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A general unified approach to chaos synchronization in continuous-time systems (with or without equilibrium points) as well as in discrete-time systems

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By analyzing the issue of chaos synchronization, it can be noticed the lack of a *general* approach, which would enable *any type of synchronization* to be achieved. Similarly, there is the lack of a *unified* method for synchronizing both continuous-time and discrete-time systems via a *scalar* signal. This paper aims to bridge all these gaps by presenting a novel *general unified* framework to synchronize chaotic (hyperchaotic) systems via a scalar signal. By exploiting nonlinear observer design, the approach enables any type of synchronization defined to date to be achieved for both continuous-time and discrete-time systems. Referring to discrete-time systems, the method assures any type of *dead beat* synchronization (i.e., exact synchronization in finite time), thus providing additional value to the conceived framework. Finally, the topic of synchronizing special type of systems, such as those characterized by the absence of equilibrium points, is also discussed.

Key words: chaos synchronization and control, scalar synchronizing signal, continuous-time system (with or without equilibrium points), discrete-time systems, observer-based synchronization, dead beat control, synchronization in finite time

1. Introduction

Chaotic phenomena are strange random aggregates of responses to internal and external stimuli in dynamical systems [1, 2]. Being highly sensitive to initial conditions, chaotic systems are characterized by trajectories that separate exponentially in the course of the time, even when they start from two nearby initial states. As a result, chaotic systems seem to intrinsically defy synchroniza-

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tion. The setting of some collective (synchronized) behavior in coupled chaotic systems has therefore a great importance and interest. To this purpose, theories and methods developed for controlling nonlinear systems could be utilized for synchronizing of chaotic systems. Chaos synchronization was first observed in 1991, when it was proved that a drive system and a driven (response) system achieve synchronized dynamics, provided that the Lyapunov exponents of the driven system are negative (*identical synchronization*) [2]. From then on, several typical synchronization phenomena have been identified, including those where a *scalar signal* is exploited for synchronizing chaotic (hyperchaotic) systems [3–8]. Among the different types of synchronization for both continuous-time and discrete-time systems [9–17], *projective synchronization* is characterized by response system states that are scaled replicas of the drive system states [11–13]. The projective synchronization turns into the *anti-phase synchronization* [13] when the scaling factor equals -1. On the other hand, when the scaling factors are different for each state variable (i.e., when the scaling matrix is a diagonal one) the *full state hybrid projective synchronization* is obtained, for both continuous- and discrete-time systems [14–18]. These approaches, when applied to discrete-time systems, may include the exact synchronization in finite time (the so-called *dead-beat synchronization*) [18].

Despite of a considerable amount of chaotic and hyperchaotic systems introduced in literature, very recently there has been an increasing interest in studying special chaotic systems, characterized by the absence of equilibrium points [19, 20]. They are called "*chaotic systems with hidden attractors*", since the absence of equilibria makes the location of the attractors very challenging [19]. Note that the topic of synchronizing these special chaotic systems is almost unexplored, due to the fact that the discovery of these systems without equilibrium points is very recent [19, 20].

By summarizing the considerations expressed above, it can be stated that the synchronization issue in literature is characterized by the presence of a "*variety of synchronization types for a variety of continuous-time and discrete-time systems*". However, it can be noticed that these varieties (of synchronization types and systems) have not yet grouped in a systematic framework. In other words, in literature there is the lack of a *general* approach, which would enable *any type of synchronization* defined to date to be achieved. Similarly, there is the lack of a *unified* method for synchronizing both continuous-time and discrete-time systems via a *scalar* signal. Additionally, the topic of synchronizing chaotic systems characterized by the absence of equilibrium points is almost unexplored, due to the recent discovery of these systems. Based on these considerations, this paper aims to bridge all these gaps by presenting a novel *general unified* approach to synchronize chaotic (hyperchaotic) systems via a scalar signal. *The framework, based on the concept of observer, enables any type of synchronization defined to date to be achieved for both continuous-time and discrete-time systems via* *a scalar signal*. Referring to continuous-time systems, both with equilibria or without equilibria, the proposed observer-based synchronization framework is based on a structural condition related to the uncontrollable eigenvalues of the error system. Referring to discrete-time systems, the proposed observer-based synchronization framework exploits a structural condition related to the controllability matrix of the error system. This condition assures any type of *dead beat* synchronization to be achieved, thus providing additional value to the conceived framework.

The paper is organized as follows. Sections 2 and 3 illustrate the proposed general unified synchronization framework. Referring to continuous-time systems (with or without equilibria) as well as discrete-time systems, two propositions are developed, which enable each drive system variable to be synchronized with any linear combination of response system variables, for any scaling matrix. A remarkable feature is that any type of synchronization defined to date is achievable (via a scalar signal) for wide classes of dynamical systems. Note that these classes comprise most of the chaotic (hyperchaotic) circuits, systems and maps discovered so far, including a number of recently introduced chaotic systems without equilibrium points. Referring to discrete-time systems, the proposed framework presents an additional features, i.e., it enables *exact* synchronization to be achieved in *finite time* (*dead-beat* synchronization). In Section 4 the advantages of the conceived framework are illustrated in detail, using a Table that summarizes the main steps to get synchronized dynamics in a systematic way. Finally, Sections 5 and 6 present examples of different types of synchronization for both continuous-time and discrete-time systems, including a chaotic system without equilibrium point that presents the feature of having one variable with the freedom of offset boosting [19]. The considered examples include some generalizations of the full state hybrid projective synchronization, the dislocated synchronization, the identical (complete) synchronization, the projective synchronization and the anti-phase synchronization.

2. General unified framework for synchronization: continuous-time systems (with or without equilibria)

This Section introduces the part of the framework related to continuous-time systems, with or without equilibrium points. Specifically, this Section focuses on the class of drive systems defined by the following continuous-time state and output equations, respectively:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}f(\boldsymbol{x}(t)) + \boldsymbol{c}\,, \tag{1}$$

$$y(t) = f(\boldsymbol{x}(t)) + \boldsymbol{k}\boldsymbol{x}(t), \tag{2}$$

where $\boldsymbol{x}(t) \in \Re^{n\times 1}$ is the state vector, $\boldsymbol{A} \in \Re^{n\times n}$, $\boldsymbol{b} \in \Re^{n\times 1}$ and $\boldsymbol{c} \in \Re^{n\times 1}$ are constant matrices, $f : \Re^n \rightarrow \Re$ is a scalar nonlinear function, $y(t)$ is a scalar synchronizing output and $\boldsymbol{k} \in \Re^{1\times n}$ is a gain vector to be determined. As pointed out in [5], the class described by (1) includes several well-known chaotic (hyperchaotic) systems (Chua's circuit, Rössler's system, the Matsumoto-Chua-Kobayashi circuit, the higher dimensional Chua's circuit and the hyperchaotic oscillator implemented in [8]). All these systems are characterized by the presence of one or more unstable equilibrium points. However, the class (1) also includes a number of special chaotic systems without equilibrium points, known as "chaotic systems with hidden attractor" [19, 20].

Now a *general* type of synchronization error is defined, which holds for any type of synchronization defined to date. Namely, the drive system (1) and the n-dimensional state vector $\hat{\boldsymbol{x}}(t)$, describing the response system dynamics, are said to be synchronized (according to the proposed *general* sense) when it results:

$$\|e_i(t)\| = \left\| \left(x_i(t) - \sum_{j=1}^{n} \alpha_{ij}\hat{x}_j(t) \right) \right\| \rightarrow 0 \quad \text{as} t \rightarrow \infty, \quad i = 1, 2, \ldots n, \tag{3}$$

where α_{ij} ($i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$) represent the elements of a scaling matrix and, consequently, the weights of the linear combination in (3). The *general* definition of synchronization error proposed herein, i.e. $e_i(t) = \left(x_i(t) - \sum\limits_{j=1}^{n} \alpha_{ij}\hat{x}_j(t) \right)$, includes any type of synchronization defined so far. Namely, when $\alpha_{11} = \alpha_{22} = \ldots = \alpha_{nn} = 1$ and $\alpha_{ij} = 0$, $i \neq j$, the condition (3) represents the *identical* (*complete*) *synchronization*, whereas for $\alpha_{11} = \alpha_{22} = \ldots = \alpha_{nn} = -1$ and $\alpha_{ij} = 0$, $i \neq j$, it represents the *anti-phase synchronization*. Moreover, when $\alpha_{11} = \alpha_{22} = \ldots = \alpha_{nn} = \alpha$ and $\alpha_{ij} = 0$, $i \neq j$, the *projective synchronization* as proposed in [11–13] is obtained. On the other hand, when $\alpha_{11} \neq \alpha_{22} \neq \ldots \neq \alpha_{nn}$ and $\alpha_{ij} = 0$, $i \neq j$, the condition (3) represents the *full state hybrid projective synchronization* as defined in [14–16], whereas for $\alpha_{11} = \alpha_{22} = \ldots = \alpha_{nn} = 0$ and $\alpha_{ij} \neq 0$, $i \neq j$ it describes the *dislocated synchronization* [21]. Finally, when $\alpha_{11} \neq \alpha_{22} \neq \ldots \neq \alpha_{nn}$ and $\alpha_{ij} \neq 0$, $i \neq j$, a more general type of synchronization can be obtained (more general than the types reported in [14–16]). Since in this case each drive system variable $x_i(t)$ synchronizes with a linear combination of response system variables $(\alpha_{i1}\hat{x}_1(t) + \alpha_{i2}\hat{x}_2(t) + \ldots\ \alpha_{in}\hat{x}_n(t))$, this synchronization type has been recently referred as *arbitrary full state hybrid projective synchronization* [22].

Given the drive system (1), the observed-based approach [5] enables the state $\boldsymbol{x}(t)$ to be reconstructed by a response system (described by state vector $\hat{\boldsymbol{x}}(t)$) via a scalar function of the states $y(t)$ and its prediction $\hat{y}(t)$. The following proposition is now given.

Proposition 1 *Given the drive system (1)–(2), let*

$$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{\alpha}^{-1}\boldsymbol{A}\boldsymbol{\alpha}\hat{\boldsymbol{x}}(t) + \boldsymbol{\alpha}^{-1}\boldsymbol{b}f(\hat{\boldsymbol{x}}(t)) + \boldsymbol{\alpha}^{-1}\boldsymbol{c} + \boldsymbol{\alpha}^{-1}\boldsymbol{b}\,(y(t) - \hat{y}(t)), \tag{4}$$

$$\hat{y}(t) = f(\hat{\boldsymbol{x}}(t)) + \boldsymbol{k}\boldsymbol{\alpha}\hat{\boldsymbol{x}}(t) \tag{5}$$

be the response system in the observer form, where $\boldsymbol{\alpha} = [\alpha_{ij}]$, $i = 1, \ldots n$, $j = 1, \ldots n$ *is the n-th order invertible matrix derived from (3). The response system (4)–(5) and the drive system (1)–(2) achieve synchronization for any scaling matrix* $\boldsymbol{\alpha} = [\alpha_{ij}]$, *provided that all the uncontrollable eigenvalues of the synchronization error system, if any, have negative real parts.*

Proof By exploiting the scaling matrix $\boldsymbol{\alpha} = [\alpha_{ij}]$, the condition (3) can be written in matrix form as

$$\|\boldsymbol{e}(t)\| = \|(\boldsymbol{x}(t) - \boldsymbol{\alpha}\hat{\boldsymbol{x}}(t))\|. \tag{6}$$

By taking into account the drive system (1)–(2) and response system (4)–(5), the synchronization error can be written as:

$$\begin{aligned}
\dot{\boldsymbol{e}}(t) &= \left(\dot{\boldsymbol{x}}(t) - \boldsymbol{\alpha}\dot{\hat{\boldsymbol{x}}}(t)\right) \\
&= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}\,f(\boldsymbol{x}(t)) + \boldsymbol{c} - \boldsymbol{A}\boldsymbol{\alpha}\hat{\boldsymbol{x}}(t) - \boldsymbol{b}f(\hat{\boldsymbol{x}}(t)) - \boldsymbol{c} - \boldsymbol{b}\,(y(t) - \hat{y}(t)) \\
&= \boldsymbol{A}\,(\boldsymbol{x}(t) - \boldsymbol{\alpha}\hat{\boldsymbol{x}}(t)) + \boldsymbol{b}f(\boldsymbol{x}(t)) - \boldsymbol{b}f(\hat{\boldsymbol{x}}(t)) \\
&\quad - \boldsymbol{b}\,(f(\boldsymbol{x}(t)) + \boldsymbol{k}\boldsymbol{x}(t) - f(\hat{\boldsymbol{x}}(t)) - \boldsymbol{k}\boldsymbol{\alpha}\hat{\boldsymbol{x}}(t)) \\
&= \boldsymbol{A}\,(\boldsymbol{x}(t) - \boldsymbol{\alpha}\hat{\boldsymbol{x}}(t)) - \boldsymbol{b}\boldsymbol{k}\,(\boldsymbol{x}(t) - \boldsymbol{\alpha}\hat{\boldsymbol{x}}(t)) = \boldsymbol{A}\,\boldsymbol{e}(t) - \boldsymbol{b}\,\boldsymbol{k}\,\boldsymbol{e}(t).
\end{aligned} \tag{7}$$

Thus, the error dynamics is described by the linear time invariant system

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{A}\,\boldsymbol{e}(t) + \boldsymbol{b}u(t), \tag{8}$$

where $u(t) = -\boldsymbol{k}\,\boldsymbol{e}(t)$ plays the role of a state feedback. The error system (8) can be stabilized at the origin by properly computing the gain vector $\boldsymbol{k} \in \Re^{1\times n}$. To this purpose, note that for (8) a coordinate transformation $\boldsymbol{e} = [\boldsymbol{T}_1 \;\; \boldsymbol{T}_2]\overline{\boldsymbol{e}}$ can be found [5], where $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ generate the controllable-state and uncontrollable-state subspaces, respectively. By defining $\overline{\boldsymbol{A}}_c = \boldsymbol{T}_1^T\boldsymbol{A}\boldsymbol{T}_1$, $\overline{\boldsymbol{A}}_{12} = \boldsymbol{T}_1^T\boldsymbol{A}\boldsymbol{T}_2$, $\overline{\boldsymbol{A}}_{nc} = \boldsymbol{T}_2^T\boldsymbol{A}\boldsymbol{T}_2$ and $\overline{\boldsymbol{b}}_c = \boldsymbol{T}_1^T\boldsymbol{b}$, system (8) can be transformed to the Kalman controllable canonical form [5]:

$$\begin{bmatrix} \dot{\overline{\boldsymbol{e}}}_c(t) \\ \dot{\overline{\boldsymbol{e}}}_{nc}(t) \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{A}}_c & \overline{\boldsymbol{A}}_{12} \\ \boldsymbol{0} & \overline{\boldsymbol{A}}_{nc} \end{bmatrix} \begin{bmatrix} \overline{\boldsymbol{e}}_c(t) \\ \overline{\boldsymbol{e}}_{nc}(t) \end{bmatrix} + \begin{bmatrix} \overline{\boldsymbol{b}}_c \\ \boldsymbol{0} \end{bmatrix} u(t), \tag{9}$$

where the eigenvalues of $\overline{\boldsymbol{A}}_c$ are controllable, i.e., they can be placed anywhere by state feedback $u(t) = -\boldsymbol{k}\boldsymbol{e}(t)$, whereas the eigenvalues of $\overline{\boldsymbol{A}}_{nc}$ are uncontrollable, i.e., they are not affected by the introduction of any state feedback. Therefore, system (9) can globally asymptotically stabilized by suitable $\boldsymbol{k}$, provided that the eigenvalues of $\overline{\boldsymbol{A}}_{nc}$ lie in the left half plane [5]. Since $\|\overline{\boldsymbol{e}}(t)\| \to 0$ implies $\|\boldsymbol{e}(t)\| \to 0$, this means that the response system (4) and the drive system (1) are synchronized for any weight matrix $\boldsymbol{\alpha}$ via the scalar synchronizing signal (2). □

Now the applicability of the approach is discussed. As pointed out earlier, the class (1) includes several well-known chaotic (hyperchaotic) systems with equilibrium points (Chua's circuit, Rössler's system, the Matsumoto-Chua-Kobayashi circuit, the oscillator in [8]) as well as a number of chaotic systems without equilibrium points [19]. By considering, for each of these systems, the matrices $\boldsymbol{A}$ and $\boldsymbol{b}$ in (1), and by computing the matrices

$$\begin{bmatrix} \overline{\boldsymbol{A}}_c & \overline{\boldsymbol{A}}_{12} \\ \boldsymbol{0} & \overline{\boldsymbol{A}}_{nc} \end{bmatrix} \tag{10}$$

it can be readily shown that $\overline{\boldsymbol{A}}_{nc}$ is the null matrix for each of these systems. As a consequence, the eigenvalues of the error system (9) are all controllable, i.e., they can be placed anywhere by suitable $\boldsymbol{k}$. This clearly indicates the wide applicability of the approach, which can be successfully applied to several chaotic and hyperchaotic systems, with or without equilibrium points (see also the examples in Section 5).

3. General unified framework for synchronization: discrete-time systems

This Section describes the part of the framework related to the synchronization of discrete-time systems. Similarly to the previous Section, the starting point is a *general* definition of synchronization error. To this purpose, given the discrete-time drive and response systems described by the n-dimensional state vectors $\boldsymbol{x}(k)$ and $\hat{\boldsymbol{x}}(k)$, respectively, those two systems are said to be synchronized when, for an initial condition $\hat{\boldsymbol{x}}(0)$, it results

$$\|e_i(k)\| = \left\| \left(x_i(k) - \sum_{j=1}^{n} \alpha_{ij}\hat{x}_j(k) \right) \right\| \to 0 \quad \text{as} \quad k \to \infty, \;\; i = 1, 2, \ldots, n. \tag{11}$$

where α_{ij} ($i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$) represent the weights of the linear combination in (11). Similarly to the previous Section, a remarkable advantage of the *general* definition of synchronization error provided herein, i.e. $e_i(k) = \left(x_i(k) - \sum_{j=1}^{n} \alpha_{ij}\hat{x}_j(k) \right)$, consists in the fact that it includes any type of synchronization defined so far, i.e., *identical* (*complete*) *synchronization, anti-phase synchronization, projective synchronization, full state hybrid projective synchronization* and *dislocated synchronization*. Finally, when $\alpha_{11} \neq \alpha_{22} \neq \ldots \neq \alpha_{nn}$ and $\alpha_{ij} \neq 0$, $i \neq j$, a more general type of synchronization can be obtained, where each drive system variable $x_i(k)$ synchronizes with a linear combination of response system variables $(\alpha_{i1}\hat{x}_1(k) + \alpha_{i2}\hat{x}_2(k) + \ldots \alpha_{in}\hat{x}_n(k))$, $i = 1, 2, \ldots, n$. In discrete-time systems this synchronization type has been recently referred as *arbitrary full state hybrid projective synchronization* [23].

Referring to the systems to be synchronized, this Section focuses on the class of discrete-time systems defined by the following state and output equations, respectively:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}f(\boldsymbol{x}(k)) + \boldsymbol{c}, \tag{12}$$

$$y(k) = f(\boldsymbol{x}(k)) + \boldsymbol{k}\boldsymbol{x}(k), \tag{13}$$

where $\boldsymbol{x}(k) \in \Re^{n\times 1}$, $\boldsymbol{A} \in \Re^{n\times n}$, $\boldsymbol{b} \in \Re^{n\times 1}$, $\boldsymbol{c} \in \Re^{n\times 1}$, $f : \Re^n \to \Re$, $y(k)$ is a scalar synchronizing output, and $\boldsymbol{k} \in \Re^{1\times n}$ is a gain vector to be determined. The class described by (12) includes several chaotic and hyperchaotic discrete-time systems, that is, the logistic map, the cubic map, the Duffing map, the Gauss map, the Lozi map and the generalized Henon map [24]. The class (12) also includes the so-called Grassi-Miller map [16], the gingerbreadman map [18] and the double scroll map [25].

In order to obtained a *unified* framework for synchronization (i.e., applicable to both continuous-time and discrete-time systems), this Section will exploit an observer-based approach similar to that developed in the previous Section.

Proposition 2 *Given the matrices* $\boldsymbol{A}$ *and* $\boldsymbol{b}$ *in (12), if the matrix*

$$\begin{bmatrix}\boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \boldsymbol{A}^2\boldsymbol{b} & \ldots & \boldsymbol{A}^{n-1}\boldsymbol{b}\end{bmatrix} \tag{14}$$

is full rank, then the following response system in the observer form

$$\hat{\boldsymbol{x}}(k+1) = \boldsymbol{\alpha}^{-1}\boldsymbol{A}\boldsymbol{\alpha}\hat{\boldsymbol{x}}(k) + \boldsymbol{\alpha}^{-1}\boldsymbol{b}f(\hat{\boldsymbol{x}}(k)) + \boldsymbol{\alpha}^{-1}\boldsymbol{c} + \boldsymbol{\alpha}^{-1}\boldsymbol{b}\left(y(k) - \hat{y}(k)\right), \tag{15}$$

$$\hat{y}(k) = f(\hat{\boldsymbol{x}}(k)) + \boldsymbol{k}\boldsymbol{\alpha}\hat{\boldsymbol{x}}(k) \tag{16}$$

and the drive system (12)–(13) achieve dead-beat synchronization for any invertible scaling matrix $\boldsymbol{\alpha} = \left[\alpha_{ij}\right]$, $i = 1, \ldots n$, $j = 1, \ldots n$, *provided that the eigenvalues of* $[\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}]$ *are placed at zero by suitable* $\boldsymbol{k}$.

Proof First of all, it is worth noting that the condition (11), applied to drive and response systems, can be written in matrix form as

$$\|\boldsymbol{e}(k)\| = \|(\boldsymbol{x}(k) - \boldsymbol{\alpha}\hat{\boldsymbol{x}}(k))\|. \tag{17}$$

By taking into account equations (12)–(13) and (15)–(16), it follows that the error dynamics is described by:

$$\begin{aligned}\boldsymbol{e}(k+1) &= (\boldsymbol{x}(k+1) - \boldsymbol{\alpha}\hat{\boldsymbol{x}}(k+1)) \\ &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}f(\boldsymbol{x}(k)) + \boldsymbol{c} - \boldsymbol{A}\boldsymbol{\alpha}\hat{\boldsymbol{x}}(k) - \boldsymbol{b}f(\hat{\boldsymbol{x}}(k)) - \boldsymbol{c} - \boldsymbol{b}\,(y(k) - \hat{y}(k)) \\ &= \boldsymbol{A}\,(\boldsymbol{x}(k) - \boldsymbol{\alpha}\hat{\boldsymbol{x}}(k)) - \boldsymbol{b}\,\boldsymbol{k}\,(\boldsymbol{x}(k) - \boldsymbol{\alpha}\hat{\boldsymbol{x}}(k)) \\ &= \boldsymbol{A}\boldsymbol{e}(k) - \boldsymbol{b}\,\boldsymbol{k}\,\boldsymbol{e}(k).\end{aligned} \tag{18}$$

The error dynamics (18) can be written as

$$\boldsymbol{e}(k+1) = \boldsymbol{A}\boldsymbol{e}(k) + \boldsymbol{b}u(k) \tag{19}$$

with $u(k) = -\boldsymbol{k}\boldsymbol{e}(k)$. Since the matrix (14), which represents the controllability matrix of the linear time-invariant error system (19), is full rank by assumption, all the eigenvalues of (19) can be placed at the origin by suitable $\boldsymbol{k}$, indicating that dead beat control is achieved [26]. Thus, the error dynamics will reach exactly zero in n steps. This means that drive and response systems will exactly synchronize for any scaling matrix $\boldsymbol{\alpha}$, confirming that any type of synchronization is achievable in n steps. □

Referring to the applicability of the method, it can be readily verified that the matrix (14) is full rank for several well-known chaotic (hyperchaotic) discrete-time systems. Specifically, all the maps mentioned at the beginning of this Section satisfy the condition that the matrix (14) is full rank. According to *Proposition 2*, all these chaotic (hyperchaotic) discrete-time systems achieve dead beat synchronization for any scaling matrix $\boldsymbol{\alpha}$, confirming the wide applicability of the conceived technique.

4. Comments on the proposed general unified framework for synchronization

The aim of this Section is to show that the proposed observer-based approach represents a *general unified* framework for synchronizing chaotic (hyperchaotic) systems via a scalar signal. By "*general*" it is meant that any type of synchronization defined to date is achievable using the proposed scheme. By "*unified*" it is meant that the framework holds for both continuous-time systems (with or without equilibrium points) as well as discrete-time systems.

In order to summarize the main features of the proposed framework, a Table has been built (see Table 1).

By looking at Table 1, it should be clear that the proposed observer-based approach truly represents a unified framework for achieving any type of synchronization defined to date. The most important advantages of the framework can be summarized as follows:

Table 1: Main features of the proposed general unified framework for synchronization

	CONTINUOUS-TIME SYSTEMS (WITH OR WITHOUT EQUILIBRIA)	DISCRETE-TIME SYSTEMS
Drive system	$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}f(\boldsymbol{x}(t)) + \boldsymbol{c}$	$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{b}f(\boldsymbol{x}(k)) + \boldsymbol{c}$
Response system	$\dot{\hat{\boldsymbol{x}}}(t) = \boldsymbol{\alpha}^{-1}\boldsymbol{A}\boldsymbol{\alpha}\hat{\boldsymbol{x}}(t) + \boldsymbol{\alpha}^{-1}\boldsymbol{b}f(\hat{\boldsymbol{x}}(t)) + \boldsymbol{\alpha}^{-1}\boldsymbol{c} + \boldsymbol{\alpha}^{-1}\boldsymbol{b}(y(k) - \hat{y}(k))$	$\hat{\boldsymbol{x}}(k+1) = \boldsymbol{\alpha}^{-1}\boldsymbol{A}\boldsymbol{\alpha}\hat{\boldsymbol{x}}(k) + \boldsymbol{\alpha}^{-1}\boldsymbol{b}f(\hat{\boldsymbol{x}}(k)) + \boldsymbol{\alpha}^{-1}\boldsymbol{c} + \boldsymbol{\alpha}^{-1}\boldsymbol{b}(y(t) - \hat{y}(t))$
Scalar synchronizing signal	$y(t) = f(\boldsymbol{x}(t)) + \boldsymbol{k}\boldsymbol{x}(t)$	$y(k) = f(\boldsymbol{x}(k)) + \boldsymbol{k}\boldsymbol{x}(k)$
Synchronization error	$\boldsymbol{e}(t) = (\boldsymbol{x}(t) - \boldsymbol{\alpha}\hat{\boldsymbol{x}}(t))$	$\boldsymbol{e}(k) = (\boldsymbol{x}(k) - \boldsymbol{\alpha}\hat{\boldsymbol{x}}(k))$
Error dynamics	$\dot{\boldsymbol{e}}(t) = \boldsymbol{A}\boldsymbol{e}(t) + \boldsymbol{b}u(t), \quad u(t) = -\boldsymbol{k}\boldsymbol{e}(t)$	$\boldsymbol{e}(k+1) = \boldsymbol{A}\boldsymbol{e}(k) + \boldsymbol{b}u(k), \quad u(k) = -\boldsymbol{k}\boldsymbol{e}(k)$
Structural condition	The eigenvalues of $\overline{\boldsymbol{A}}_{nc}$ must lie in the left half plane	The matrix $\left[\boldsymbol{b}\ \ \boldsymbol{A}\boldsymbol{b}\ \ \boldsymbol{A}^2\boldsymbol{b}\ \ \ldots\ \ \boldsymbol{A}^{n-1}\boldsymbol{b}\right]$ must be full rank
Applicability	Wide class of well-known chaotic systems (with or without equilibrium points)	Wide class of well-known chaotic maps
Design parameter	Choose $\boldsymbol{k}$ in the synchronizing signal so that the eigenvalues of the error system are placed in the left half plane	Choose $\boldsymbol{k}$ in the synchronizing signal so that the eigenvalues of the error system are placed at the origin
Synchronization types	Any type defined to date is achievable	Any type defined to date is achievable
Time to synchronization	Asymptotic ($\lVert\boldsymbol{e}(t)\rVert \to 0$ for $t \to \infty$)	Dead beat ($\lVert\boldsymbol{e}(k)\rVert = 0$ at most for $k = n$)

i) it can be applied to several chaotic (hyperchaotic) continuous-time systems, including those without equilibria;

ii) it can be applied to several chaotic (hyperchaotic) maps;

iii) it enables any type of synchronization defined to date to be achievable;

iv) it adopts a scalar synchronizing signal only;

v) it represents a *rigorous* approach to synchronization (being the method based on two propositions);

vi) it represents a *systematic* approach to synchronization (being the technique based on some specified steps, see Table 1);

vii) it can be readily applied, since the only design parameter is $\boldsymbol{k}$ (to be computed according to Table 1).

As a final remark, we would pointed that, regardless of the fact that a system has equilibrium or not, the control problem remains the same. Consequently, through the paper we do not claim that we are introducing "for the first time chaos synchronization in systems without equilibria". Rather than this, we *are applying* the proposed design tool to systems without equilibria, showing that synchronization can be effectively achieved in these systems, provided that the uncontrollable eigenvalues of the error system, if any, have negative real parts.

5. Examples of different types of synchronization: continuous-time systems with and without equilibria

This Section illustrates some examples of different types of synchronization that can be achieved in continuous-time systems with equilibrium points as well as without equilibrium points.

5.1. Systems with equilibria: generalization of full state hybrid projective synchronization and dislocated synchronization

The 4-D hyperchaotic Rössler's system [5] is a well-known example of continuous-time system characterized by (unstable) equilibrium points. Its hyperchaotic dynamics are represented in the form (1) as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_1(t)x_3(t) + \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}. \tag{20}$$

Given the matrices $\boldsymbol{A}$ and $\boldsymbol{b}$ in (20), it can be readily shown that $\overline{\boldsymbol{A}}_{nc}$ is the null matrix. As a consequence, all the eigenvalues of the error system (9) are controllable, i.e., they can be placed anywhere by suitable $\boldsymbol{k}$. By placing them at -1, the output of (12) becomes the scalar signal

$$y(t) = x_1(t)x_3(t) + [-3.3712 \quad -0.9561 \quad 4.3 \quad -5.8126] \times [x_1(t) \;\; x_2(t) \;\; x_3(t) \;\; x_4(t)]^T, \tag{21}$$

whereas the dynamics of the observer (4) are described by

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \\ \dot{\hat{x}}_3(t) \\ \dot{\hat{x}}_4(t) \end{bmatrix} = \boldsymbol{\alpha}^{-1} \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0.25 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0.05 \end{bmatrix} \boldsymbol{\alpha} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \\ \hat{x}_4(t) \end{bmatrix} + \boldsymbol{\alpha}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_1(t)x_3(t) + \boldsymbol{\alpha}^{-1} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \boldsymbol{\alpha}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} (y(t) - \hat{y}(t)), \tag{22}$$

$$\hat{y}(t) = \hat{x}_1(t)\hat{x}_3(t) + [-3.3712 \quad -0.9561 \quad 4.3 \quad -5.8126] \times \boldsymbol{\alpha} [\hat{x}_1(t) \;\; \hat{x}_2(t) \;\; \hat{x}_3(t) \;\; \hat{x}_4(t)]^T, \tag{23}$$

where $\boldsymbol{\alpha}$ is any invertible fourth-order scaling matrix whereas (23) is the observer prediction of the output (21). Proposition 1 guarantees that the response system (22) and the drive system (20) achieve any type of synchronization via the scalar control signal (21).

In order to show the effectiveness of the approach, simulations have been carried out by choosing the following scaling matrix:

$$\boldsymbol{\alpha} = \begin{bmatrix} 1 & 3 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \tag{24}$$

Figure 1 depicts the attractor of the drive system (20) in the (x_1, x_2)-plane (left) and the attractor of the response system (22) in the $(\hat{x}_1, \hat{x}_2)$-plane (right). According to (24), the variable $x_1(t)$ is synchronized with the linear combination of response system variables given by $\hat{x}_1(t) + 3\hat{x}_2(t) + 4\hat{x}_4(t)$, whereas the variable $x_2(t)$ is synchronized with $2\hat{x}_2(t)$. Clearly, the shape of the response system attractor in the $(\hat{x}_1, \hat{x}_2)$-plane is completely different from the shape of the drive system attractor in the (x_1, x_2)-plane, although the two systems achieve synchronized dynamics via the scalar signal (21). This type of synchronization represents a *generalization* of the *full state hybrid projective synchronization* (recently referred as *arbitrary full state hybrid projective synchronization* [22]).

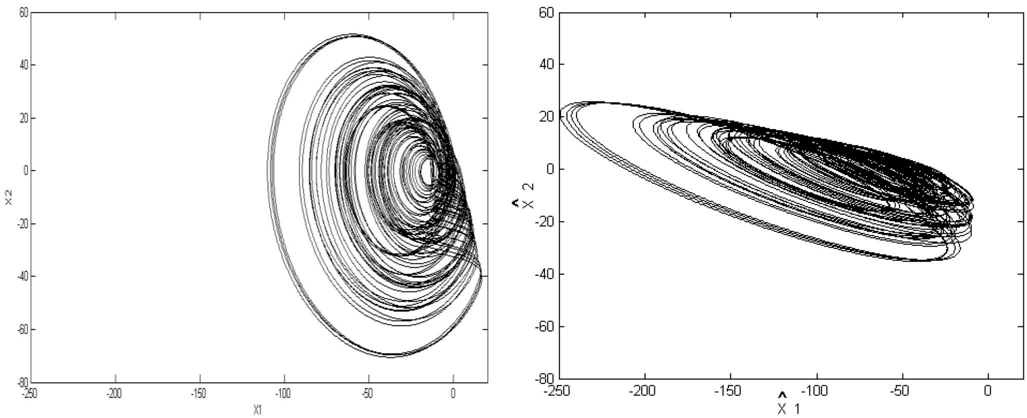

Figure 1: Generalization of the full state hybrid projective synchronization: attractor of the drive system (20) in the (x_1, x_2)-plane (left) and attractor of the response system (22) in the $(\hat{x}_1, \hat{x}_2)$-plane (right). Note that the shapes of the attractors in the two plots are completely different

Now an example of *dislocated synchronization* [21] is shown. Simulations have been carried out by considering the drive system (20)–(21) and by taking the response system (22)–(23) with the following scaling matrix:

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 4 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ -2 & -4 & -6 & 1 \end{bmatrix}. \tag{25}$$

By looking at (25), it is clear that the drive variables $x_1(t)$, $x_2(t)$ and $x_3(t)$ are allowed to synchronize with all the response system variables except $\hat{x}_1(t)$, $\hat{x}_2(t)$ and $\hat{x}_3(t)$, respectively.

From (25) it follows that $x_2(t)$ is synchronized with $4\hat{x}_1(t)$, whereas the variable $x_3(t)$ is synchronized with $5\hat{x}_2(t)$. Figure 2 plots the drive system attractor in the (x_2, x_3)-plane (left) and the response system attractor in the $(\hat{x}_2, \hat{x}_3)$-plane (right).

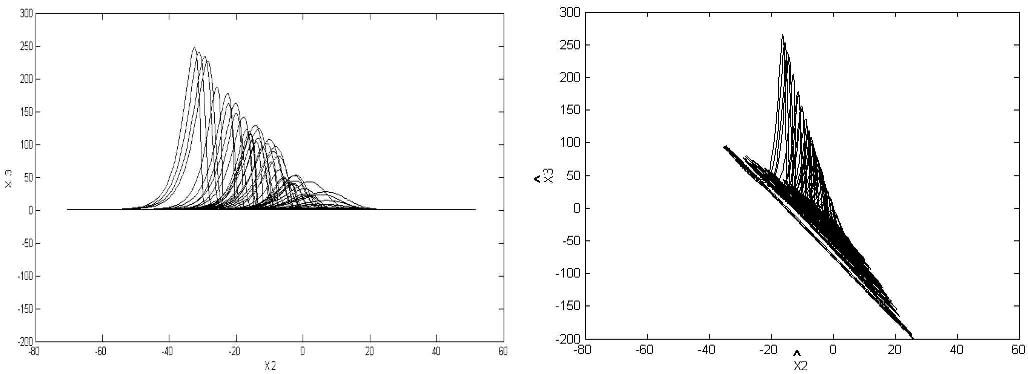

Figure 2: Dislocated synchronization: drive system attractor in the (x_2, x_3)-plane (left) and response system attractor in the $(\hat{x}_2, \hat{x}_3)$-plane (right)

5.2. Systems without equilibria: identical (complete) synchronization

In recent years, there has been an increasing interest in special chaotic systems without equilibrium points [19–20]. They are called "chaotic systems with hidden attractors", since the absence of equilibria makes difficult to identify the location of the attractors [19–20]. Herein, attention is focused on a recently reported system, which presents the additional feature of having one variable with the freedom of offset boosting [19]. The system dynamics can be written in the form (1) as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(-0.8(x_1(t))^2 + (x_3(t))^2\right) + \begin{bmatrix} a \\ 0 \\ 2 \end{bmatrix}. \tag{26}$$

It can be readily shown that system (26) has no equilibrium point for any positive value of a. Moreover, note that the variable $x_2(t)$ can be boosted [19] by varying the parameter a. Despite the absence of equilibria, system (26) possesses chaotic attractors for any positive value of a, as shown in reference [19].

Since the field of synchronizing chaotic systems without equilibrium points is almost unexplored, now it will be shown that the conceived tool can be also applied to these systems. By choosing $a = 1$, it can be readily shown that the matrix $\overline{\boldsymbol{A}}_{nc}$ derived from (26) is the null matrix, indicating that all the eigenvalues of the error system (9) are controllable. By placing them at $[-1 \quad -2 \quad -3]$, the output of (26) becomes the scalar control signal

$$y(t) = (-0.8(x_1(t))^2 + (x_3(t))^2) + [0\ \ 10\ \ 6]\,[x_1(t)\ \ x_2(t)\ \ x_3(t)]^T. \tag{27}$$

The observer of (26) can be written in the form (4) as

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \\ \dot{\hat{x}}_3(t) \end{bmatrix} = \boldsymbol{\alpha}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{\alpha} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \end{bmatrix} + \boldsymbol{\alpha}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(-0.8(x_1(t))^2 + (x_3(t))^2\right)$$
$$+\ \boldsymbol{\alpha}^{-1} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \boldsymbol{\alpha}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (y(t) - \hat{y}(t)), \tag{28}$$

$$\hat{y}(t) = \left(-0.8(\hat{x}_1(t))^2 + (\hat{x}_3(t))^2\right) + [0\ \ 10\ \ 6]\boldsymbol{\alpha}[\hat{x}_1(t)\ \ \hat{x}_2(t)\ \ \hat{x}_3(t)]^T, \tag{29}$$

where equation (29) represents the observer prediction of the control signal (27). Although $\boldsymbol{\alpha}$ can be any invertible third-order scaling matrix, herein the identity matrix is selected:

$$\boldsymbol{\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30}$$

with the aim to show that identical (complete) synchronization can be readily achieved. To this purpose, Figure 3 (left) displays the time-behavior of the variables $x_1(t)$ and $\hat{x}_1(t)$, whereas Figure 3 (right) plots the time-behavior of the variables $x_2(t)$ and $\hat{x}_2(t)$. From these pictures it can be seen that the drive system variables quickly track the response system variables, indicating that identical (complete) synchronization is effectively achieved between the drive system (26) and the response system (28).

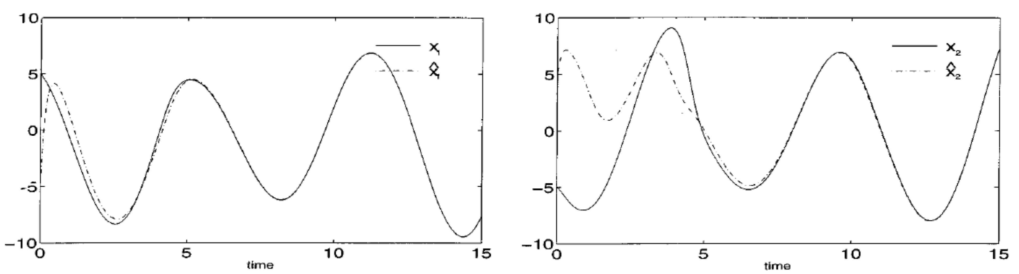

Figure 3: Identical (complete) synchronization in chaotic systems without equilibria: time-behaviors for the variables $x_1(t)$ and $\hat{x}_1(t)$ (left) and for the variables $x_2(t)$ and $\hat{x}_2(t)$ (right). The time-scale has been selected with the aim to show how the drive system variables track the response system variables

6. Examples of different types of synchronization: discrete-time systems

Examples of generalization of full state hybrid projective synchronization as well as anti-phase synchronization will be illustrated herein. The approach is applied to a novel piecewise-linear map, which is capable of displaying a hyperchaotic attractor referred as "the discrete hyperchaotic double-scroll" [25]. The system is described by the following difference equations [25]:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} -a \\ 0 \end{bmatrix} h(x_2(k)), \tag{31}$$

$$h(x_2(k)) = (2m_1x_2(k) + (m_0 - m_1)(|x_2(k)+1| - |x_2(k)-1|))/2, \tag{32}$$

where a and b are bifurcation parameters whereas (32) is the piecewise linear function originally introduced in Chua's circuit. System (31) displays the hyperchaotic double scroll (see Fig. 4a) when $a = 3.36$, $b = 1$, $m_0 = -0.43$ and $m_1 = 0.41$. Since the matrix (14) is full rank, the eigenvalues of $[\boldsymbol{A} - \boldsymbol{bk}]$ are placed at zero by $\boldsymbol{k} = [-0.2976 \;\; 0]$, indicating that the scalar control signal (13) can be written as:

$$y(k) = h(x_2(k)) - 0.2976x_1(k). \tag{33}$$

Thus, the nonlinear observer (15)-(16) is described by

$$\begin{aligned} \begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \end{bmatrix} = \boldsymbol{\alpha}^{-1} \begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix} \boldsymbol{\alpha} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} + \boldsymbol{\alpha}^{-1} \begin{bmatrix} -a \\ 0 \end{bmatrix} h(\hat{x}_2(k)) \\ + \boldsymbol{\alpha}^{-1} \begin{bmatrix} -a \\ 0 \end{bmatrix} (y(k) - \hat{y}(k)), \end{aligned} \tag{34}$$

$$\hat{y}(k) = h(\hat{x}_2(k)) + [-0.2976 \;\; 0]\, \boldsymbol{\alpha}\, [\hat{x}_1(k) \;\; \hat{x}_2(k)]^T. \tag{35}$$

According to *Proposition 2*, any type of dead-beat synchronization is achieved between systems (31) and (34), that is, the error dynamics will reach exactly zero after two steps for any choice of the invertible scaling matrix $\boldsymbol{\alpha}$. In order to show the effectiveness of the approach, some simulations have been carried out for different scaling matrices. For example, by taking the following full matrix:

$$\boldsymbol{\alpha} = \begin{bmatrix} 0.5 & -1 \\ 1 & 2 \end{bmatrix}, \tag{36}$$

the results are shown in Figure 4, which depicts the hyperchaotic attractor of the drive system (31) in the (x_1, x_2)-plane and the attractor of the response system (34) in the $(\hat{x}_1, \hat{x}_2)$-plane. Note that this type of synchronization represents a *generalization* of the full state hybrid projective synchronization, where the matrix is a diagonal one. This generalization has been recently referred as *arbitrary full state hybrid projective synchronization* [23]. The error dynamics for the variable $x_1(k)$ clearly indicate that the error is *exactly zero* after *two steps* (see Fig. 4c).

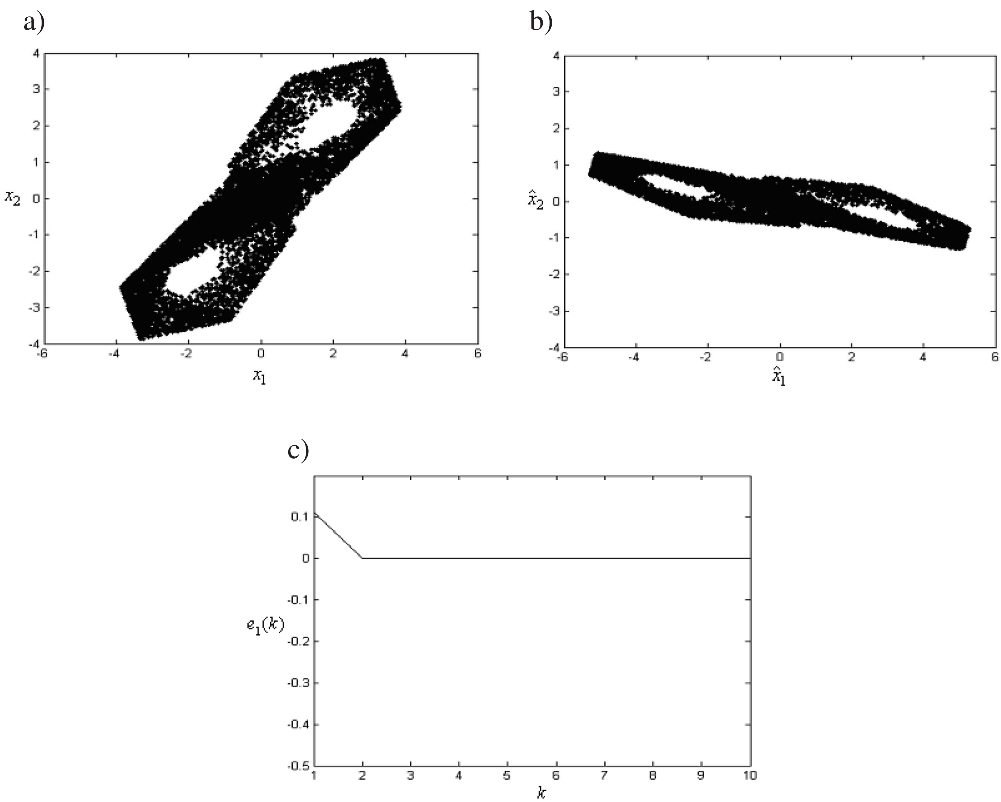

Figure 4: Generalization of full state hybrid projective synchronization: a) hyperchaotic attractor of the drive system (31) in the (x_1, x_2)-plane; b) attractor of the observer (34) in the $(\hat{x}_1, \hat{x}_2)$-plane (right) for the scaling matrix (36); c) the error $e_1(k)$ as a function of k

By properly selecting the matrix $\boldsymbol{\alpha}$, any type of synchronization can be achieved. For example, by taking (36) as $-\boldsymbol{I}$, that is:

$$\boldsymbol{\alpha} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{37}$$

the *anti-phase synchronization* is obtained. This synchronization type is displayed in Fig. 5, where the variables $x_1(k)$ and $\hat{x}_1(k)$ are plotted as a function of the time k. From Fig. 5 it can be observed that, for $k = 1$, $x_1(k)$ and $\hat{x}_1(k)$ assume different values, reported in black color and red color, respectively. However, starting from $k = 2$, the variable $x_1(k)$ assumes values that are the opposite of those assumed by the variable $\hat{x}_1(k)$ (and vice versa), indicating that the error is exactly zero after two steps and *anti-phase synchronization* is effectively achieved.

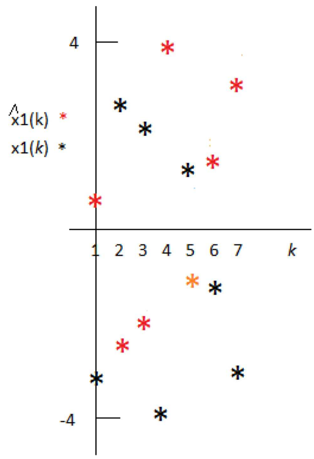

Figure 5: Anti-phase synchronization: the values of the variables $x_1(k)$ and $\hat{x}_1(k)$ are plotted as a function of k (in black color and red color, respectively). The error is exactly zero for k =2, indicating that dead-beat anti-phase synchronization is achieved

7. Conclusion

This paper has presented a novel *general unified* approach to synchronize chaotic systems via a scalar signal. The framework presented in Table 1 (Section 4) is the only one to **simultaneously** include the following remarkable features: i) it can be applied to several chaotic (hyperchaotic) continuous-time systems, including those without equilibria; ii) it can be applied to several chaotic (hyperchaotic) maps; iii) it enables any type of synchronization defined to date to be achievable; iv) it adopts a scalar synchronizing signal only; v) it represents a rigorous approach to synchronization (being the method based on two propositions); vi) it represents a systematic approach to synchronization (being the technique based on some specified steps); vii) it can be readily applied, since the only design parameter is the gain vector $\boldsymbol{k}$. We would stress that the original contribution of the present manuscript consists in providing a tool to manage any synchronization types, for both continuous-time systems (including those without equilibria) and discrete-time systems, in a general unified fashion. Several synchronization examples have been reported, for continuous-time systems (including a chaotic system with no-equilibrium and a variable with the freedom of offset boosting) as well as for discrete-time systems.

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