Central European Journal of Economic Modelling and Econometrics

Complex Dynamics in a Bertrand Duopoly Game with Heterogeneous Players

Tomasz Dubiel-Teleszyński*

Submitted: 17.08.2010, Accepted: 15.01.2011

Abstract

A heterogeneous Bertrand duopoly game with bounded rational and adaptive players manufacturing differentiated products is subject of investigation. The main goal is to demonstrate that participation of one bounded rational player in the game suffices to destabilize the duopoly. The game is modelled with a system of two difference equations. Evolution of prices over time is obtained by iteration of a two dimensional nonlinear map. Equilibria are found and local stability properties thereof are analyzed. Complex behavior of the system is examined by means of numerical simulations. Region of stability of the Nash equilibrium is demonstrated in the plane of the speeds of adjustment. Period doubling route to chaos is presented on the bifurcation diagrams and on the largest Lyapunov characteristic exponent graph. Lyapunov time is calculated. Chaotic attractors are depicted and their fractal dimensions are computed. Sensitive dependence on initial conditions is evidenced.

Keywords: Bertrand duopoly, heterogeneous expectations, nonlinear dynamics, chaos

JEL Classification: C72, D43, D84.

*Warsaw School of Economics, email: td23149@sgh.waw.pl

1 Introduction

Duopoly is a market structure characterized by a domination of two firms, which manufacture homogeneous products and completely control trade. It is this characteristic of interdependence that makes duopolist consider reaction of the competitor. Duopolist has to pay attention both to the market demand, that is the behavior of consumers, as well as to the strategy of the other firm, that is it forms expectations concerning how its rival will act. In a duopolistic competition both firms are either output or price setters; see Cournot (1838), Bertrand (1883).

The most widely used and simultaneously the first formal model of the duopoly market was proposed in 1838 by a French mathematician Antoine Augustin Cournot, who investigated a duopoly case with two output setters. Each company was assumed to adjust its quantity of production to that of its rival and there was no retaliation at all, so that in every step duopolist perceived the latest step taken by the competitor to remain his last. Thus, in the Cournot model each firm assumes that the other firm holds its output constant. In order to maximize profit, duopolist selects quantity to produce observing output of the other firm. Later, in 1883 another French mathematician Joseph Louis Francois Bertrand modified Cournot game suggesting that firms actually choose prices rather than quantities. Hence, in the Bertrand model each firm assumes that the competitor holds its price constant. Duopolist maximizes profit setting a price that undercuts competitor's price, until marginal cost has been reached.

Originally Cournot and Bertrand models were based on the premise that all players follow naive expectations under perfect information; see Gibbons (1992). However, in real market conditions such an assumption is very unlikely since not all players share naive beliefs. Therefore, different approaches to firm behavior were proposed. Players were not only perceived to be naive, but also bounded rational and adaptive; see Bischi and Naimzada (1999), Bischi and Kopel (2001). Numerous works on complex dynamics of the Cournot model were done under different expectations, both with homogeneous; see Puu (1991), (1998), Kopel (1996), Agiza (1998), (1999), Agiza, Hegazi, Elsadany (2001), (2002) and with heterogeneous players; see Agiza and Elsadany (2003), (2004), Leonard and Nishimura (1999), Den Haan (2001), Zhang, Da, Wang (2007). Published results confirm that under different expectations Cournot model reveals a complex dynamical behavior, such as cyclical and chaotic.

In turn, much less has been done in terms of investigating dynamical properties of the price duopoly by Bertrand under different expectations, especially heterogeneous. One of the very few works in this field is the recent article by Zhang, Da, Wang (2009). In the paper, authors applied the technique of Onazaki, Sieg, Yokoo (2003) to study dynamics of a homogeneous Bertrand duopoly model with two bounded rational players. Assuming differentiated products they examined the existence and the stability conditions of equilibria resulting from the game and by means of numerical simulations investigated complex dynamics of the model, showing period doubling route to chaos, strange chaotic attractors and sensitive dependence on initial conditions of the resulting two dimensional nonlinear map, among others. The reasoning behind analyzing dynamics of price duopoly games further is that Bertrand model is different than Cournot's. Strategy spaces and the payoff functions are different, thus behavior in the Nash equilibria of the two models varies as well (see Gibbons (1992)).

In this paper, an attempt is made to further fill the gap mentioned above, thereby contributing to the literature of the field. In particular, complex dynamics of a Bertrand duopoly game with heterogeneous players, having bounded rational and adaptive expectations, is investigated under differentiated products scheme. The main goal is to show that participation of one bounded rational player in the game suffices to destabilize the duopoly. The paper is organized in the following way. The starting point of the analysis constitutes a theoretical framework embracing forms of best replies resulting from general assumptions of a product differentiated price duopoly game detailed in Zhang, Da, Wang (2009), along with three diverse expectations concepts - naive, bounded rational and adaptive - all that outlined in section 2. Section 3 is a study of a heterogeneous Bertrand duopoly game with bounded rational and adaptive players, modelled with a two dimensional nonlinear map, where the existence of fixed points and local stability thereof are examined. Section 4 includes numerical simulations, applied so as to demonstrate the complexity of the obtained dynamical system. Region of stability, period doubling route to chaos, attracting chaotic sets along with the fractal dimensions as well as sensitive dependence of the map on initial conditions are presented. Finally, section 5 concludes obtained results.

2 Theoretical framework

Original Bertrand duopoly game is a model of price competition between two non-cooperating firms, which produce homogeneous products under linear demand and constant marginal costs equal for both players. Duopolists compete solely in price and choose their respective prices simultaneously, then supply the quantity demanded. Consumers buy everything from the cheaper producer or half from each, if the prices are equal. Each firm tries to reduce the price of its good, until the good sells at no profit. In the literature of game theory such a surprising result is called Bertrand paradox. The paradox seldom appears in practice since real products are almost always subject to differentiation in some way other than price. Hence, one way of avoiding the paradox is to enable firms sell differentiated products (see Bierman and Fernandez (1998)).

2.1 Best replies

Best replies, also named reaction functions, are obtained through equating partial derivatives of the expected profit functions to zero. Reaction functions can be monotonic, e.g. linear (Rassenti, Reynolds, Smith, Szidarovszky (2000)) and

nonmonotonic, e.g. in the form of a logistic map (Bischi and Kopel (2001)). Their monotonicity depends directly on the functional form resulting from combining demand and cost functions (Dana and Montrucchio (1986)) and indirectly on various economic conditions, which may give rise to nonmonotonicity (see Kopel (1996)). While Dana and Montrucchio (1986) assumed that quantity demanded is reciprocal to price and firms operate under constant marginal costs, Kopel (1996) applied linear inverse demand and bivariate cost functions, reflecting interfirm externalities. Both sets of assumptions led to unimodal reaction curves. Agiza and Elsadany (2003), (2004) used linear and decreasing inverse demand with linear cost functions and obtained monotonic, linear best replies. According to Puu (2005) the simplest conditions under which the Cournot quantity duopoly game leads to complex behavior, are isoelastic inverse demand and linear cost functions. Obvious advantage of such a mix is the feasibility of solving for reaction functions in a simple closed form. Zhang, Da, Wang (2009) found chaotic behavior in a homogeneous bounded rational Bertrand price duopoly game under linear demand and cost functions.
In this paper two firms X and Y produce differentiated goods, priced x_t and y_t respectively. Firms set prices at discrete time periods $t = 0, 1, 2, ...$ on a shared market. Both inverse demand and cost functions are assumed linear (see Zhang, Da, Wang (2009)). The quantities $Q_{X,t}$ and $Q_{Y,t}$ that firms X and Y sell accordingly are determined by the following equations

$$\begin{aligned} Q_{X,t} &= a - b \cdot x_t + d \cdot y_t \\ Q_{Y,t} &= a - b \cdot y_t + d \cdot x_t, \end{aligned} \tag{1}$$

where $a > 0$, $b > 0$ and $d > 0$. Since parameter d reflects the extent to which two products are substitutes for each other, its presence in equations (1) assures products differentiation and thus lets avoid Bertrand paradox. The cost functions are

$$\begin{aligned} C_X(x_t) &= c_X \cdot Q_{X,t} \\ C_Y(y_t) &= c_Y \cdot Q_{Y,t}, \end{aligned} \tag{2}$$

where $c_X > 0$ and $c_Y > 0$ are shift parameters to cost functions of firms X and Y respectively and simultaneously their corresponding marginal costs, which may differ. Under the assumptions made so far, single period profit functions for both players have the form

$$\begin{aligned} \pi^X(x_t, y_t) &= x_t \cdot Q_{X,t} - C_X(x_t) \\ \pi^Y(x_t, y_t) &= y_t \cdot Q_{Y,t} - C_Y(y_t). \end{aligned} \tag{3}$$

Inserting equations (1) and (2) into (3) yields

$$\begin{aligned} \pi^X(x_t, y_t) &= (x_t - c_X) \cdot (a - b \cdot x_t + d \cdot y_t) \\ \pi^Y(x_t, y_t) &= (y_t - c_Y) \cdot (a - b \cdot y_t + d \cdot x_t). \end{aligned} \tag{4}$$

Partial derivations of the first equation in (4) with respect to x_t and of the second equation in (4) with respect to y_t allow to obtain local estimates of the marginal profits in period t of firms X and Y respectively

$$\begin{aligned} \Phi_{X,t} &= \frac{\partial \pi^X(x_t, y_t)}{\partial x_t} = a + b \cdot c_X - 2 \cdot b \cdot x_t + d \cdot y_t \\ \Phi_{Y,t} &= \frac{\partial \pi^Y(x_t, y_t)}{\partial y_t} = a + b \cdot c_Y - 2 \cdot b \cdot y_t + d \cdot x_t. \end{aligned} \tag{5}$$

Optimization problems in (5) have unique solutions of the form

$$\begin{aligned} f(y_t) &= \tfrac{1}{2 \cdot b} \cdot (a + b \cdot c_X + d \cdot y_t) \\ g(x_t) &= \tfrac{1}{2 \cdot b} \cdot (a + b \cdot c_Y + d \cdot x_t), \end{aligned} \tag{6}$$

which let obtain linear best replies of firms X and Y accordingly.

2.2 Expectations

Three expectations rules are most often observed in analyses of diverse duopoly games. Basic naive on one hand, versus sophisticated adaptive and bounded rational beliefs on the other hand. These tenets can be found in both homogeneous and heterogeneous agents paradigms. Homogeneous Bertrand duopoly setup serves here as a starting point. Next, two naive players evolve into a bounded rational and an adaptive agent respectively.
At each period t the companies X and Y form expectations of each other's price in period $t + 1$. Firms' prices for the latter period are decided by solving the following optimization problem

$$\begin{cases} x_{t+1} = \arg\max \pi^X(x_t, y^e_{t+1}) \\ y_{t+1} = \arg\max \pi^Y(x^e_{t+1}, y_t) \end{cases} \tag{7}$$

where $\pi^X(\cdot)$ and $\pi^Y(\cdot)$ symbolize expected profit functions of firms X and Y accordingly, and x^e_{t+1}, y^e_{t+1} represent their respective beliefs about the competitor's future pricing decision. Unique solutions to (7), if exist, are denoted by

$$\begin{cases} x_{t+1} = f(y^e_{t+1}) \\ y_{t+1} = g(x^e_{t+1}) \end{cases} \tag{8}$$

where f and g are best replies to y^e_{t+1} and x^e_{t+1} accordingly. Naive expectations rule assumes that each firm expects the rival to offer the good for sale at the same price in the current period, as it did in the preceding one. Hence, $y^e_{t+1} = y_t$ and $x^e_{t+1} = x_t$ for company X and Y respectively. After implementing Bertrand theory, (8) becomes

$$\begin{cases} x_{t+1} = f(y_t) \\ y_{t+1} = g(x_t) \end{cases} \tag{9}$$

which describes a homogeneous duopoly game with naive agents, where Bertrand reaction functions f and g are linear (Bertrand (1883)).
In opposition to naive players, other firms may rationally utilize information from the market. Thus, notion of bounded rationality can be employed in dynamic duopoly models (see Bischi and Naimzada (1999)). Bounded or not fully rational players possess incomplete knowledge about the market. For instance, if firm X is assumed to be a bounded rational player, though ignorant about the market demand, it decides to update its pricing strategy by an adjustment mechanism based on a local estimate of the marginal profit:

$$\Phi_{X,t} = \frac{\partial \pi^X(x_t, y_t)}{\partial x_t}$$

(Bischi, Galletgatti, Naimzada (1999)), denoted also as gradient dynamics (see Bischi and Lamantia (2005)), or myopic adjustment mechanism (see Dixit (1986)). Then, agent X is said to behave as a local profit maximizer. At each period t firm X decides to increase or decrease the price for period $t+1$ if it experiences positive or negative marginal profit on the basis of information held at period t, according to the following dynamic adjustment mechanism

$$x_{t+1} = x_t + \alpha(x_t) \cdot \Phi_{X,t}, \tag{10}$$

where $\alpha(x_t)$ is a positive function, which gives the extent of price variation for product of firm X following a given profit signal. Linear function $\alpha(x_t) = u \cdot x_t$ is assumed, since it captures the fact that relative price variations are proportional to marginal profits, that is

$$\frac{x_{t+1} - x_t}{x_t} = u \cdot \frac{\partial \pi^X(x_t, y_t)}{\partial x_t},$$

where $u > 0$ is the speed of adjustment of the bounded rational firm X. Final equation for bounded rational player X exhibits the following structure

$$x_{t+1} = x_t + u \cdot x_t \cdot \frac{\partial \pi^X(x_t, y_t)}{\partial x_t}, \quad t = 0, 1, 2, \ldots \quad . \tag{11}$$

The third theory of firm's beliefs is adaptive expectations principle (see Bischi and Kopel (2001)). In such a case duopolist does not instantaneously jump to the new optimal position, but adjusts the previous decisions in the direction of the best reply. For example, if firm Y forms its expectations in an adaptive mode, it calculates the price for period $t+1$ weighing current period price y_t and its reaction function $g(x_t)$. Dynamic equation for agent Y is obtained from the expression of the form

$$y^e_{t+1} = y^e_t + v \cdot [y_t - y^e_t] \tag{12}$$

First, the lagged equation for company Y from (8) is inserted into (12). Subsequently, expectations are dropped because player Y ultimately determines his price for period

$t+1$ at period t, hence all information up to period t is available and can be utilized in the decision process. Simple conversion yields the final equation for the adaptive agent Y

$$y_{t+1} = (1-v) \cdot y_t + v \cdot g(x_t), \quad t = 0, 1, 2, \ldots \quad . \tag{13}$$

where $0 \leq v \leq 1$ is the speed of adjustment of company Y. Setting $v = 1$, naive expectations rule for agent Y shown in (9) is obtained from (13). Thus, naive player is a special case within the adaptive agent framework. When $v = 0$, duopolist never revises any taken decision. Intermediate values, that is $0 < v < 1$, bring in a host of new possibilities and are therefore considered in this paper.

3 The model

Heterogeneous Bertrand duopoly game with bounded rational and adaptive players is derived through coalescing equation (11) with equation (13) in the following way

$$\begin{cases} x_{t+1} = x_t + u \cdot x_t \cdot \dfrac{\partial \pi^X(x_t, y_t)}{\partial x_t} \\ y_{t+1} = (1-v) \cdot y_t + v \cdot g(x_t), \end{cases} \tag{14}$$

where $\frac{\partial \pi^X(x_t,y_t)}{\partial x_t} = \Phi_{X,t}$ is the local estimate of firm's X marginal profit, $g(x_t)$ is firm's Y reaction function and $t = 0, 1, 2, ...$ Substituting the first equation from (5) and the second equation from (6) into (14) yields

$$\begin{cases} x_{t+1} = x_t + u \cdot x_t \cdot (a + b \cdot c_X - 2 \cdot b \cdot x_t + d \cdot y_t) \\ y_{t+1} = (1-v) \cdot y_t + \frac{v}{2 \cdot b} \cdot (a + b \cdot c_Y + d \cdot x_t), \end{cases} \tag{15}$$

which can be expressed as a discrete time dynamical system of the form

$$(x_{t+1}, y_{t+1}) = T(x_t, y_t), \tag{16}$$

with a two dimensional nonlinear map $T : R^2 \to R^2$

$$T : \begin{pmatrix} x_t \\ y_t \end{pmatrix} \to \begin{pmatrix} x_t + u \cdot x_t \cdot (a + b \cdot c_X - 2 \cdot b \cdot x_t + d \cdot y_t) \\ (1-v) \cdot y_t + \frac{v}{2 \cdot b} \cdot (a + b \cdot c_Y + d \cdot x_t), \end{pmatrix} \tag{17}$$

where $t = 0, 1, 2, \ldots$. Starting from the initial condition (x_0, y_0) iteration of map T uniquely determines the trajectory of states of firms X and Y prices

$$(x_{t+1}, y_{t+1}) = T^t(x_0, y_0), \quad t = 0, 1, 2, \ldots \quad . \tag{18}$$

The map depends on seven parameters, a, b and d of demand functions (1), c_X and c_Y of cost functions (2), u and v being the speeds of adjustment of the bounded rational player X and adaptive player Y respectively.

3.1 Fixed points

Since Bertrand game is an economic setting, equilibrium points of the analysed price duopoly model are defined as nonnegative fixed points of map T, in particular nonnegative solutions of a nonlinear algebraic system of equations of the form

$$\begin{cases} x \cdot (a + b \cdot c_X - 2 \cdot b \cdot x + d \cdot y) = 0 \\ \frac{1}{2 \cdot b} \cdot (a + b \cdot c_Y + d \cdot x) - y = 0, \end{cases} \tag{19}$$

which is derived by setting $x_{t+1} = x_t = x$ and $y_{t+1} = y_t = y$ in (17). Two stationary points are obtained

$$\begin{aligned} E_1 &= \left(0, \tfrac{a+b \cdot c_Y}{2 \cdot b}\right) \\ E_2 &= \left(\tfrac{2 \cdot b \cdot (a+b \cdot c_X)+d \cdot (a+b \cdot c_Y)}{(2 \cdot b+d) \cdot (2 \cdot b-d)}, \tfrac{2 \cdot b \cdot (a+b \cdot c_Y)+d \cdot (a+b \cdot c_X)}{(2 \cdot b+d) \cdot (2 \cdot b-d)}\right). \end{aligned} \tag{20}$$

The first fixed point E_1 is always nonnegative under the assumptions made. It is located at the y coordinate axis and thus denoted as a monopoly or a boundary equilibrium (see Bischi and Naimzada (1999), Bischi and Lamantia (2005), Zhang, Da, Wang (2009)). The x coordinate axis is an invariant submanifold, that is if $x_t = 0$ then $x_{t+1} = 0$. This implies that starting from a monopoly initial condition on the x coordinate axis, the dynamics is confined in the same axis for each t, thus yielding monopoly bahavior, governed by the restriction of map T to that axis. Such a restriction is given by the following one dimensional map $M : R \to R$, obtained from (17) with $x_t = 0$

$$M: \; y_t \to (1 - v) \cdot y_t + \frac{v \cdot (a + b \cdot c_Y)}{2 \cdot b}. \tag{21}$$

Map M generates a fixed point of the form

$$y^*_M = \frac{a + b \cdot c_Y}{2 \cdot b},$$

which reflects the y coordinate of the monopoly equilibrium point E_1 of map T.
The second fixed point E_2 is the unique Nash equilibrium, which is nonnegative and has economic meaning if $2 \cdot b - d > 0$. It is located at the intersection of two reaction curves which represent the locus of points of vanishing marginal profits in (5).

3.2 Local stability

The study of local stability of equilibria is based on localization of eigenvalues of the Jacobian matrix of the two dimensional nonlinear map T on a complex plane. Jacobian matrix along the pricing strategy (x, y) is following

$$\boldsymbol{J}(x, y) = \begin{bmatrix} 1 + u \cdot (a - 4 \cdot b \cdot x + d \cdot y + b \cdot c_X) & u \cdot d \cdot x \\ \frac{v \cdot d}{2 \cdot b} & 1 - v \end{bmatrix} \tag{22}$$

Proposition 1 *The monopoly equilibrium E_1 is unstable.*

Proof. Matrix $\boldsymbol{J}(x,y)$ evaluated at the boundary equilibrium point E_1 has the form

$$\boldsymbol{J}(E_1) = \begin{bmatrix} 1 + u \cdot [a + \frac{d\cdot(a+b\cdot c_Y)}{2\cdot b} + b \cdot c_X] & 0 \\ \frac{v\cdot d}{2\cdot b} & 1 - v \end{bmatrix} \tag{23}$$

Due to lower triangularity of matrix $\boldsymbol{J}(E_1)$ its eigenvalues are easily obtained as the diagonal entries. Eigenvalues are $\lambda_{E_1,1} = 1+u\cdot[a+\frac{d\cdot(a+b\cdot c_Y)}{2\cdot b}+b\cdot c_X]$ with eigenvector: $\mathbf{r}_{E_1,1} = (1,0)$ and $\lambda_{E_1,2} = 1 - v$ with eigenvector

$$\mathbf{r}_{E_1,2} = \left(-\frac{v \cdot d}{2 \cdot b \cdot [v + u \cdot (a + \frac{d\cdot(a+b\cdot c_Y)}{2\cdot b} + b \cdot c_X)]}, 1\right).$$

Both eigenvalues are real and nonnegative under the assumptions made. Since also $|\lambda_{E_1,1}| > 1$ and $|\lambda_{E_1,2}| < 1$ holds, the boundary equilibrium point E_1 is a saddle point, what means that solutions to map T are not sequences of points monotonically approaching the equilibrium as $t \to \infty$, except for the case when they originate from points on the eigenvectors associated with $\lambda_{E_1,1}$ or $\lambda_{E_1,2}$; see Medio and Lines (2001). This completes the proof of the proposition. ■

Proposition 2 *The Nash equilibrium E_2 is stable if:*

$$u < \frac{4 \cdot b \cdot (2 - v) \cdot (2 \cdot b + d) \cdot (2 \cdot b - d)}{[4 \cdot b^2 \cdot (2 - v) + v \cdot d^2] \cdot [2 \cdot b \cdot (a + b \cdot c_X) + d \cdot (a + b \cdot c_Y)]}$$

Proof. Investigation of local stability of the second fixed point of the two dimensional nonlinear map T is based on inspection of eigenvalues of the Jacobian matrix evaluated at E_2. At the Nash equilibrium point, matrix $\boldsymbol{J}(x,y)$ becomes

$$\boldsymbol{J}(E_2) = \begin{bmatrix} 1 - 2 \cdot u \cdot b \cdot x^* & u \cdot d \cdot x^* \\ \frac{v\cdot d}{2\cdot b} & 1 - v \end{bmatrix} \tag{24}$$

where $x^* = \frac{2\cdot b\cdot(a+b\cdot c_X)+d\cdot(a+b\cdot c_Y)}{(2\cdot b+d)\cdot(2\cdot b-d)}$, and its eigenvalues

$$\begin{aligned} \lambda_{E_2,1} &= \frac{b\cdot(1-v)+2\cdot(1-2\cdot b\cdot u\cdot x^*)+\sqrt{(v\cdot b-2\cdot b^2\cdot u\cdot x^*)^2+2\cdot b\cdot d^2\cdot u\cdot v\cdot x^*}}{b} \\ \lambda_{E_2,2} &= \frac{b\cdot(1-v)+2\cdot(1-2\cdot b\cdot u\cdot x^*)-\sqrt{(v\cdot b-2\cdot b^2\cdot u\cdot x^*)^2+2\cdot b\cdot d^2\cdot u\cdot v\cdot x^*}}{b} \end{aligned} \tag{25}$$

are real under the assumptions made. Analysis of discriminant of the characteristic equation of matrix $\boldsymbol{J}(E_2)$ confirms the real character of the eigenvalues. Characteristic equation of the Jacobian matrix evaluated at the Nash equilibrium point E_2 has the form

$$P_{\boldsymbol{J}(E_2)}(\lambda) = \lambda^2 - tr\ \boldsymbol{J}(E_2) \cdot \lambda + \det \boldsymbol{J}(E_2), \tag{26}$$

where $tr\ \boldsymbol{J}(E_2)$ is the trace and $\det \boldsymbol{J}(E_2)$ is the determinant of matrix $\boldsymbol{J}(E_2)$, which are following

$$\begin{aligned} tr\ \boldsymbol{J}(E_2) &= 2 - v - 2 \cdot b \cdot u \cdot x^* \\ \det \boldsymbol{J}(E_2) &= (1 - v) \cdot (1 - 2 \cdot b \cdot u \cdot x^*) - \tfrac{u \cdot v \cdot d^2 \cdot x^*}{2 \cdot b}. \end{aligned} \tag{27}$$

Discriminant $\Delta_{P_{J(E_2)}(\lambda)} = [tr\ \boldsymbol{J}(E_2)]^2 - 4 \cdot \det \boldsymbol{J}(E_2)$ of the characteristic equation is positive under the assumptions made because

$$(2 \cdot b \cdot u \cdot x^* - v)^2 + \frac{2 \cdot u \cdot v \cdot d^2 \cdot x^*}{b} > 0, \tag{28}$$

what sustains the real nature of eigenvalues of the Jacobian matrix evaluated at the Nash equilibrium. Yet, local stability analysis of the second fixed point through direct inspection of the eigenvalues may be troublesome.
In a two dimensional case conditions for which $|\lambda_{E_2,i}| < 1$, $i = 1, 2$ and E_2 is a stable node have a simple representation in terms of trace and determinant of the constant matrix $\boldsymbol{J}(E_2)$; see Medio and Lines (2005). The so-called Jury's conditions for matrix $\boldsymbol{J}(E_2)$ are following

$$\begin{aligned} &(i) && 1 + tr\ \boldsymbol{J}(E_2) + \det \boldsymbol{J}(E_2) > 0 \\ &(ii) && 1 - tr\ \boldsymbol{J}(E_2) + \det \boldsymbol{J}(E_2) > 0 \\ &(iii) && 1 - \det \boldsymbol{J}(E_2) > 0. \end{aligned} \tag{29}$$

If inequalities $(i) - (iii)$ hold, sufficient and necessary conditions for local stability of the second fixed point are met, that is eigenvalues of matrix $\boldsymbol{J}(E_2)$ lie inside the unit circle of the complex plane. Substituting (27) into (29) yields

$$\begin{aligned} &(i) && 3 - v - 2 \cdot b \cdot u \cdot x^* + (1 - v) \cdot (1 - 2 \cdot b \cdot u \cdot x^*) - \tfrac{u \cdot v \cdot d^2 \cdot x^*}{2 \cdot b} > 0 \\ &(ii) && v - 1 + 2 \cdot b \cdot u \cdot x^* + (1 - v) \cdot (1 - 2 \cdot b \cdot u \cdot x^*) - \tfrac{u \cdot v \cdot d^2 \cdot x^*}{2 \cdot b} > 0 \\ &(iii) && 1 - (1 - v) \cdot (1 - 2 \cdot b \cdot u \cdot x^*) + \tfrac{u \cdot v \cdot d^2 \cdot x^*}{2 \cdot b} > 0. \end{aligned} \tag{30}$$

While condition (ii) is always satisfied under the assumptions made, because after transformations it reduces to

$$(ii) \quad 2 \cdot b - d > 0, \tag{31}$$

conditions (i) and (iii) require further reasoning. After modifications inequality (iii) takes the form

$$(iii) \quad v + 2 \cdot b \cdot u \cdot x^* \cdot (1 - v) + \tfrac{u \cdot v \cdot d^2 \cdot x^*}{2 \cdot b} > 0 \tag{32}$$

and holds when $u > \frac{2 \cdot b \cdot v}{[4 \cdot b^2 \cdot (v-1) - v \cdot d^2] \cdot x^*}$. However, from the beginning of the analysis

$u > 0$ and in addition $4 \cdot b^2 \cdot (v-1) - v \cdot d^2 < 0$, what makes initial assumption about the nonnegativity of the speed of adjustment u of the bounded rational player X stronger, hence condition (iii) is always satisfied in the given game. Following transformations inequality (i) becomes

$$(i) \quad 2 \cdot (2-v) - 2 \cdot b \cdot u \cdot x^* \cdot (2-v) - \tfrac{u \cdot v \cdot d^2 \cdot x^*}{2 \cdot b} > 0 \tag{33}$$

and holds when $u < \frac{4 \cdot b \cdot (2-v)}{[4 \cdot b^2 \cdot (2-v) + v \cdot d^2] \cdot x^*]}$, which after substituting for x^* leads to

$$u < \frac{4 \cdot b \cdot (2-v) \cdot (2 \cdot b + d) \cdot (2 \cdot b - d)}{[4 \cdot b^2 \cdot (2-v) + v \cdot d^2] \cdot [2 \cdot b \cdot (a + b \cdot c_X) + d \cdot (a + b \cdot c_Y)]} \tag{34}$$

what completes the proof of the proposition. ■

4 Numerical simulations

Since local stability analysis of fixed points of the Bertrand duopoly game with bounded rational and adaptive players reveals that Nash equilibrium is stable only if certain condition is met, its local stability properties subject to violation of this stability condition are numerically explored. First, region of stability of the Nash equilibrium is shown in the plane of the speeds of adjustment. Next, period doubling route to chaos is presented on bifurcation diagrams and on the largest Lypanunov characteristic exponent graph. Then, chaotic attractors are depicted and their fractal dimensions are calculated. Finally, sensitive dependence of the model on initial conditions is evidenced. It is convenient to commence the numerical simulations with the following values of the parameters $(a, b, d, c_X, c_Y) = (2, 0.5, 0.3, 0.2, 0.8)$ analogously to Zhang, Da, Wang (2009), arbitrarily choose the speeds of adjustment and keep initial values fixed at $(x_0, y_0) = (3, 3)$.

4.1 Region of stability

Condition for stability of the Nash equilibrium in the heterogeneous Bertrand duopoly game with bounded rational and adaptive players represents an unbounded region R in the plane of the speeds of adjustment (v, u)

$$R: \; 4 \cdot b \cdot v + 8 \cdot b^2 \cdot x^* \cdot u - (4 \cdot b^2 - d^2) \cdot x^* \cdot v \cdot u - 8 \cdot b < 0, \tag{35}$$

where $x^* = \frac{2 \cdot b \cdot (a + b \cdot c_X) + d \cdot (a + b \cdot c_Y)}{(2 \cdot b + d) \cdot (2 \cdot b - d)}$. Region R is constrained by a hyperbola of the form

$$H: \; 4 \cdot b \cdot v + 8 \cdot b^2 \cdot x^* \cdot u - (4 \cdot b^2 - d^2) \cdot x^* \cdot v \cdot u - 8 \cdot b = 0. \tag{36}$$

Inequality determining region R lets obtain two pairs of simultaneously binding conditions, which are following

$$I: \begin{cases} v < \frac{8 \cdot b^2}{4 \cdot b^2 - d^2} \\ u < \frac{4 \cdot b \cdot (2-v)}{[4 \cdot b^2 \cdot (2-v) + v \cdot d^2] \cdot x^*} \end{cases} \quad II: \begin{cases} v > \frac{8 \cdot b^2}{4 \cdot b^2 - d^2} \\ u > \frac{4 \cdot b \cdot (2-v)}{[4 \cdot b^2 \cdot (2-v) + v \cdot d^2] \cdot x^*} \end{cases} \tag{37}$$

thus asymptote AS of hyperbola H is

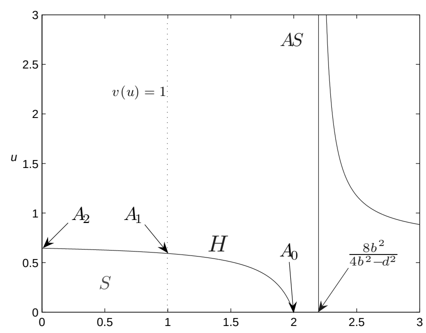
$$AS: \; v = \frac{8 \cdot b^2}{4 \cdot b^2 - d^2} \tag{38}$$

Along with the assumptions made until now, pair I uniquely determines the region of stability S of the Nash equilibrium of map T

$$S: \begin{cases} 0 < v < 1 \\ 0 < u < \frac{4 \cdot b \cdot (2-v)}{[4 \cdot b^2 \cdot (2-v) + v \cdot d^2] \cdot x^*} \end{cases} \tag{39}$$

shown in figure 1.

Figure 1: Region of stability S in the (v, u) plane, $(a, b, d, c_X, c_Y) = (2, 0.5, 0.3, 0.2, 0.8)$



Hyperbola H redefined as

$$H: \begin{cases} u = \frac{4 \cdot b \cdot (2-v)}{[4 \cdot b^2 \cdot (2-v) + v \cdot d^2] \cdot x^*} \\ 0 < v < 1 \end{cases} \tag{40}$$

represents a bifurcation curve, at which Nash equilibrium of map T looses its stability; see Gandolfo (1997). Originally, hyperbola H intersects the axes v and u at points A_0 and A_2 respectively, coordinates of which are following

$$\begin{aligned} A_0 &= (2, 0) \\ A_2 &= (0, \tfrac{1}{b \cdot x^*}) \end{aligned} \tag{41}$$

Point A_0 is located at the border of the region R (fig. 1). Constant function $v(u) = 1$, which forms the rightmost part of the border of the region of stability S intersects hyperbola H at point $A_1 = (1, \frac{4 \cdot b}{(4 \cdot b^2 + d^2) \cdot x^*})$.

4.2 Period doubling route to chaos

One of the ways how stability of a dynamical system is lost and complex dynamics, followed by a chaotic zone, appears as a result of a change in value of one of the model parameters, is a period doubling route to chaos ($PDRC$) or in the Russian literature notation a safe boundary bifurcation scenario (Medio and Lines (2001)). $PDRC$ is the basic route to chaos, thus the most common in applications.
Let $T \equiv \boldsymbol{T}(\boldsymbol{z}_t, \boldsymbol{\mu}) \equiv \boldsymbol{T}(\boldsymbol{z}_t)$, where $\boldsymbol{T} = (T_1, T_2)'$ is a 2×1 function vector, $\boldsymbol{z}_t = (x_t, y_t)'$ is a 2×1 state vector and $\boldsymbol{\mu} = (u, v, a, b, d, c_X, c_Y)'$ is a 7×1 parameter vector. Safe boundary route to chaos begins with a flip bifurcation (bif) of a fixed point of a map $\boldsymbol{T}(\boldsymbol{z}_t, \boldsymbol{\mu})$, which occurs when a single eigenvalue passes through minus one, for instance at $u = u^{bif,1}$. The fixed point loses its stability and a stable period-2 cycle is born. Next, period-2 cycle loses its stability at $u = u^{bif,2}$ and gives rise to a stable period-4 cycle. As u increases further, the scenario repeats itself over and over again: each time a period-2^k cycle of the map T loses stability through a flip bifurcation of the map T^{k+1}, which gives rise to an initially stable period-2^{k+1} cycle, and so on and so forth. The sequence $\{u^{bif,k}\}$ of values of u, at which cycles of period 2^k appear, has a finite accumulation point $u^{bif,\infty}$. At $u = u^{bif,\infty}$ there exists infinite number of periodic orbits with periods equal to powers of 2, all of them unstable. The convergence of u to $u^{bif,\infty}$ is controlled by the universal parameter $\delta = 4.67$, known also as the Feigenbaum constant, defined for $k \geq 2$ as

$$\delta = \lim_{k \to \infty} \frac{\Delta_k}{\Delta_{k+1}} \tag{42}$$

where $\Delta_k = u^{bif,k} - u^{bif,k-1}$ and $\Delta_{k+1} = u^{bif,k+1} - u^{bif,k}$ (see Feigenbaum (1978)). In this paper, a simplified formula of the form $\hat{\delta} = \frac{\Delta_3}{\Delta_4}$ is used to obtain an approximation of the associated constant.
Existence of flip bifurcation of the equilibrium point E_2 of map T observed on the bifurcation diagram is confirmed on the graph of the largest Lyapunov characteristic exponent $L_1(\boldsymbol{z}_0)$ as well. Lyapunov characteristic exponents of vector $\boldsymbol{w}$ in the tangent space at $\boldsymbol{z}_0$ are defined as

$$L(\boldsymbol{z}_0, \boldsymbol{w}) = \lim_{t \to \infty} \frac{1}{t} \cdot \ln \frac{\left(\boldsymbol{w}' \left[\boldsymbol{J}^t(\boldsymbol{z}_0)' \boldsymbol{J}^t(\boldsymbol{z}_0)\right] \boldsymbol{w}\right)^{\frac{1}{2}}}{(\boldsymbol{w}'\boldsymbol{w})^{\dfrac{1}{2}}} \tag{43}$$

where $\boldsymbol{J}(\boldsymbol{z}_t)$ is the known Jacobian matrix of map $\boldsymbol{T}(\boldsymbol{z}_t)$ and

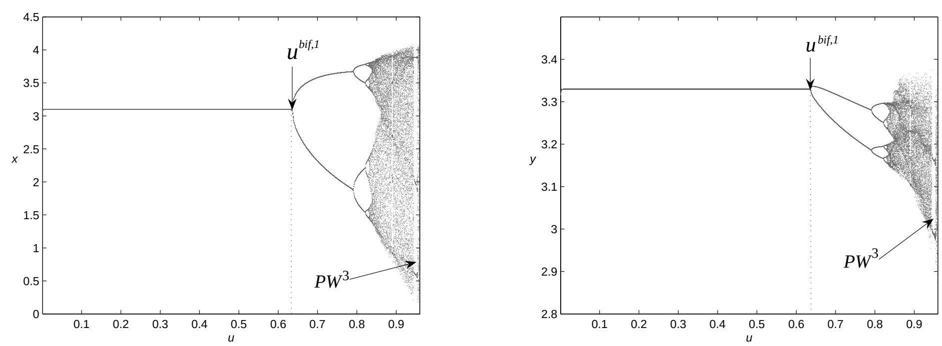
$$\boldsymbol{J}^t(\boldsymbol{z}_0) = \prod_{i=0}^{t-1} \boldsymbol{J}\left(\boldsymbol{T}^i(\boldsymbol{z}_0)\right) \tag{44}$$

For simplicity vector $\boldsymbol{w}$ is such that $(\boldsymbol{w}'\boldsymbol{w})^{\frac{1}{2}} = 1$, because Lyapunov characteristic exponents do not depend on the length of $\boldsymbol{w}$. Matrix $\left[\boldsymbol{J}^t(\boldsymbol{z}_0)'\boldsymbol{J}^t(\boldsymbol{z}_0)\right]^{\frac{1}{2}}$ determines to what extent vector $\boldsymbol{w}$ is stretched (contracted) under the action of matrix $\boldsymbol{J}^t(\boldsymbol{z}_0)$. Logarithms of the eigenvalues of the matrix

$$\boldsymbol{\Lambda}(\boldsymbol{z}_0) = \lim_{t\to\infty} \left[\boldsymbol{J}^t(\boldsymbol{z}_0)'\boldsymbol{J}^t(\boldsymbol{z}_0)\right]^{\frac{1}{2\cdot t}} \tag{45}$$

are Lyapunov characteristic exponents $L_1(\boldsymbol{z}_0)$ and $L_2(\boldsymbol{z}_0)$, which measure the asymptotic, average, exponential rate of stretching (contraction) of vector $\boldsymbol{w}$ (see Medio and Lines (2001)).

Figure 2: Bifurcation diagrams with respect to parameter u of map T, $(v, a, b, d, c_X, c_Y) = (0.3, 2, 0.5, 0.3, 0.2, 0.8)$

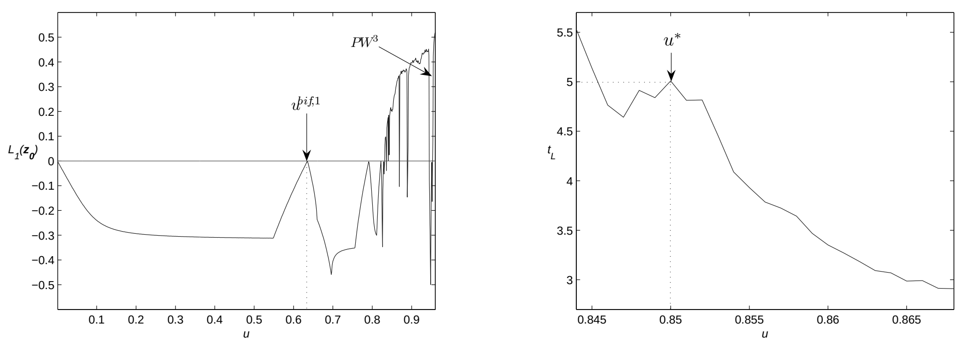


However, computation of Lyapunov characteristic exponents from (45) poses two problems. First, product matrix $\boldsymbol{J}^t(\boldsymbol{z}_0)$ is often so large for $t \to \infty$ that calculation of $\boldsymbol{\Lambda}(\boldsymbol{z}_0)$ proves impossible. Second, unless calculation of $\boldsymbol{J}^t(\boldsymbol{z}_0)$ is such that linear independence of the columns is maintained, computation leads only to the largest Lyapunov characteristic exponent $L_1(\boldsymbol{z}_0)$. To deal with these problems, orthogonal-triangular decomposition is applied to compute Lyapunov characteristic exponents; see Eckmann, Kamphorst, Ruelle, Ciliberto (1986), Brown, Bryant, Abarbanel (1991), Diect, Russel, Van Vleck (1997), Oiwa and Fiedler-Ferrara (1998). Given orthogonal 2×2 matrix $\boldsymbol{Q_0}$ chosen at random, decomposition of $\boldsymbol{J}(\boldsymbol{z}_t) \cdot \boldsymbol{Q}_t = \boldsymbol{Q}_{t+1} \cdot \boldsymbol{R}_{t+1}$ is obtained for $t = 0, 1, ..., S$, where $\boldsymbol{Q}_{t+1}$ is an orthogonal matrix and $\boldsymbol{R}_{t+1}$ is an upper triangular matrix with positive diagonal elements. Lyapunov characteristic exponents are approximated as

$$\hat{L}_j(\boldsymbol{z}_0) = \frac{1}{S} \cdot \sum_{t=1}^{S} \ln\left(R_{t,jj}\right),\ j = 1, 2 \tag{46}$$

where $R_{t,jj}$ is the j-th diagonal element of $\boldsymbol{R}_t$ and here $S = 10^4$.
Intuitive interpretation of the largest Lyapunov characteristic exponent $L_1(\boldsymbol{z}_0)$ is the following. Firstly, if $L_1(\boldsymbol{z}_0) < 0$, then orbit of $\boldsymbol{z}_0$ converges to a stable periodic orbit. Secondly, if the orbit of $\boldsymbol{z}_0$ is an unstable periodic or a chaotic orbit, then $L_1(\boldsymbol{z}_0) > 0$. Lastly, $L_1(\boldsymbol{z}_0) = 0$ at bifurcation point or if the orbit of $\boldsymbol{z}_0$ converges to a quasiperiodic orbit; see Medio and Lines (2001). Closely related to the largest Lyapunov characteristic exponent is the notion of Lyapunov time, calculated as inverse of the former, that is $t_L = \frac{1}{L_1(\boldsymbol{z}_0)}$, which designates a period when dynamical system moves beyond the bounds of precise predictability and enters a chaotic mode.

Figure 3: Largest Lyapunov characteristic exponent and Lyapunov time graphs with respect to parameter u of map T, $(v, a, b, d, c_X, c_Y) = (0.3, 2, 0.5, 0.3, 0.2, 0.8)$



Search for determination of the way in which stability of the Nash equilibrium point E_2 of map T is lost, is simplified by the use of the Jury's conditions, which guarantee that eigenvalues of the Jacobian matrix evaluated at the fixed point E_2 are less than one in modulus. Occurrence of a flip bifurcation of the stationary point E_2 when a single eigenvalue becomes equal to minus one means violation of the first Jury's condition

$$1 + tr\ \boldsymbol{J}(E_2) + \det \boldsymbol{J}(E_2) = 0 \quad \Rightarrow \quad \begin{cases} -2 < \ \ tr\ \boldsymbol{J}(E_2) < 0 \\ -1 < \det \boldsymbol{J}(E_2) < 1 \end{cases} \tag{47}$$

Combined with numerically simulated bifurcation diagrams and the largest Lyapunov

Table 1: Period doubling route to chaos of map T,
$(v, a, b, d, c_X, c_Y) = (0.3, 2, 0.5, 0.3, 0.2, 0.8)$

$u^{bif,1}$	$u^{bif,2}$	$u^{bif,3}$	$u^{bif,4}$	$\hat{\delta}$
0.635	0.791	0.822	0.829	4.429

characteristic exponent graph suggesting that a flip bifurcation does arise, this necessary albeit insufficient condition for the existence of the associated bifurcation constitutes strong evidence. In applications where analytical proofs can be difficult to derive, this procedure is the only one available (Medio and Lines (2005)).
Consequently, initial flip bifurcation point occurs at the following value of the speed of adjustment of the bounded rational player X

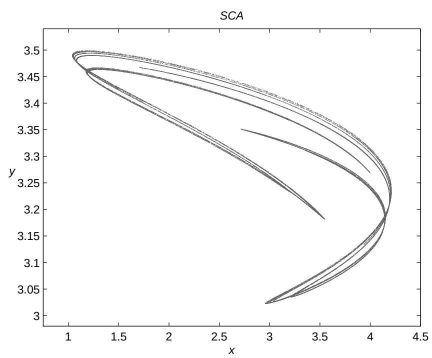
$$u^{bif,1} = \frac{4 \cdot b \cdot (2 - v) \cdot (2 \cdot b + d) \cdot (2 \cdot b - d)}{[4 \cdot b^2 \cdot (2 - v) + v \cdot d^2] \cdot [2 \cdot b \cdot (a + b \cdot c_X) + d \cdot (a + b \cdot c_Y)]} \tag{48}$$

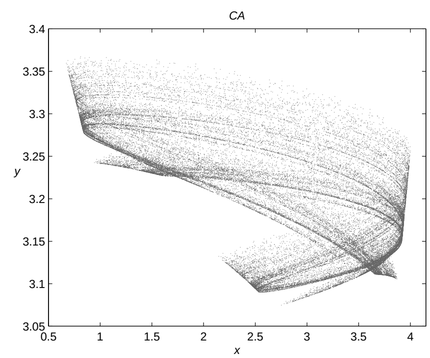
which gives $u^{bif,1} \approx 0.635$ (fig. 2-3). Numerical estimates of flip bifurcation points up to $k = 4$ and of the Feigenbaum constant are shown in table 1.
Given $(v, a, b, d, c_X, c_Y) = (0.3, 2, 0.5, 0.3, 0.2, 0.8)$, in approximate terms map T has a stable fixed point $E_2 = (3.10, 3.33)$ for $0 < u < 0.635$, while for $0.635 < u < 0.791$ a stable period-2 cycle, for $0.791 < u < 0.822$ a stable period-4 cycle, for $0.822 < u < 0.829$ a stable period-8 cycle and for $u > 0.829$ cycles of higher orders and eventually chaos arise. Moreover, for $0.945 < u < 0.953$ a narrow period-3 window PW^3, where map T has a stable period-3 cycle, followed by chaotic zone, is observed both on the bifurcation diagrams and on the largest Lyapunov characteristic exponent graph (fig. 2-3). Yet, in applications the presence of a certain amount of noise has to be assumed and practical relevance of narrow periodic windows is questionable. Eventually, for $u^* = 0.85$ prices x_t and y_t become unpredictable after $t_L = 5$ periods (fig. 3).

4.3 Strange attractors and fractal dimensions

Chaotic sets are typically characterized by a peculiar geometric structure usually denoted as fractal. Fractal sets disclose noninteger dimension. The Kolmogorov capacity and the correlation dimensions are prototypical members of a family of dimension-like quantities, which can be grouped under the label fractal dimensions and are suitable for fractal objects, such as strange attractors. However chaoticity, defined by sensitive dependence on initial conditions, and strangeness, defined by fractal structure, are independent properties in case of dynamical systems in discrete time. Hence, there are chaotic attractors that are not strange and strange attractors that are not chaotic (Medio and Gallo (1995)).

Figure 4: Strange chaotic attractor (SCA) for $(u, v, c_X) = (0.8, 0.6, 0.6)$ and chaotic attractor (CA) for map T with $(u, v, c_X) = (0.9, 0.3, 0.2)$, given $(a, b, d, c_Y) = (2, 0.5, 0.3, 0.8)$





A good numerical approximation of capacity and correlation dimensions, directly relating the dimension of an attractor and the associated Lyapunov exponents is the Lyapunov dimension (D_L) suggested by Kaplan and Yorke (1979), defined as

$$D_L = N + \frac{\sum_{n=1}^{N} L_n(\boldsymbol{z}_0)}{|L_{N+1}(\boldsymbol{z}_0)|} \tag{49}$$

where N is the largest integer for which $\sum_{n=1}^{N} L_i(\boldsymbol{z}_0) > 0$, meaning that $L_{N+1}(\boldsymbol{z}_0) < 0$, and the associated set of Lyapunov exponents is ordered from the largest $L_1(\boldsymbol{z}_0)$ to the smallest $L_m(\boldsymbol{z}_0)$. Map T is two dimensional hence $m = 2$ and the estimate of the Lyapunov dimension D_L, which is bounded by m from above, is following

$$\hat{D}_L = 1 + \frac{\hat{L}_1(\boldsymbol{z}_0)}{\left|\hat{L}_2(\boldsymbol{z}_0)\right|} \tag{50}$$

Under the assumption that $(a, b, d, c_Y) = (2, 0.5, 0.3, 0.8)$ chaotic attractor of map T with $(u, v, c_X) = (0.8, 0.6,\ 0.6)$ has noninteger Lyapunov dimension, hence exhibits fractal structure and is strange (SCA), whereas chaotic attractor (CA) of map T with $(u, v, c_X) = (0.9, 0.3, 0.2)$ is Lyapunov two dimensional, therefore is not strange (fig. 4, tab. 2).

Table 2: Lyapunov characteristic exponents and Lyapunov dimensions of strange chaotic attractor (SCA) for $(u, v, c_X) = (0.8, 0.6, 0.6)$ and chaotic attractor (CA) for $(u, v, c_X) = (0.9, 0.3, 0.2)$ of map T, given $(a, b, d, c_Y) = (2, 0.5, 0.3, 0.8)$

	$\hat{L}_1(\boldsymbol{z}_0)$	$\hat{L}_2(\boldsymbol{z}_0)$	$\hat{D}_L$
SCA	0.331	−0.808	1.409
CA	0.409	−0.328	2.000

4.4 Sensitive dependence on initial conditions

Sensitive dependence on initial conditions ($SDIC$) is one of the features of chaotic maps (Ott (1997)). After Medio and Lines (2001), map $\boldsymbol{T}(\boldsymbol{z}_t)$ on a metric space $\mathbb{R}^2_+ \cup \{0\}$ has sensitive dependence on initial conditions on $\mathbb{R}^2_+ \cup \{0\}$ if there exists a real number $\delta > 0$ such that for all $\boldsymbol{z}_0^I \in \mathbb{R}^2_+ \cup \{0\}$ and for all $\varepsilon > 0$, there exists $\boldsymbol{z}_0^{II} \in \mathbb{R}^2_+ \cup \{0\}$, $\boldsymbol{z}_0^{II} \neq \boldsymbol{z}_0^I$, and $t > 0$ such that $d(\boldsymbol{z}_0^I, \boldsymbol{z}_0^{II}) < \varepsilon$ and $d\left[\boldsymbol{T}^t(\boldsymbol{z}_0^I), \boldsymbol{T}^t(\boldsymbol{z}_0^{II})\right] > \delta$. Showing $SDIC$ of map T analytically is nontrivial, yet alternative numerical procedure is available.
Assuming $(u, v, a, b, d, c_X, c_Y) = (0.9, 0.3, 2, 0.5, 0.3, 0.2, 0.8)$, trajectories of prices x_t and y_t with marginally differentiated initial values, that is (x_0, y_0) and $(x_0+0.0001, y_0)$ in the first step and (x_0, y_0) and $(x_0, y_0 + 0.0001)$ in the second step, are simulated for $t = 0, ..., 100$. Approximately for $0 \leq t < 10$, generated time series are indistinguishable, but from $t = 10$ onwards differences between them build up rapidly (fig. 5). Hence, map T features sensitive dependence on initial conditions.

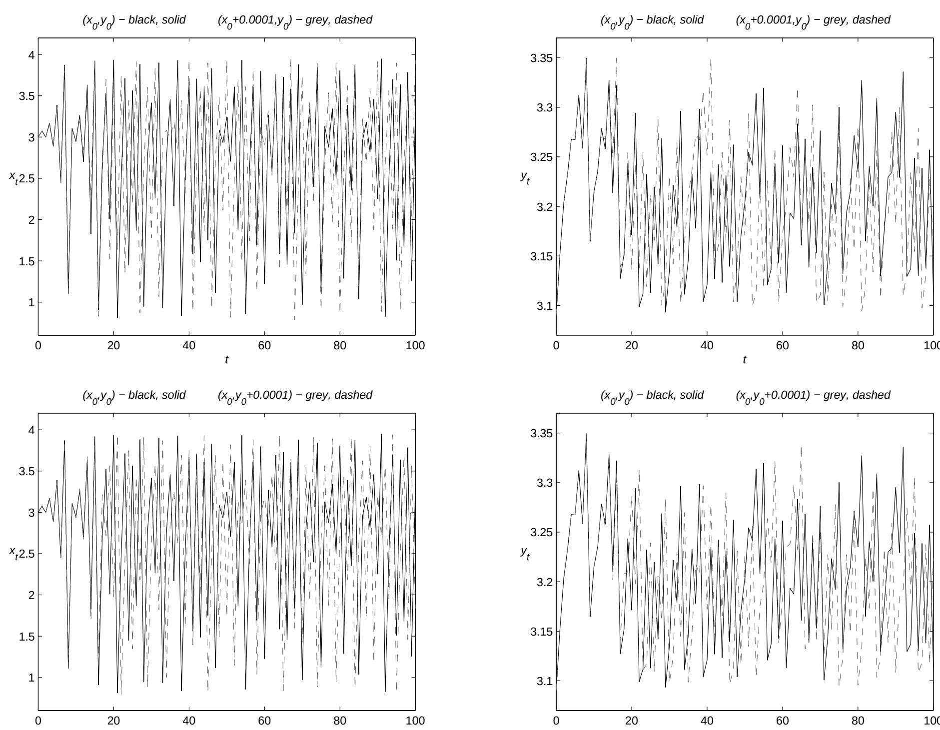
5 Conclusions

In this paper, complex dynamics of a Bertrand duopoly game with heterogeneous players and differentiated products, modelled with a two dimensional nonlinear map, was studied in detail. Bounded rational and adaptive expectations were considered. Conducted analysis revealed that higher periodic and chaotic behavior, such as similar dynamical properties already proved to exist in homogeneous and heterogeneous Cournot duopoly games and homogeneous Bertrand duopoly game, is observed in the associated heterogeneous Bertrand duopoly game as well.
In the analytical part, boundary and the Nash equilibria were found in the considered model. Next, local stability thereof was examined. Boundary equilibrium was proved unstable, whereas Nash equilibrium was conditionally stable and its exact stability condition was derived. In the numerical part, devoted to conditionally stable fixed point solely, region of stability thereof was defined and visualized in the plane of the speeds of adjustment. Next, period doubling route to chaos of the map was shown on bifurcation diagrams and on the largest Lyapunov exponents graph. Approximation of the Feigenbaum constant was obtained and Lyapunov time was calculated. Then, two

chaotic attractors were depicted, one of which was strange and Lyapunov dimensions thereof were computed. Eventually, sensitive dependence of the map on initial conditions was evidenced.

Obtained results disclosed that participation of one bounded rational player in the game, with a tendency to overshoot, synonymous to a high speed of adjustment, already suffices to destabilize the Nash equilibrium, what is a major implication for practicioners. In case of a considerable overshooting, prices in the Bertrand duopoly market become unpredictable already after a few periods, as indicated by the computed Lyapunov time.

Figure 5: $SDIC$ of map T, $(u, v, a, b, d, c_X, c_Y) = (0.9, 0.3, 2, 0.5, 0.3, 0.2, 0.8)$



Acknowledgements

The author thanks the anonymous referees of this journal for comments and useful suggestions.

References

[1] Agiza H.N., (1998), Explicit stability zones for Cournot games with 3 and 4 competitors, *Chaos, Solitons and Fractals* 23, 1955-1966.

[2] Agiza H.N., (1999), On the stability, bifurcations, chaos and chaos control of Kopel map, *Chaos, Solitons and Fractals* 11, 1909-1916.

[3] Agiza H.N., Elsadany A.A., (2003), Nonlinear dynamics in the Cournot duopoly game with heterogeneous players, *Physica A* 320, 512-524.

[4] Agiza H.N., Elsadany A.A., (2004), Chaotic dynamics in nonlinear duopoly game with heterogeneous players, *Applied Mathematics and Computation* 149, 843-860.

[5] Agiza H.N., Hegazi A.S., Elsadany A.A., (2001), The dynamics of Bowley's model with bounded rationality, *Chaos, Solitons and Fractals* 9, 1705-1717.

[6] Agiza H.N., Hegazi A.S., Elsadany A.A., (2002), Complex dynamics and synchronization of duopoly game with bounded rationality, *Mathematics and Computers in Simulation* 58, 133-146.

[7] Bertrand J., (1883), Revue de la theorie de la recherche sociale et des recherches sur les principes mathematiques de la thematiques de la theorie des richesses, *Journal des Savants* 1883, 499-508.

[8] Bierman H.S., Fernandez L., (1998), *Game Theory with Economic Applications*, Addison-Wesley.

[9] Bischi G.I., Galletgatti M., Naimzada A., (1999), Symmetry-breaking bifurcations and representative firm in dynamic duopoly games, *Annals of Operations Research* 89, 253-272.

[10] Bischi G.I., Kopel M., (2001), Equilibrium selection in a nonlinear duopoly game with adaptive expectations, *Journal of Economic Behavior and Organization* 46, 73-100.

[11] Bischi G.I., Lamantia F., (2005) Coexisting attractors and complex basins in discrete time economics models, [in:] *Nonlinear dynamical systems in economics*, [ed.:] M. Lines, CISM, SpringerWienNewYork, 187-231.

[12] Bischi G.I., Naimzada A., (1999), Global analysis of a dynamic duopoly game with bounded rationality, [in:] *Advances in Dynamic Games and Applications*, Vol. 5., [ed.:] J.A. Filar, V. Gaitsgory, K. Mizukami, Birkhauser, Basel, 361-385.

[13] Brown R., Bryant P., Abarbanel H.D.I., (1991), Computing the Lyapunov spectrum of a dynamical system from an observed time series, *Physical Review A* 43, 2787-2806.

[14] Cournot A., (1838), *Recherches sur les principes mathematiques de la theorie des richesses*, Hachette, Paris.

[15] Dana R.A., Montrucchio L., (1986), Dynamic complexity in duopoly games, *Journal of Economic Theory* 40, 40-56.

[16] Den Haan W.J., (2001), The importance of the number of different agents in a heterogeneous asset-pricing model, *Journal of Economic Dynamics and Control* 25, 721-746.

[17] Diect L., Russel R.D., Van Vleck E.S., (1997), On the computation of Lyapunov exponents from continuous dynamical systems, *SIAM Journal on Numerical Analysis* 34, 402-423.

[18] Dixit A., (1986), Comparative statics for oligopoly, *International Economic Review* 27, 107-122.

[19] Eckmann J.P., Kamphorst S.O., Ruelle D., Ciliberto S., (1986), Lyapunov exponents from time series, *Physical Review A* 33, 1134-1140.

[20] Feigenbaum M.J., (1978, Quantitative universality for a class of nonlinear transformations, *Journal od Statistical Physics* 19, 25-52.

[21] Gandolfo G., (1997), *Economic dynamics*, Springer.

[22] Gibbons R., (1992), *A Primer in Game Theory*, Prentice Hall.

[23] Kaplan J.L., Yorke Y.A., (1979), A regime observed in a fluid flow model of Lorenz, *Communications in Mathematical Physics* 67, 93-108.

[24] Kopel M., (1996), Simple and complex adjustment dynamics in Cournot duopoly models, *Chaos, Solitons and Fractals* 12, 2031-2048.

[25] Leonard D., Nishimura K., (1999), Nonlinear dynamics in the Cournot model without full information, *Annals of Operations Research* 89, 165-173.

[26] Medio A., Gallo G., (1995), *Chaotic dynamics: theory and applications to economics*, Cambridge University Press.

[27] Medio A., Lines M., (2001), *Nonlinear dynamics. A primer*, Cambridge University Press.

[28] Medio A., Lines M., (2005), Introductory notes on the dynamics of linear and linearized systems, [in:] *Nonlinear dynamical systems in economics*, [ed.:] M. Lines, CISM, SpringerWienNewYork, 1-26.

[29] Oiwa N.N., Fiedler-Ferrara N., (1998), A fast algorithm for estimating Lyapunov exponents from time series, *Physics Letters A* 246, 117-121.

[30] Onazaki T., Sieg G., Yokoo M., (2003), Stability, chaos and multiple attractors: a single agent makes a difference, *Journal of Economic Dynamics and Contro* 27, 1917-1938.

[31] Ott E., (1997), *Chaos w ukladach dynamicznych*, WNT, Warszawa.

[32] Puu T., (1991), Chaos in duopoly pricing, *Chaos, Solitons and Fractals* 1, 573-581.

[33] Puu T., (1998), The chaotic duopolists revisited, *Journal of Economic Behavior and Organization* 37, 385-394.

[34] Puu T., (2005), Complex oligopoly dynamics, [in:] *Nonlinear dynamical systems in economics*, [ed.:] M. Lines, CISM, SpringerWienNewYork, 165-186.

[35] Rassenti S., Reynolds S.S., Smith V.L., Szidarovszky F., (2000), Adaptation and convergence of behavior in repeated experimental Cournot games, *Journal of Economic Behavior and Organization* 41, 117-146.

[36] Zhang J., Da Q., Wang Y., (2007), Analysis of nonlinear duopoly game with heterogeneous players, *Economic Modelling* 24, 138-148.

[37] Zhang J., Da Q., Wang Y., (2009), The dynamics of Bertrand model with bounded rationality, *Chaos, Solitons and Fractals* 39, 2048-2055.