

Specific duality and stability of positive electrical circuits

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Abstract: The specific duality and asymptotic stability of the positive linear electrical circuits are addressed. The specific duality of positive linear electrical circuits composed of resistances, inductances, capacitances and source voltages is established. 1) The linear electrical circuits are positive if and only if the common branches between meshes with resistances and inductances and meshes with resistances and capacitances contain only source voltages; 2) In linear electrical circuits the interchanges of the inductances by the capacitances and the capacitances by inductances do not change the asymptotic stability of the electrical circuits. The asymptotic stability of the positive and nonpositive electrical circuits is analyzed.

Key words: duality, positive, electrical circuit, linear, stability

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [3, 10]. Variety of models having positive behavior can be found in engineering, especially in electrical circuits [18], economics, social sciences, biology and medicine, etc. [3, 10].

The positive electrical circuits have been analyzed in [1, 5-9, 11, 18]. The constructability and observability of standard and positive electrical circuits have been addressed in [6], the decoupling zeros in [7] and minimal-phase positive electrical circuits in [8]. A new class of normal positive linear electrical circuits has been introduced in [9]. Positive fractional linear electrical circuits have been investigated in [12], positive linear systems with different fractional orders in [13, 14] and positive unstable electrical circuits in [15]. Zeroing of state variables in descriptor electrical circuits has been addressed in [16] and the realization problem of positive linear systems in [2].

In this paper the specific duality and asymptotic stability of the positive linear electrical circuits will be analyzed.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning the positivity and asymptotic stability are recalled. The specific duality of the positive linear electrical circuits is analyzed in Section 3 and their asymptotic stability in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: \mathfrak{R} is the set of real numbers, $\mathfrak{R}^{n \times m}$ is the set of $n \times m$ real matrices, $\mathfrak{R}_+^{n \times m}$ is the set of $n \times m$ real matrices with nonnegative entries and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$, M_n is the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n \times n$ identity matrix, A^T denotes the transpose of the matrix A .

2. Positive electrical circuits and their stability

Consider the linear continuous-time electrical circuit described by the state equations:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1a)$$

$$y(t) = Cx(t) + Du(t), \quad (1b)$$

where $x(t) \in \mathfrak{R}^n$, $u(t) \in \mathfrak{R}^m$, $y(t) \in \mathfrak{R}^p$ are the state, input and output vectors and $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$, $C \in \mathfrak{R}^{p \times n}$, $D \in \mathfrak{R}^{p \times m}$.

Definition 1. [3, 11, 18] The electrical circuit (1) is called (internally) positive if $x(t) \in \mathfrak{R}_+^n$ and $y = y(t) \in \mathfrak{R}_+^p$, $t \in [0, +\infty]$ for any $x_0 = x(0) \in \mathfrak{R}_+^n$ and every $u(t) \in \mathfrak{R}_+^m$, $t \in [0, +\infty]$.

Theorem 1. [3, 11, 18] The electrical circuit (1) is positive if and only if

$$A \in M_n, \quad B \in \mathfrak{R}_+^{n \times m}, \quad C \in \mathfrak{R}_+^{p \times n}, \quad D \in \mathfrak{R}_+^{p \times m}. \quad (2)$$

It is well-known [3, 11, 18] that any linear electrical circuit composed of resistors, coils, capacitors and voltage (current) sources can be described by the state Eq. (1). Usually as the state variables $x_1(t), \dots, x_n(t)$ (the components of the state vector $x(t)$) the currents in the coils and voltages on the capacitors are chosen.

Theorem 2. The linear electrical circuit composed of resistors, coils and voltage sources is positive for any values of the resistances, inductances and source voltages if the number of coils is less or equal to the number of its linearly independent meshes and the direction of the mesh currents are consistent with the directions of the mesh source voltages.

Proof. Proof is given in [18].

Theorem 3. The R, L, C, e electrical circuits are not positive for any values of their resistances, inductances, capacitances and source voltages if at least one of their branches contains a coil and capacitor.

Proof. Proof is given in [18].

Definition 2. [18] The positive electrical circuit (1) is called asymptotically stable if

$$\lim_{t \rightarrow \infty} x(t) = 0 \text{ for all } x_0 \in \mathfrak{R}_+^n. \quad (3)$$

Theorem 4. [18] The positive electrical circuit (1) is asymptotically stable if and only if

$$\operatorname{Re} \lambda_k < 0 \text{ for } k = 1, \dots, n, \quad (4)$$

where λ_k is the eigenvalue of the matrix $A \in M_n$ and

$$\det[I_n \lambda - A] = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n). \quad (5)$$

Theorem 5. [3, 15] A symmetric matrix $A \in \mathfrak{R}^{n \times n}$ ($A^T = A$) has only real eigenvalues λ_k , $k = 1, \dots, n$.

Theorem 6. If λ_{1k} , $k = 1, \dots, n$ are the eigenvalues of the matrix $A_1 \in \mathfrak{R}^{n \times n}$ and λ_{2k} , $k = 1, \dots, n$ are the eigenvalues of the matrix $A_2 \in \mathfrak{R}^{n \times n}$ then $\lambda_k = \lambda_{1k} \lambda_{2k}$, $k = 1, \dots, n$ are the eigenvalues of the matrix $A = A_1 A_2$.

Proof. Proof follows from the Lagrange-Sylvester formula for the product of two square matrices [4, 16].

3. Specific duality

We shall show the essence of the problem with simple examples of electrical circuits.

Example 1. Consider the electrical circuit shown in Fig. 1 with given resistances R_1, R_2, R_3 , inductances L_1, L_2 and source voltages e_1, e_2 .

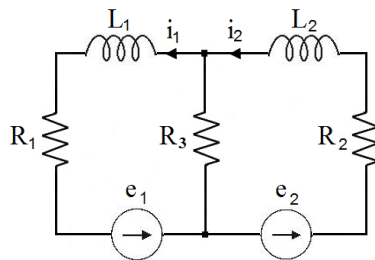


Fig. 1. Electrical circuit with inductances

Using Kirchhoff's laws we may write the equations:

$$e_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + R_3 (i_1 - i_2), \quad (6a)$$

$$e_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + R_3 (i_2 - i_1), \quad (6b)$$

which can be written in the form

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A_1 \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_1 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (7a)$$

where

$$A_1 = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \quad (7b)$$

The electrical circuit is positive since the matrix A_1 is a Metzler matrix and the matrix B_1 has nonnegative entries.

Now let us consider the electrical circuit shown in Fig. 2 with given resistances R_1 , R_2 , R_3 , capacitances C_1 , C_2 and source voltages e_1 , e_2 .

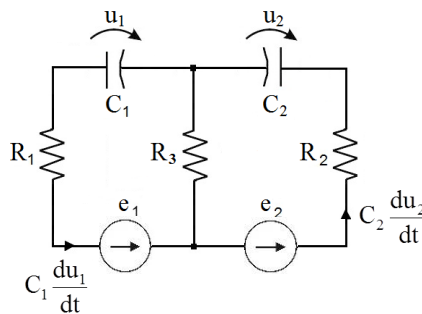


Fig. 2. Electrical circuit with capacitances

Note that the electrical circuit shown in Fig. 2 has been obtained from the electrical circuit shown in Fig. 1 by interchange of inductances L_1 , L_2 by capacitances C_1 , C_2 . It will be shown that the electrical circuit shown in Fig. 2 is not positive for all values of the resistances, capacitances and source voltages.

Using the Kirchhoff's laws for the electrical circuit shown in Fig. 2 we obtain the equations:

$$e_1 = (R_1 + R_3)C_1 \frac{du_1}{dt} - R_3C_2 \frac{du_2}{dt} + u_1, \quad (8a)$$

$$e_2 = -R_3C_1 \frac{du_1}{dt} + (R_2 + R_3)C_2 \frac{du_2}{dt} + u_2, \quad (8b)$$

which can be written in the form

$$\begin{bmatrix} (R_1 + R_3)C_1 & -R_3C_2 \\ -R_3C_1 & (R_2 + R_3)C_2 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \quad (9)$$

Premultiplying (9) by the matrix

$$\begin{bmatrix} (R_1 + R_3)C_1 & -R_3C_2 \\ -R_3C_1 & (R_2 + R_3)C_2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} \\ \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & \frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} \end{bmatrix} \quad (10)$$

we obtain

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A_2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B_2 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad (11a)$$

where

$$A_2 = - \begin{bmatrix} \frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} \\ \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & \frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} \end{bmatrix}, \quad (11b)$$

$$B_2 = \begin{bmatrix} \frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} \\ \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & \frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} \end{bmatrix}.$$

Note that the matrix A_2 is not a Metzler matrix. Therefore, the electrical circuit shown in Fig. 2 is not a positive one and we have the following important conclusion:

Conclusion 1. By interchanging the inductances L_1, L_2 by the capacitances C_1, C_2 from the positive electrical circuit shown in Fig. 1 we obtain the electrical circuit shown in Fig. 2 which is not positive.

Example 2. Consider the electrical circuit shown in Fig. 3 with given resistances R_1, R_2, R_3 , inductances L_1, L_2 and source voltage e .

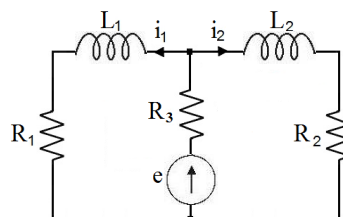


Fig. 3. Electrical circuit with inductances

Using Kirchhoff's laws we obtain the equations:

$$e = (R_1 + R_3)i_1 + R_3i_2 + L_1 \frac{di_1}{dt}, \quad (12a)$$

$$e = R_3 i_1 + (R_2 + R_3) i_2 + L_2 \frac{di_2}{dt}, \quad (12b)$$

which can be written in the form

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A_3 \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_3 e, \quad (13a)$$

where

$$A_3 = - \begin{bmatrix} \frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & \frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B_3 = \begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \end{bmatrix}. \quad (13b)$$

The electrical circuit shown in Fig. 3 is not positive since matrix A_3 is not a Metzler matrix.

Now let us consider the electrical circuit shown in Fig. 4 with given resistances R_1, R_2, R_3 , capacitances C_1, C_2 and source voltage e .

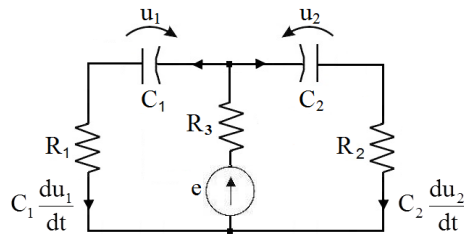


Fig. 4. Electrical circuit with capacitances

The electrical circuit shown in Fig. 4 has been obtained from the electrical circuit shown in Fig. 3 by interchanging of inductances L_1, L_2 by capacitances C_1, C_2 .

It will be shown that the electrical circuit shown in Fig. 4 is positive for all values of the resistances, capacitances and source voltage e .

Using Kirchhoff's laws for the electrical circuit shown in Fig. 4 we obtain:

$$e = (R_1 + R_3) C_1 \frac{du_1}{dt} + R_3 C_2 \frac{du_2}{dt} + u_1, \quad (14a)$$

$$e = R_3 C_1 \frac{du_1}{dt} + (R_2 + R_3) C_2 \frac{du_2}{dt} + u_2, \quad (14b)$$

which can be written in the form

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A_4 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B_4 e, \quad (15a)$$

where:

$$A_4 = \begin{bmatrix} -\frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} \\ \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & -\frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} \end{bmatrix}, \quad (15b)$$

$$B_4 = \begin{bmatrix} \frac{R_2}{[R_1(R_2 + R_3) + R_2R_3]C_1} \\ \frac{R_1}{[R_1(R_2 + R_3) + R_2R_3]C_2} \end{bmatrix}.$$

Note that matrix A_4 is a Metzler matrix and $B_4 \in \mathfrak{R}_+^2$. Therefore, the electrical circuit shown in Fig. 4 is positive and we have the following conclusion:

Conclusion 2. By interchanging inductances L_1, L_2 by capacitances C_1, C_2 from the nonpositive electrical circuit shown in Fig. 3 we obtain the positive electrical circuit shown in Fig. 4.

Example 3. Consider the electrical circuit shown in Fig. 5 with given resistances $R_k, k = 1, \dots, 4$, inductances L_1, L_2 , capacitance C and source voltages $e_j, j = 1, 2, 3$.

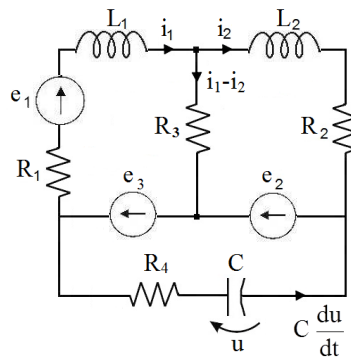


Fig. 5. Electrical circuit of Example 3

Using Kirchhoff's laws we can write the equations:

$$\begin{aligned} e_1 + e_3 - (R_1 + R_3)i_1 + R_3i_2 &= L_1 \frac{di_1}{dt}, \\ e_2 - (R_2 + R_3)i_2 + R_3i_1 &= L_2 \frac{di_1}{dt}, \\ e_2 + e_3 - u &= R_4C \frac{du}{dt}, \end{aligned} \quad (16)$$

which can be written in the form

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} = A_5 \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} + B_5 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad (17a)$$

where:

$$A_5 = \begin{bmatrix} -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} & 0 \\ \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} & 0 \\ 0 & 0 & -\frac{1}{R_4 C} \end{bmatrix}, \quad B_5 = \begin{bmatrix} \frac{1}{L_1} & 0 & \frac{1}{L_1} \\ 0 & \frac{1}{L_2} & 0 \\ 0 & \frac{1}{R_4 C} & \frac{1}{R_4 C} \end{bmatrix}. \quad (17b)$$

The electrical circuit is positive since $A_5 \in M_3$ and $B_5 \in \mathfrak{R}_+^{3 \times 3}$ for all values of the resistances, inductances and capacitance.

Now let us consider the electrical circuit show in Fig. 6 with given resistances R_k , $k = 1, \dots, 4$, inductance L , capacitances C_1, C_2 and source voltages e_j , $j = 1, 2, 3$.

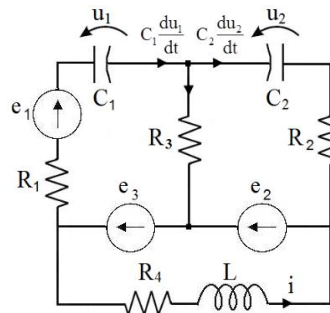


Fig. 6. Electrical circuit of Example 3

Using Kirchhoff's laws we obtain the equations:

$$\begin{aligned} e_1 + e_3 &= u_1 + (R_1 + R_3)C_1 \frac{du_1}{dt} - R_3 C_2 \frac{du_2}{dt}, \\ e_2 &= u_2 + (R_2 + R_3)C_2 \frac{du_2}{dt} - R_3 C_1 \frac{du_1}{dt}, \\ e_2 + e_3 &= R_4 i + L \frac{di}{dt}, \end{aligned} \quad (18)$$

which can be written in the form

$$\begin{bmatrix} (R_1 + R_3)C_1 & -R_3 C_2 & 0 \\ -R_3 C_1 & (R_2 + R_3)C_2 & 0 \\ 0 & 0 & L \end{bmatrix} \frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \\ i \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -R_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ i \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \quad (19)$$

Premultiplying (19) by the inverse matrix

$$\begin{aligned} & \begin{bmatrix} (R_1 + R_3)C_1 & -R_3C_2 & 0 \\ -R_3C_1 & (R_2 + R_3)C_2 & 0 \\ 0 & 0 & L \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & 0 \\ \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & \frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & 0 \\ 0 & 0 & \frac{1}{L} \end{bmatrix} \end{aligned} \quad (20)$$

we obtain

$$\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \\ i \end{bmatrix} = A_6 \begin{bmatrix} u_1 \\ u_2 \\ i \end{bmatrix} + B_6 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad (21a)$$

where

$$\begin{aligned} A_6 &= - \begin{bmatrix} \frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & 0 \\ \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & \frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & 0 \\ 0 & 0 & \frac{R_4}{L} \end{bmatrix}, \\ B_6 &= \begin{bmatrix} \frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & \frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} \\ \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & \frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} \\ 0 & \frac{1}{L} & \frac{1}{L} \end{bmatrix}. \end{aligned} \quad (21b)$$

The matrix A_6 is not a Metzler matrix since $-A_6 \in \mathfrak{R}_+^{3 \times 3}$. Therefore, the electrical circuit shown in Fig. 6 is not positive and we have the following conclusion:

Conclusion 3. By interchanging inductances L_1 , L_2 by capacitances C_1 , C_2 and capacitance C by inductance L we obtain from the positive electrical circuit shown in Fig. 5 the nonpositive electrical circuit shown in Fig. 6.

From analysis of the procedures of obtaining matrices A_2 and A_6 we have the following lemma:

Lemma 1. To obtain matrices A_2 and A_6 of the linear electrical circuits with capacitors we have to premultiply Eqs. (9) and (19) by suitable inverse matrices with nonnegative entries. As the result we obtain the matrices with nonpositive entries A_2 and A_6 .

If we add additional resistance R_5 in the branch with e_3 of the positive electrical circuit shown in Fig. 5 then we obtain the electrical circuit shown in Fig. 7.

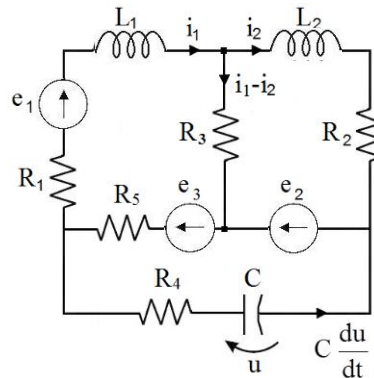


Fig. 7. Electrical circuit with additional resistance

We shall show that the electrical circuit shown in Fig. 7 is positive if and only if $R_5 = 0$. Using Kirchhoff's laws for the electrical circuit shown in Fig. 7 we obtain:

$$\begin{aligned}
 e_1 + e_3 - (R_1 + R_3 + R_5)i_1 + R_3i_2 &= R_5C \frac{du}{dt} + L_1 \frac{di_1}{dt}, \\
 e_2 - (R_2 + R_3)i_2 + R_3i_1 &= L_2 \frac{di_2}{dt}, \\
 e_2 + e_3 - u &= (R_4 + R_5)C \frac{du}{dt} + R_5i_1,
 \end{aligned} \tag{22}$$

which can be written in the form

$$\begin{bmatrix} L_1 & 0 & R_5C \\ 0 & L_2 & 0 \\ 0 & 0 & R_{33}C \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} = \begin{bmatrix} -R_{11} & R_3 & 0 \\ R_3 & -R_{22} & 0 \\ -R_5 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{23}$$

where: $R_{11} = R_1 + R_3 + R_5$, $R_{22} = R_2 + R_3$, $R_{33} = R_4 + R_5$.

From (23) we have

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} = A_7 \begin{bmatrix} i_1 \\ i_2 \\ u \end{bmatrix} + B_7 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{24a}$$

where:

$$A_7 = \begin{bmatrix} L_1 & 0 & R_5 C \\ 0 & L_2 & 0 \\ 0 & 0 & R_{33} C \end{bmatrix}^{-1} \begin{bmatrix} -R_{11} & R_3 & 0 \\ R_3 & -R_{22} & 0 \\ -R_5 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1} \left(\frac{R_5^2}{R_{33}} - R_{11} \right) & \frac{R_3}{L_1} & \frac{R_5}{L_1 R_{33}} \\ \frac{R_3}{L_2} & -\frac{R_{22}}{L_2} & 0 \\ -\frac{R_5}{R_{33} C} & 0 & -\frac{1}{R_{33} C} \end{bmatrix}, \quad (24b)$$

$$B_7 = \begin{bmatrix} L_1 & 0 & R_5 C \\ 0 & L_2 & 0 \\ 0 & 0 & R_{33} C \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1} & -\frac{R_5}{L_1 R_{33}} & \frac{1}{L_1} \left(1 - \frac{R_5}{R_{33}} \right) \\ 0 & \frac{1}{L_2} & 0 \\ 0 & \frac{1}{R_{33} C} & \frac{1}{R_{33} C} \end{bmatrix}.$$

From (24b) it follows that the electrical circuit is positive if and only if $R_5 = 0$.

In a similar way we can show that the electrical circuit shown in Fig. 7 is positive if and only if the common branch between the meshes with current i_2 and Cdu/dt contains only source voltage e_2 .

In a general case we have the following theorems:

Theorem 7. Let the positive linear electrical circuit composed of resistances, inductances, capacitances and source voltages be positive. Then the corresponding linear electrical circuit obtained from the positive electrical circuit by interchanging the inductances by the capacitances and the capacitances by the inductances (the remaining elements of the electrical circuit are the same) is not positive if and only if the first electrical circuit is positive.

Proof. Proof follows immediately from Lemma 1.

Theorem 8. Electrical circuits are positive if and only if the common branches between meshes with resistances and inductances and meshes with resistances and capacitances contain only source voltages.

Proof. Proof is similar to the one presented for the electrical circuit shown in Fig. 7.

4. Stability of electrical circuits

In this section the asymptotic stability of the positive and nonpositive electrical circuits will be addressed.

Theorem 9. Let $\bar{\lambda}_k$, $k = 1, \dots, n$ be real eigenvalues of symmetric matrix $\bar{A} \in \mathfrak{R}^{n \times n}$. Then the matrix

$$A = D\bar{A} \in \mathfrak{R}^{n \times n}, \quad D = \text{diag}[d_1 \ \cdots \ d_n], \quad d_k > 0, \quad k = 1, \dots, n \quad (25)$$

has also only real eigenvalues

$$\lambda_k = d_k \bar{\lambda}_k \text{ for } k = 1, \dots, n. \quad (26)$$

Proof. By Theorem 5 the symmetric matrix \bar{A} has only real eigenvalues $\bar{\lambda}_k$, $k = 1, \dots, n$. Note that

$$\det[I_n \lambda - D\bar{A}] = \det\{D[D^{-1}\lambda - \bar{A}]\} = \det D \det[D^{-1}\lambda - \bar{A}] \quad (27)$$

and

$$\det[I_n \lambda - D\bar{A}] = 0 \text{ if and only if } \det[D^{-1}\lambda - \bar{A}] = \det[I_n \bar{\lambda} - \bar{A}] = 0. \quad (28)$$

The equality (28) is equivalent to (26)

From (26) for $d_k > 0$, $k = 1, \dots, n$ we have the following corollary:

Corollary 4. If $d_k > 0$, $k = 1, \dots, n$ then matrix A is Hurwitz ($\text{Re } \lambda_k < 0$, $k = 1, \dots, n$) if and only if matrix \bar{A} is Hurwitz.

Example 4. (Continuation of Example 1) Consider the positive electrical circuit shown in Fig. 1 for $R_1 = R_2 = 2$, $R_3 = 1$ and $L_1 = L_2 = 2$.

In this case

$$\bar{A}_1 = - \begin{bmatrix} -(R_1 + R_3) & R_3 \\ R_3 & -(R_2 + R_3) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \quad (29)$$

and

$$\det[I_2 \bar{\lambda} - \bar{A}_1] = \begin{vmatrix} \bar{\lambda} + 3 & -1 \\ -1 & \bar{\lambda} + 3 \end{vmatrix} = \bar{\lambda}^2 + 6\bar{\lambda} + 8 = (\bar{\lambda} + 2)(\bar{\lambda} + 4). \quad (30)$$

The eigenvalues of the matrix (29) are: $\bar{\lambda}_1 = -2$, $\bar{\lambda}_2 = -4$.

Using (7b) we obtain:

$$A_1 = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \quad (31)$$

and

$$\det[I_2 \lambda - A_1] = \begin{vmatrix} \lambda + \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda + \frac{3}{2} \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2). \quad (32)$$

The eigenvalues of the matrix (31) are: $\lambda_1 = -1$, $\lambda_2 = -2$.

In this case

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad (33)$$

and from (26) we have

$$\lambda_1 = \frac{\bar{\lambda}_1}{L_1} = -1, \quad \lambda_2 = \frac{\bar{\lambda}_2}{L_2} = -2.$$

The positive electrical circuit is asymptotically stable.

Now let us consider the nonpositive electrical circuit shown in Fig. 2 with $R_1 = R_2 = 2$, $R_3 = 1$ and $C_1 = C_2 = 2$.

Using (11b) we obtain:

$$A_2 = - \begin{bmatrix} \frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} \\ \frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & \frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} \end{bmatrix} = - \begin{bmatrix} \frac{3}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{3}{16} \end{bmatrix} \quad (34)$$

and

$$\det[I_2\lambda - A_2] = \begin{vmatrix} \lambda + \frac{3}{16} & \frac{1}{16} \\ \frac{1}{16} & \lambda + \frac{3}{16} \end{vmatrix} = \lambda^2 + \frac{3}{8}\lambda + \frac{1}{32} = \left(\lambda + \frac{1}{8}\right)\left(\lambda + \frac{1}{4}\right). \quad (35)$$

Therefore, the nonpositive electrical circuit is also asymptotically stable and the interchange of inductances L_1 , L_2 by capacitances C_1 , C_2 does not change the asymptotic stability of the electrical circuit.

Using (31) and (34) we obtain:

$$A_{12} = A_1A_2 = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix} \times \begin{bmatrix} -\frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} & -\frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_1} \\ -\frac{R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} & -\frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2R_3]C_2} \end{bmatrix} \quad (36)$$

and after substitution $R_1 = R_2 = 2$, $R_3 = 1$, $L_1 = L_2 = 2$ and $C_1 = C_2 = 2$

$$A_{12} = A_1A_2 = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{3}{16} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}. \quad (37)$$

By Theorem 6 the eigenvalues $\tilde{\lambda}_1 = \lambda_{11}\lambda_{21} = 1/4$, $\tilde{\lambda}_2 = \lambda_{21}\lambda_{22} = 1/4$ of the matrix (4.13) are the product of the eigenvalues $\lambda_{11} = -1$, $\lambda_{12} = -2$ of the matrix (31) and the eigenvalues $\lambda_{21} = -1/4$, $\lambda_{22} = -1/8$ of the matrix (34).

Therefore, we have the following important lemma.

Lemma 2. The real eigenvalues of matrices A_1 and A_2 are negative and matrices A_1 and A_2 are Hurwitz since the eigenvalues of matrix $A_{12} = A_1 A_2$ are positive.

Example 5. (Continuation of Example 3) Consider the positive electrical circuit shown in Fig. 5 for $R_1 = R_2 = R_3 = R_4 = 1$, $L_1 = L_2 = 2$ and $C = 2$.

Using (17b) we obtain:

$$\bar{A}_5 = \begin{bmatrix} -(R_1 + R_3) & R_3 & 0 \\ R_3 & -(R_2 + R_3) & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (38)$$

and

$$\det[I_3 \bar{\lambda} - \bar{A}_5] = \begin{vmatrix} \bar{\lambda} + 2 & -1 & 0 \\ -1 & \bar{\lambda} + 2 & 0 \\ 0 & 0 & \bar{\lambda} + 1 \end{vmatrix} = (\bar{\lambda} + 1)(\bar{\lambda}^2 + 4\bar{\lambda} + 3) = (\bar{\lambda} + 1)(\bar{\lambda} + 1)(\bar{\lambda} + 3). \quad (39)$$

The eigenvalues of the matrix (38) are: $\bar{\lambda}_1 = \bar{\lambda}_2 = -1$, $\bar{\lambda}_3 = -3$.

In this case matrix A_5 has the form

$$A_5 = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} & 0 \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} & 0 \\ 0 & 0 & -\frac{1}{R_4 C} \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad (40)$$

and

$$\det[I_3 \lambda - A_5] = \begin{vmatrix} \lambda + 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \lambda + 1 & 0 \\ 0 & 0 & \lambda + \frac{1}{2} \end{vmatrix} = \left(\lambda + \frac{1}{2}\right) \left(\lambda^2 + 2\lambda + \frac{3}{4}\right) = \left(\lambda + \frac{1}{2}\right) \left(\lambda + \frac{1}{2}\right) \left(\lambda + \frac{3}{2}\right). \quad (41)$$

The eigenvalues of the matrix (40) are: $\lambda_1 = \lambda_2 = -1/2$, $\lambda_3 = -3/2$.

In this case

$$D = \text{diag}[d_1 \quad d_2 \quad d_3] = \text{diag}[L_1 \quad L_2 \quad R_4 C] = \text{diag}[2 \quad 2 \quad 2] \quad (42)$$

and from (26) we have

$$\lambda_1 = \frac{\bar{\lambda}_1}{L_1} = -\frac{1}{2}, \quad \lambda_2 = \frac{\bar{\lambda}_2}{L_2} = -\frac{1}{2}, \quad \lambda_3 = \frac{\bar{\lambda}_3}{R_4 C} = -\frac{3}{2}.$$

The positive electrical circuit is asymptotically stable.

Now let us consider the nonpositive electrical circuit shown in Fig. 6 with $R_1 = R_2 = R_3 = R_4 = 1$, $C_1 = C_2 = 2$ and $L = 2$.

Using (21b) we obtain:

$$A_6 = - \begin{bmatrix} \frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2 R_3]C_1} & \frac{R_3}{[R_1(R_2 + R_3) + R_2 R_3]C_1} & 0 \\ \frac{R_3}{[R_1(R_2 + R_3) + R_2 R_3]C_2} & \frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2 R_3]C_2} & 0 \\ 0 & 0 & \frac{R_4}{L} \end{bmatrix} = - \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (43)$$

and

$$\det[I_3 \lambda - A_6] = \begin{vmatrix} \lambda + \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & \lambda + \frac{1}{3} & 0 \\ 0 & 0 & \lambda + \frac{1}{2} \end{vmatrix} = \left(\lambda + \frac{1}{2} \right) \left(\lambda^2 + \frac{2}{3} \lambda + \frac{1}{12} \right) = \left(\lambda + \frac{1}{2} \right) \left(\lambda + \frac{1}{2} \right) \left(\lambda + \frac{1}{6} \right). \quad (44)$$

Therefore, the nonpositive electrical circuit is also asymptotically stable and the interchange of inductances L_1 , L_2 by capacitances C_1 , C_2 and capacitance C by inductance L does not change the asymptotic stability of the electrical circuit.

Using (40) and (43) we obtain:

$$A_{56} = A_5 A_6 = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} & 0 \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} & 0 \\ 0 & 0 & -\frac{1}{R_4 C} \end{bmatrix} \quad (45)$$

$$\times \begin{bmatrix} -\frac{R_2 + R_3}{[R_1(R_2 + R_3) + R_2 R_3]C_1} & -\frac{R_3}{[R_1(R_2 + R_3) + R_2 R_3]C_1} & 0 \\ \frac{R_3}{[R_1(R_2 + R_3) + R_2 R_3]C_2} & \frac{R_1 + R_3}{[R_1(R_2 + R_3) + R_2 R_3]C_2} & 0 \\ 0 & 0 & -\frac{R_4}{L} \end{bmatrix}$$

and after substitution $R_1 = R_2 = R_3 = R_4 = 1$, $L = L_1 = L_2 = 2$ and $C = C_1 = C_2 = 2$

$$A_{56} = A_5 A_6 = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & -\frac{1}{6} & 0 \\ -\frac{1}{6} & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}. \quad (46)$$

By Theorem 6 the eigenvalues $\tilde{\lambda}_1 = \lambda_{51}\lambda_{61} = 1/4$, $\tilde{\lambda}_2 = \lambda_{52}\lambda_{62} = 1/4$, $\tilde{\lambda}_3 = \lambda_{53}\lambda_{63} = 1/4$ of the matrix (46) are the product of the eigenvalues of the matrices (40) and (43). Therefore, we have the following important lemma:

Lemma 3. The real eigenvalues of matrices A_5 and A_6 are negative and matrices A_5 and A_6 are Hurwitz since the eigenvalues of matrix A_{56} are positive.

In a general case we have the following theorem:

Theorem 10. In linear electrical circuits composed of the resistances, inductances and capacitances the interchanges of the inductances by capacitances and the capacitances by inductances do not change the asymptotic stability of the electrical circuits.

Proof. Proof follows immediately from the Lemmas 2 and 3.

5. Concluding remarks

The specific duality and asymptotic stability of the positive linear electrical circuits have been addressed. It has been shown:

- 1) The linear electrical circuits composed of resistances, inductances, capacitances and source voltages obtained by interchanging the inductances by capacitances and the capacitances by inductances is positive if and only if the first electrical circuit is not positive (Theorem 6).
- 2) The linear electrical circuits are positive if and only if the common branches between meshes with resistances and inductances and meshes with resistances and capacitances contain only source voltages (Theorem 7).
- 3) In linear electrical circuits the interchanges of the inductances by the capacitances and the capacitances by inductances do not change the asymptotic stability of the electrical circuits (Theorem 8).

The theorems have been illustrated by examples of linear electrical circuits. The considerations can be easily extended to positive fractional linear electrical circuits.

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