premultiplication of (20) with $\mathbf{W}_\alpha$ and expressing the twists with help of the spatial Jacobians yields closure condition

$$\mathbf{0} = \mathbf{W}_\alpha \mathbf{Ad}^{-1}_{\mathbf{S}_{\alpha,k'}} \mathbf{Ad}^{-1}_{\mathbf{C}_{k'}} (\mathbf{V}^{\mathrm{s}}_{l''} - \mathbf{V}^{\mathrm{s}}_{k'}). \tag{21}$$

Expressing the relative twists in terms of the spatial Jacobians leads to the velocity constraints represented in the JFR on body k'

$$\mathbf{0} = \mathbf{H}_\alpha (\mathbf{q})\, \dot{\mathbf{q}} \tag{22}$$

where the columns of the constraint Jacobian are

$$\mathbf{H}_{\alpha i} := \mathbf{W}_\alpha \mathbf{Ad}^{-1}_{\mathbf{S}_{\alpha,k'}} \mathbf{Ad}^{-1}_{\mathbf{C}_{k'}} \times \begin{cases} \mathbf{J}_i, & i \in \mathcal{K}_{l''} \\ -\mathbf{J}_i, & i \in \mathcal{K}_{k'} \end{cases} \tag{23}$$

and $\mathbf{0}$ otherwise. This is the standard formulation of the loop closure constraints in MBS dynamics in relative coordinate formulations. The selection matrix appears under different names. In [14] the basis matrices for the joint subgroups are called 'free motion maps' and the $\mathbf{W}_\alpha$ are called their orthogonal complements.

5. Motion Group associated to a Kinematic Loop

The crucial point and motivation for using the Lie group approach is that it allows for identification of the space of relative motions of a kinematic chain, and thus for estimating those relative motions that must be constrained.

An important characteristics of a kinematic chain is the set of configurations that its members can attain, and equivalently the vector spaces of possible velocities. At a given configuration, using the spatial representation of velocities, this is the image space of the Jacobian denoted $D^{\mathrm{s}}_i(\mathbf{q})$, where

$$D^{\mathrm{s}}_i := \mathrm{span}\ (\mathbf{J}^{\mathrm{s}}_1, \ldots, \mathbf{J}^{\mathrm{s}}_i). \tag{24}$$

In screw theory this is called the screw system generated by the joint screws. The vector space $D^{\mathrm{s}}_i(\mathbf{q})$ is the space of spatial twists of body i that can be possibly generated by joint rates $\dot{\mathbf{q}}$. The smallest vector space comprising all possible spatial twists that is invariant w.r.t. the motion is the Lie subalgebra generated by the joint screws

$$\overline{D}^{\mathrm{s}}_i = \mathrm{Lie}\ (\mathbf{Y}_1, \ldots, \mathbf{Y}_i) = \mathrm{span}\ \left(\mathbf{Y}_j, [\mathbf{Y}_j, \mathbf{Y}_k], [\mathbf{Y}_j, [\mathbf{Y}_k, \mathbf{Y}_l]], j, k, l = 1, \ldots, i\right). \tag{25}$$

That is, $D_i^s(\mathbf{q}) \subseteq \overline{D}_i^s$ for any $\mathbf{q} \in \mathbb{V}^n$. In the kinematic context this is called *spatial motion algebra* of the kinematic chain $\mathcal{K}_i$. Recall that the Lie bracket is given by the screw product of two screws [23]:

$$[\mathbf{Y}_j, \mathbf{Y}_k] = \left(\boldsymbol{\omega}_j \times \boldsymbol{\omega}_k, \boldsymbol{\omega}_j \times \mathbf{v}_k + \mathbf{v}_j \times \boldsymbol{\omega}_k\right). \tag{26}$$

$\overline{D}_{k'}^s$ and $\overline{D}_{l''}^s$ are the smallest $se(3)$ subalgebras comprising possible spatial twists of the terminal bodies of the two chains. The corresponding subgroups are $\exp \overline{D}_{k'}^s, \exp \overline{D}_{l''}^s \subset SE(3)$, which are called the 'completion groups' of the partial screw system.

In order to estimate the space of relative motions of the terminal bodies connected by cut-joint, and thus the required constraints, it is necessary to determine vector space of relative velocities that is invariant during the MBS motion. It can be shown that the relative twists of body l'' w.r.t. k' expressed in the cut-joint frame on body k' are always in the vector space

$$\begin{aligned} \Delta_\alpha^{k'} &:= \mathbf{Ad}^{-1}_{\mathbf{m}_{k'}\mathbf{S}_{\alpha,k'}}(\overline{D}_{k'}^s + \overline{D}_{l''}^s + \text{span}\,(\mathbf{Y}_\alpha)) \\ &= \mathbf{Ad}^{-1}_{\mathbf{m}_{k'}\mathbf{S}_{\alpha,k'}}(\overline{D}_{k'}^s + \overline{D}_{l''}^s) + \text{span}\,(\mathbf{Z}_\alpha). \end{aligned} \tag{27}$$

This is a conservative estimate for the vector space of infeasible relative twists of body k' and l'' due to cut-joint α. Δ_α^k is invariant w.r.t. the motion of the loop provided that geometric constraints are satisfied, i.e. the loop is closed.

6. An Algebraic Reduction Method

The above classification of kinematic loops gives rise to a method for the reduction of loop constraints. The vector space $\Delta_\alpha^{k'}$ is an estimate for the vector space of relative velocities of the two bodies connected by cut-joint α, and is invariant under the motion of the closed loop. It thus allows estimating those relative velocities that must be constrained. The basic idea is to eliminate the components of the velocity (and acceleration) constraints that are not in $\Delta_\alpha^{k'}$. It is important that this elimination does not require computationally expensive decompositions for the evaluation of the constraints.

A basis for $\Delta_\alpha^{k'}$ is obtained from (25). That is, it can be determined by nested Lie brackets of the spatial joint screws $\mathbf{Y}_j$ in the reference configuration. This is the very simple algebraic operation (26). The dimension of $\Delta_\alpha^{k'}$ is $d_\alpha := \dim \Delta_\alpha^{k'} \leq \dim \overline{D}_{k'}^s + \dim \overline{D}_{l''}^s + 1$. The d_α linearly independent vectors obtained from the Lie brackets together with $\mathbf{Z}_\alpha$ can be summarized in a basis matrix $\mathbf{B}_\alpha$. It is, however, computationally simpler to use all vectors obtained in (25) to construct a basis matrix $\mathbf{B}_\alpha \in \mathbb{R}^{6,N}$, with $N \geq d_\alpha$.

Multiplication of this matrix with the selection matrix $\mathbf{W}_\alpha$ (which removes the free motion components of the cut-joint) yields a conservative estimate of the image of the constraint Jacobian (23). Since this is motion invariant a numerical decomposition can be applied that is valid in all configurations.

A singular value decomposition (SVD)

$$\mathbf{W}_\alpha \mathbf{B}_\alpha = \mathbf{U}^T \boldsymbol{\Sigma} \mathbf{V} \tag{28}$$

with $\mathbf{U}^T\mathbf{U} = \mathbf{I}_5, \mathbf{V}^T\mathbf{V} = \mathbf{I}_{d_\alpha}$, and the matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_N & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & & 0 \end{pmatrix} \in \mathbb{R}^{5,d_\alpha}$$

composed of the non-zero singular values $\sigma_1, \ldots, \sigma_{N-1}$ of $\mathbf{W}_\alpha \mathbf{B}_\alpha$. The first N rows of $\mathbf{U}$ form a basis for the estimated image space of (23). Hence the velocity constraints are equivalent to

$$\mathbf{0} = \overline{\mathbf{U}} \mathbf{H}_\alpha (\mathbf{q})\, \dot{\mathbf{q}} \tag{29}$$

where $\overline{\mathbf{U}} \in \mathbb{R}^{N,5}$ is submatrix of $\mathbf{U}$ consisting of the first N rows. This is a reduced system of $N - 1 \leq d_\alpha - 1 \leq 5$ velocity constraints that have the same solution as the original constraints (22). The preconditioning matrix $\overline{\mathbf{U}}$ can also be applied to reduce the acceleration constraints.

This gives rise to an elimination algorithm that can be summarized by the following steps:

- Input:
  - Joint screw coordinate vectors $\mathbf{Y}_i$ w.r.t. IFR
  - Relative configurations $\mathbf{S}_{i,i-1}$ and $\mathbf{S}_{i,i}$ of JFRs
- FOR all possible choices of cut-joint α DO
  - Introduce indexing $\mathcal{K}_{k'} = \{1', \ldots, k'\}$ and $\mathcal{K}_{l''} = \{1'', \ldots, l''\}$
  - Determine basis vectors for motion algebras $\overline{D}^{\mathrm{s}}_{k'}$ and $\overline{D}^{\mathrm{s}}_{l''}$ by constructing nested Lie brackets up to second order
  - Assemble the basis matrix $\mathbf{B}_\alpha$
  - Perform SVD of $\mathbf{B}_\alpha$
  - Use submatrix $\overline{\mathbf{U}}$ to determine rank $\overline{\mathbf{U}}\mathbf{W}_\alpha$
- Output: Select the cut-joint for which rank $\overline{\mathbf{U}}\mathbf{W}_\alpha$ is minimal (this may not be unique)

Some remarks are in order now:

1. The elimination method is 'semialgebraic' in the sense that it involves an algebraic determination of the basis matrix, which is subsequently decomposed numerically. This decomposition is only required *once* prior to actual application of the model. This preconditioning step is thus part of the preprocessing of the model data. During simulation (time integration) only the premultiplication of the constraints with the *preconditioning matrix* $\overline{\mathbf{U}}$ is required but no computationally expensive decomposition as for in all other methods proposed in the literature.
2. The basis matrix $\mathbf{B}_\alpha$ is given in terms of screw coordinates and their Lie brackets, i.e. in terms of unit vectors, finite position vectors, and their cross products. Consequently this matrix is *well-conditioned*. This is beneficial for the numerical SVD.
3. The motion space of the kinematic loop is estimated upon the involutive closure of the screw systems of the partial kinematic chains. The elimination procedure is hence conservative as it *always retains a sufficient number of constraints*. In terms of the established terminology [7] the method overestimates the number of independent constraints for so-called 'trivial' and 'paradoxical' mechanisms.

7. Examples

7.1. Spherical 4-Bar Mechanism

The spherical 4-bar mechanism in Fig. 1b) is a simple but instructive example for the occurrence of redundant closure constraints [16]. The spatial joint screws must be expressed in the IFR. The reference frames can be located at arbitrary positions, but it is apparent from the kinematics that locating the IFR at intersection point of the revolute joint axes (as shown in Fig. 6) leads to the simplest expressions. In fact then, $\mathbf{Y}_i = (\mathbf{e}_i, \mathbf{0})^T$, $i = 1, \ldots, 4$. For any selection of cut-joint is immediately apparent that $\overline{D}^{\mathrm{s}}_{k'} = \overline{D}^{\mathrm{s}}_{l''} = so\,(3)$. Hence $\Delta^{k'}_{\alpha}$ is conjugate to the rotation algebra and thus 3-dimensional.

The geometry is chosen so that the JFR of the revolute joints are located at a unit distance from the center of rotation. Joint 3 is used as cut-joint. Body 2 is the terminal body k' of the first chain and body 3 is the terminal body l'' of the second chain. The flattened configuration in Fig. 6 is used as reference configuration $\mathbf{q} = \mathbf{0}$, where the mechanism exhibits a singularity.

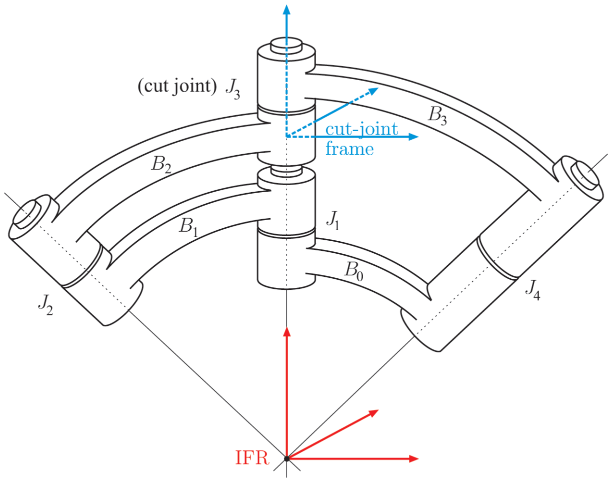

Fig. 6. Cut-joint and reference frame for spherical 4-bar mechanism. Shown is the reference configuration

Then the constraint Jacobian in (22) is

$$\mathbf{H}_\alpha(\mathbf{q}) = \begin{pmatrix} \frac{\sqrt{2}}{9}\left(\sqrt{3}s_2 - 1 - 5c_2\right) & \frac{\sqrt{2}}{27}(1 - 4\sqrt{3}s_1 + c_2(5 + \sqrt{3}s_1) - (\sqrt{3} + 9s_1)s_2 + c_1(-4 + 7c_2 + 7\sqrt{3}s_2)) & -\frac{\sqrt{2}}{3} \\ \frac{1}{9}\left(\sqrt{6}(c_2 - 1) - 3\sqrt{2}s_2\right) & -\frac{\sqrt{2}}{27}(-\sqrt{3} + 4\sqrt{3}c_1 + \sqrt{3}c_{1-2} + \sqrt{3}c_2 + 4\sqrt{3}c_{1+2} + 12s_1 - 3s_{1-2} - 3s_2) & -\sqrt{\frac{2}{3}} \\ (\frac{\sqrt{2}}{3}(c_2 - 1 - \sqrt{3}s_2) & -\frac{\sqrt{2}}{9}(-1 + 4c_1 + c_{1-2} + c_2 + 4c_{1+2} + 4\sqrt{3}s_1 - \sqrt{3}s_{1-2} - \sqrt{3}s_2) & -\sqrt{2} \\ \frac{\sqrt{2}}{9}\left(\sqrt{3}(1 + 5c_2) - 3s_2\right) & -\frac{\sqrt{2}}{27}(\sqrt{3} + 5\sqrt{3}c_2 - 3s_2 + c_1(-4\sqrt{3} + 7\sqrt{3}c_2 + 21s_2) + 3s_1(-4 + c_2 - 3\sqrt{3}s_2)) & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 \end{pmatrix}$$

where $s_{1+2} = \sin\left(q^1 + q^2\right)$ etc. Algebraic determination of the rank shows that this matrix has rank 2 except at singular points. A simple selection of 2 constraint equations would not be globally valid. Even more determination in the reference configuration $\mathbf{q} = \mathbf{0}$ would suggest only one independent equation. The proposed elimination method on the other hand is robust against singularities. Computing the basis matrix $\mathbf{B}$ for $\Delta_\alpha^{k'}$ and application of SVD yields the submatrix

$$\overline{\mathbf{U}} = \begin{pmatrix} 0.193497 & 0.461041 & 0.798547 & -0.335146 & 0. \\ -0.461041 & 0.193497 & 0.335146 & 0.798547 & 0. \end{pmatrix}$$

in (28). Hence (29) is a system of 2 independent constraints. This reduction is valid in all configurations. Notice that the method does not suffer from singularities. It yields the correct number of constraints although the reference configuration is singular.

7.2. Multi-Loop Parallel Manipulator

It may appear most straightforward to apply the elimination method to the (topologically) independent fundamental loops of a multi-loop MBS. The second example is to show the problem when simply adopting the elimination method to multi-loop MBS. Details are omitted due to space limitation.

The 3-DOF parallel manipulator in Fig. 7 is an example for so-called lower-mobility parallel kinematic machines (PKM). This 3$\underline{\text{P}}$RRR PKM in particular allows for pure translations of the moving platform and exhibits good transmission properties. It was proposed and analyzed in several publications [4, 8, 11, 12]. From an MBS modeling perspective this mechanism is problematic since it leads to redundant loop constraints. One possible choice of fundamental loops is shown in Fig. 7a). Any selection of cut-joints yields 5 constraints for each loop, i.e. altogether 10 constraints imposed to the 10 joint angles of the tree-topology system. Apparently 3 of these constraints are redundant so that the 10×10 Jacobian is rank deficient. This can be explained geometrically by looking at the motion groups generated by the tree-topology MBS model in 7b). The chain including actuator 1 connecting the platform to the ground can perform planar motions in the plane with normal parallel to the axis of actuator 1 together with translations along this normal. The corresponding subalgebra $\overline{D}^{s}_{k'}$ is conjugate to $se(2) \times \mathbb{R}$. Also the motion algebra of the chain including actuator 2 is conjugate to $se(2) \times \mathbb{R}$ with plane normal parallel to the actuator 2 axis. Hence the vector space $\Delta^{k'}_{\alpha}$ for cut-joint $\alpha = 1$ is the 5-dimensional $se(3)$ subspace of spatial translation and rotations about the axis of actuators 1 and 2. The loop closure of cut-joint 1 imposes constraints on these motions except for the components that belong to the cut-joint rotation (achieved via the selection matrix $\mathbf{W}_\alpha$) giving rise to 4 constraints for the 7 joint variables of this loop. The same argument applies to the second loop. Now the constraints for each individual loop are correctly reduced to 4, so that each loop alone would have 3 DOF. However,

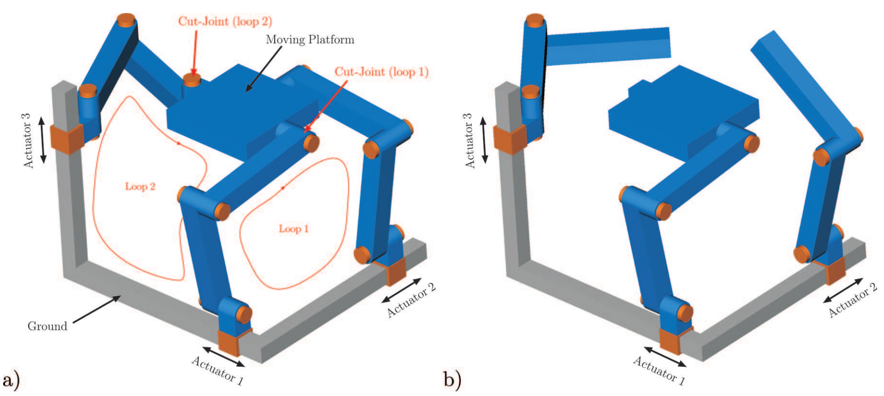

Fig. 7. a) Independent fundamental loops for the 3$\underline{\text{P}}$RRR PKM. b) Tree-topology MBS model with 10 generalized coordinates

whereas the constraints for the individual loop are regular, the overall system of 8 constraint for the 10 joint variables contains one redundant constraint. This is because the elimination is carried out independently for the loops. This problem can be solved by considering the intersection of motion spaces of the loops. The necessary extension to multi-loop MBS will be presented in a forthcoming paper.

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Implementacja geometrycznej regularyzacji więzów w układach wieloczłonowych

Streszczenie

Nadmiarowe więzy w układach wieloczłonowych (MBS) poważnie pogarszają wydajność obliczeniową i dokładność numerycznych metod symulacji systemów MBS. Klasycznym podejściem do rozwiązania tego problemu jest numeryczna dekompozycja Jakobianu więzów w kolejnych krokach całkowania cyfrowego. Dekompozycje takie są jednak kosztowne obliczeniowo. W artykule zaprezentowano metodę eliminacji, która wymaga tylko pojedynczej dekompozycji na etapie wstępnego przetwarzania modelu, a nie w trakcie integracji czasowej. Metoda jest oparta na wyznaczaniu przestrzeni ruchu przy wykorzystaniu koncepcji grup Liego. Pozwala ona zredukować zbiór więzów pętli dla szerokiej klasy systemów technicznych, przy czym w każdym przypadku zachowuje ona dostateczną liczbę więzów. Metoda została wyprowadzona i zilustrowana dla pojedynczych pętli kinematycznych.