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DUAL QUATERNIONS AS A TOOL FOR RIGID BODY MOTION ANALYSIS: A TUTORIAL WITH AN APPLICATION TO BIOMECHANICS

Dual quaternions and dual quaternion interpolation are powerful mathematical tools for the spatial analysis of rigid body motions. In this paper, after a review of some basic results and formulas, it will be presented an attempt to use these tools for the the kinematic modeling of human joints. In particular, the kinematic parameters extracted from experimentally acquired data are compared with those theoretically computed from dual quaternions rigid body motion interpolation.

1. Introduction

Rigid body spatial rotations and screw motions can be described by means of different mathematical entities such as matrices, quaternions, dual vectors and dual quaternions [1, 2].

In this paper the use of dual quaternions for the description of screw displacements will be discussed with an application to human motion analysis.

The mathematical definition of relative motion between human body segments is a complex task. This is due to the peculiar shape of the kinematic elements forming cartilagineous and synovial joints. For this reason, in the development of mathematical models, a common approach is the replacement of such joints with those whose modeling is simplified. For instance, the knee joint is usually substituted by an hinge joint, as shown in Fig. 1[1]*.

This investigation has the purpose of exploring the use of rigid motion interpolation for the mathematical modeling of elbow joints.

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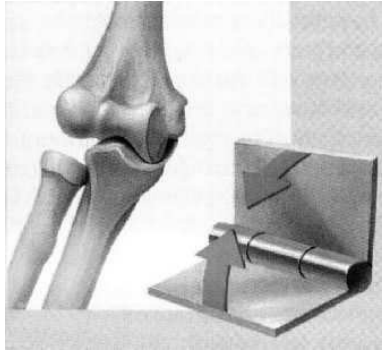* [1]Adapted from http://www.shockfamily.net/skeleton/JOINTS.HTML

Fig. 1. Kinematic modeling of a knee-joint

In particular, the approach herein presented can be applied to locate the rotation axis of the relative motion between human segments. The method is valid for a three dimensional motion and does not require the hypothesis of planar motion, as those proposed in [3, 4].

This paper is mainly divided into the following parts:

- tutorial on dual quaternions with the definition of the main algebraic operations;
- review and numerical comparison of some algorithms of rigid motion interpolation based on dual quaternions;
- application of these algorithms to human motion analysis.

2. Dual quaternion operators

Quaternions are four terms real numbers $\begin{pmatrix} q_r & q_x & q_y & q_z \end{pmatrix}$ which include a three-term vector with components q_x, q_y and q_z. Quaternions are usually represented in the form

$$\mathbf{Q} = q_r + q_x\vec{i} + q_y\vec{j} + q_z\vec{k} = q_r + \vec{q} \tag{1}$$

where q_r and $\vec{q}$ are the real and vector parts, respectively, and $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors associated with the axes of a Cartesian coordinate system.

The use of quaternions for the description of spatial rotation is reported in several bibliographical sources (e.g. Ref.[5, 6]).

A dual quaternion can be used to define a rigid body rotation of an angle θ about an axis $\vec{u}$ through the origin

$$\mathbf{Q} = \cos\frac{\theta}{2} + u_x \sin\frac{\theta}{2}\vec{i} + u_y \sin\frac{\theta}{2}\vec{j} + u_z \sin\frac{\theta}{2}\vec{k} = \cos\frac{\theta}{2} + \vec{u}\sin\frac{\theta}{2} \tag{2}$$
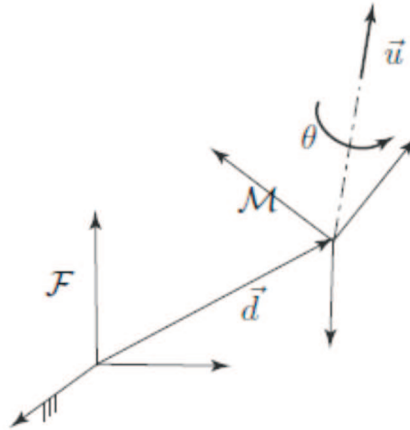
Fig. 2. Translation and rotation of a rigid body

Dual quaternions are mathematical entities [7, 8] whose four components are dual numbers. Therefore, introduced the dual unit ε such that $\varepsilon^2 = \varepsilon^3 = \ldots = 0$, they can be expressed as follows

$$\widehat{\mathbf{Q}} = \mathbf{Q} + \varepsilon \mathbf{Q}_O , \tag{3}$$

where

$$\mathbf{Q} = q_r + \vec{q} , \tag{4}$$
$$\mathbf{Q}_O = q_{rO} + \vec{q}_O , \tag{5}$$

Alternatively, dual quaternions can also be interpreted as a quaternion whose four terms are dual numbers

$$\widehat{\mathbf{Q}} = \widehat{q_r} + \widehat{q_x}\vec{i} + \widehat{q_y}\vec{j} + \widehat{q_z}\vec{k} . \tag{6}$$

By letting $\widehat{q_r} = 0$, a dual quaternion is transformed into a dual vector. Dual numbers, and dual vectors are special cases of a dual quaternion. In fact a dual number is a dual quaternion with zero vector part. A dual vector is a dual quaternion with zero scalar part. The dual quaternions follow the rules of quaternion algebra with the condition $\varepsilon^2 = 0$. As it will be discussed in this paper, dual quaternions allow a coincise representation of a screw rigid body motion.

Through the scalar part $\widehat{q_r}$ of the dual quaternion one can express the rotation angle θ and the translation distance s along the screw axis $\vec{h}$ as follows:

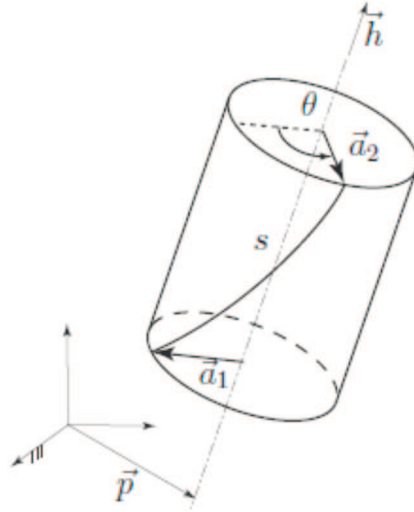$$\widehat{\theta} = 2\cos^{-1}\widehat{q_r} = \theta + \varepsilon s \tag{7}$$

Fig. 3. Screw motion of a rigid body

From the dual vector part of $\widehat{\mathbf{Q}}$

$$Vec(\widehat{\mathbf{Q}}) = \widehat{q_x}\vec{i} + \widehat{q_y}\vec{j} + \widehat{q_z}\vec{k} \tag{8}$$

one can retrieve the screw axis $\vec{h}$ and the vector $\vec{p}$ pointing from the origin to any point of such axis (see Fig. [3]).

In particular, with reference to the geometry of Fig. [2], these two motions are respectively characterized by the following quaternions [6]:

- Translation:

$$\widehat{\mathbf{Q}}_T = \left(2 + \varepsilon\vec{d}\right) , \tag{9}$$

where $\vec{d}$ is the vector oriented from the origin of the fixed Cartesian coordinate system $\mathcal{F}$ to the origin of the moving one $\mathcal{M}$;

- Rotation

$$\widehat{\mathbf{Q}}_R = \left(\cos\frac{\theta}{2} + \vec{u}\sin\frac{\theta}{2}\right) = e_0 + e_1\vec{i} + e_2\vec{j} + e_3\vec{k} , \tag{10}$$

with e_0, e_1, e_2 and e_3 the set of Euler parameters.

Denoting with $*$ the product between quaternions, the dual number quaternion which describe the combined screw motion is

$$\widehat{\mathbf{Q}} = \widehat{\mathbf{Q}}_T * \widehat{\mathbf{Q}}_R \tag{11}$$

whose components are

$$\widehat{q_r} = 2\cos\frac{\theta}{2} - \varepsilon \vec{u} \cdot \vec{d} \sin\frac{\theta}{2}$$

$$\widehat{q_x} = 2u_x \sin\frac{\theta}{2} + \varepsilon \left[d_x \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left(u_y d_z - d_y u_z \right) \right]$$

$$\widehat{q_y} = 2u_y \sin\frac{\theta}{2} + \varepsilon \left[d_y \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left(u_z d_x - d_z u_x \right) \right]$$

$$\widehat{q_z} = 2u_z \sin\frac{\theta}{2} + \varepsilon \left[d_z \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left(u_x d_y - d_x u_y \right) \right]$$

Unit dual quaternions represent rigid body motions only if the following constraint is satisfied [5]

$$\widehat{\mathbf{Q}} \cdot \widehat{\mathbf{Q}} = 1 \; , \tag{12}$$

therefore the components need to be normalized such that

$$\widehat{q_r}^2 + \widehat{q_x}^2 + \widehat{q_y}^2 + \widehat{q_z}^2 = 1 \tag{13}$$

If we interpret $\widehat{\mathbf{Q}}$ as a point in the four dimension dual space, the equation [12] represents a dual hypersphere. From these components one immediately obtains, by means of equation [7], the dual rotation angle $\widehat{\theta}$ and the line versor which defines the spatial location and direction of the screw axis

$$\widehat{\mathbf{h}} = \frac{Vec(\widehat{\mathbf{Q}})}{\sin\frac{\widehat{\theta}}{2}} = \vec{h} + \varepsilon \vec{p} \times \vec{h} = \left\{ \begin{array}{ccc} \widehat{h_x} & \widehat{h_y} & \widehat{h_z} \end{array} \right\}^T \tag{14}$$

whose components are

$$\widehat{h_x} = \frac{\widehat{q_x}}{\sin\frac{\widehat{\theta}}{2}} \; , \quad \widehat{h_y} = \frac{\widehat{q_y}}{\sin\frac{\widehat{\theta}}{2}} \; , \quad \widehat{h_z} = \frac{\widehat{q_z}}{\sin\frac{\widehat{\theta}}{2}} \tag{15}$$

Given the dual quaternion $\widehat{\mathbf{Q}}$, $\widehat{\mathbf{Q}}_T$ and $\widehat{\mathbf{Q}}_R$ can be retrieved, respectively, as follows [6]:

$$\widehat{\mathbf{Q}}_T = \widehat{\mathbf{Q}} * Q^* \tag{16a}$$

$$\widehat{\mathbf{Q}}_R = \frac{1}{Q * Q^*} Q \tag{16b}$$

where Q^* is the conjugate of the quaternion Q (see Appendix).

Since any spatial finite motion is decomposed into a translation and rotation of an angle θ about an axis $\vec{h}$, dual quaternions can be used as a operators to describe a sequence of finite motions.
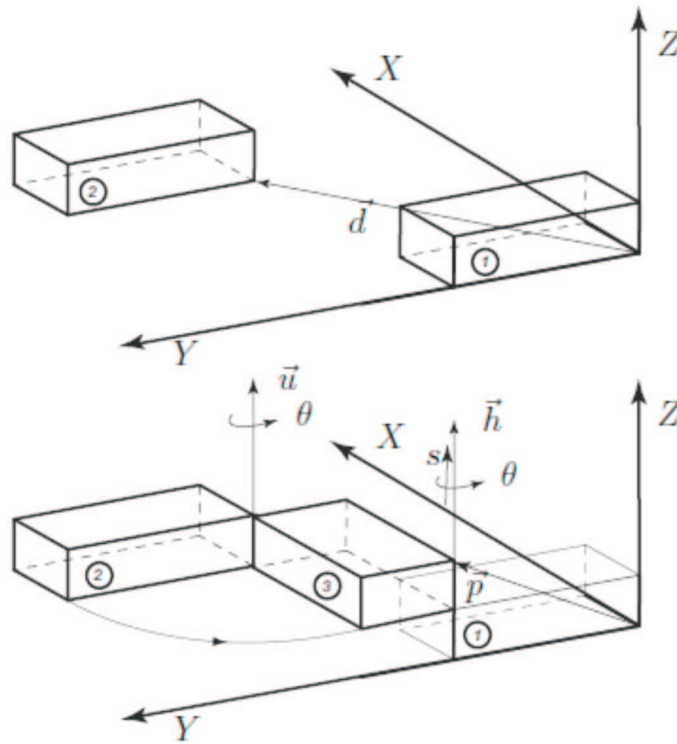
Fig. 4. Numerical example: Nomenclature

Numerical Example

A rigid body is subjected first to a translation of a vector $\vec{d} = \{\begin{array}{ccc} 2 & 2 & 1 \end{array}\}$ and then to a rotation of 90° about a vertical axis $\vec{u} = \{\begin{array}{ccc} 0 & 0 & 1 \end{array}\}$ passing through the origin of the Cartesian coordinate system attached to the body. We want compute the screw parameters of the overall finite motion by means of dual quaternions.

From [11], [7] and [14] one, respectively, obtains:

$$\widehat{\mathbf{Q}} = (1.4142 - \varepsilon 0.7071) + \varepsilon 2.8284\vec{i} + (1.4142 + \varepsilon 0.7071)\vec{k}$$

$$\widehat{\theta} = 1.570796 + \varepsilon$$

$$\widehat{\mathbf{h}} = \vec{k} + \varepsilon 2\vec{i}$$

Hence the screw axis $\vec{h}$ is parallel to $\vec{u}$ and contains the point $\vec{p} = \{\begin{array}{ccc} 0 & 2 & 0 \end{array}\}$ (see Fig. 4). The translation displacement of the body along the screw axis is $s = 1$.

3. Coordinate tranform by means of dual quaternions

In this section it will be discussed how to transform the coordinates of points and line vectors attached to a rigid body whose screw motion is defined by the dual quaternion $\widehat{\mathbf{Q}}$.

Let the vector $\{r_0\} \equiv \{\begin{array}{cccc} r_{0x} & r_{0y} & r_0z & 1 \end{array}\}^T$ contain the homogeneous coordinates of a generic point on a rigid body in its initial position.

Given $\widehat{\mathbf{Q}}$ by means of [16], [9], [10] one can extract $\vec{d}$, e_0, e_1, e_2 and e_3. Thus the transform matrix

$$\left[{}_0^1 A\right] = \begin{bmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2\left(e_1 e_2 - e_0 e_3\right) & 2\left(e_0 e_2 + e_1 e_3\right) & d_x \\ 2\left(e_0 e_3 + e_1 e_2\right) & e_0^2 + e_2^2 - e_1^2 - e_3^2 & 2\left(e_2 e_3 - e_0 e_1\right) & d_y \\ 2\left(e_1 e_3 - e_0 e_2\right) & 2\left(e_0 e_1 + e_2 e_3\right) & e_0^2 + e_3^2 - e_1^2 - e_2^2 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

of real elements can be formed.

The vector of homogeneous coordinates of the point in its final position is

$$\{r_1\} = \left[{}_0^1 A\right]\{r_0\} \tag{18}$$

where $\{r_1\} \equiv \left\{\begin{array}{cccc} r_{1x} & r_{1y} & r_{1z} & 1 \end{array}\right\}^T$.
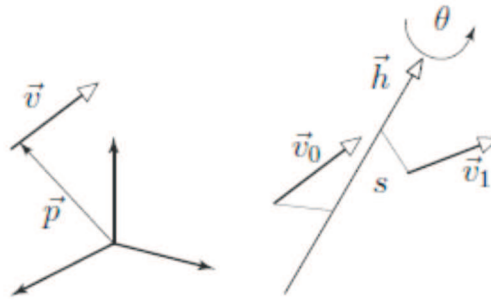

Fig. 5. Definition of line vector (left) and screw displacement of a line vector (right)

With reference to the geometry of Fig. 5, a line vector $\vec{v}$ can be represented by means of the dual vector

$$\widehat{\mathbf{v}} = \vec{v} + \varepsilon\left(\vec{p} \times \vec{v}\right) \tag{19}$$

where $\vec{p}$ is the vector joining the origin of the Cartesian reference system with any point of the line.

The new line vector $\widehat{\mathbf{v_1}}$ of a line vector $\widehat{\mathbf{v_0}}$ after a screw motion defined by a dual quaternion $\widehat{\mathbf{Q}}$ is obtained through the matrix product

$$\widehat{\mathbf{v_1}} = \left[{}_0^1 \widehat{A}\right]\widehat{\mathbf{v_0}} \tag{20}$$

where

$$\left[{}_0^1 \widehat{A}\right] = \begin{bmatrix} \widehat{q}_r^2 + \widehat{q}_x^2 - \widehat{q}_y^2 - \widehat{q}_z^2 & 2\left(\widehat{q}_x \widehat{q}_y - \widehat{q}_r \widehat{q}_z\right) & 2\left(\widehat{q}_r \widehat{q}_y + \widehat{q}_x \widehat{q}_z\right) \\ 2\left(\widehat{q}_r \widehat{q}_z + \widehat{q}_x \widehat{q}_y\right) & \widehat{q}_r^2 + \widehat{q}_y^2 - \widehat{q}_x^2 - \widehat{q}_z^2 & 2\left(\widehat{q}_y \widehat{q}_z - \widehat{q}_r \widehat{q}_x\right) \\ 2\left(\widehat{q}_x \widehat{q}_z - \widehat{q}_r \widehat{q}_y\right) & 2\left(\widehat{q}_r \widehat{q}_x + \widehat{q}_y \widehat{q}_z\right) & \widehat{q}_r^2 + \widehat{q}_z^2 - \widehat{q}_x^2 - \widehat{q}_y^2 \end{bmatrix} \tag{21}$$

is an orthogonal matrix.

4. How to obtain screw parameters from point coordinates

Dual algebra can be conveniently used to retrieve the screw parameters of a rigid body motion from the measurement of point coordinates [9, 10].

Let us denote with $\vec{r}_{0i}$ and $\vec{r}_i$ $(i = 1, 2, \ldots, n)$, $(n \geq 4)$ respectively, the initial and final coordinates of points attached to a body subjected to a screw motion. These coordinates can be collected through different experimental techniques such as photogrammetry, magnetic sensors, laser sensors, etc. The initial and final positions of the centroid of the points are given by

$$\vec{c}_{0i} = \frac{1}{n} \sum_i \vec{r}_{0i} \tag{22a}$$

$$\vec{c}_i = \frac{1}{n} \sum_i \vec{r}_i \; . \tag{22b}$$

The initial and final positions of line vectors attached to the moving body are expressed, respectively, by the following dual vectors

$$\widehat{\mathbf{r}}_{0i} = \vec{r}_{0i} - \vec{c}_{0i} + \varepsilon \vec{c}_{0i} \times (\vec{r}_{0i} - \vec{c}_{0i}) \tag{23a}$$

$$\widehat{\mathbf{r}}_i = \vec{r}_i - \vec{c}_i + \varepsilon \vec{c}_i \times (\vec{r}_i - \vec{c}_i) \tag{23b}$$

In the absence of errors, the following equality would hold:

$$\left[{}_0^1\widehat{A}\right] \widehat{\mathbf{r}}_{0i} = \widehat{\mathbf{r}}_i \tag{24}$$

with $\left[{}_0^1\widehat{A}\right]$ expressed by [21]. However, due to the presence of measurement errors

$$\left[{}_0^1\widehat{A}\right] \widehat{\mathbf{r}}_{0i} \approx \widehat{\mathbf{r}}_i \; , \tag{25}$$

where $\left[{}_0^1\widehat{A}\right]$ is in this case an unknown matrix to be computed trying to minimize the differences with the least squares optimality criterion.

In this section a two step method is proposed:
1. After forming the matrices

$$\left[\widehat{\mathsf{R}}_0\right] = \left[\begin{array}{cccc} \widehat{\mathbf{r}}_{01} & \widehat{\mathbf{r}}_{02} & \ldots & \widehat{\mathbf{r}}_{0n} \end{array}\right] \tag{26a}$$

$$\left[\widehat{\mathsf{R}}_1\right] = \left[\begin{array}{cccc} \widehat{\mathbf{r}}_1 & \widehat{\mathbf{r}}_2 & \ldots & \widehat{\mathbf{r}}_n \end{array}\right] \tag{26b}$$

a dual tranform $\left[{}_0^1\widehat{A}\right]$ matrix is simply obtained as follows

$$\left[{}_0^1\widehat{A}\right] = \left[\widehat{\mathsf{R}}_1\right]\left[\widehat{\mathsf{R}}_0\right]^+ \tag{27}$$

where $\left[\widehat{\mathsf{R}}_0\right]^+$ denotes the pseudoinverse of $\left[\widehat{\mathsf{R}}_0\right]$.
Then the dual QR decomposition is applied

$$\left[{}^1_0\widehat{A}\right] = \left[\widehat{Q}\right]\left[\widehat{R}\right] \tag{28}$$

and we let

$$\left[{}^1_0\widehat{A}\right] = \left[\widehat{Q}\right] . \tag{29}$$

In this way it is ensured that $\left[{}^1_0\widehat{A}\right]$ is orthogonal.

Under ideal conditions the matrix $\left[\widehat{R}\right]$ will result into an identity matrix. Hence, its elements can be used as a rough estimate of the deviation from rigid body condition.

2. From $\left[{}^1_0\widehat{A}\right]$ the screw motion parameters $\widehat{\theta}$ and $\widehat{\mathbf{h}}$ are then retrieved [10].

Numerical Example

Let us denote respectively with

$$\{r_{01}\} = \left\{\begin{array}{ccc} 1 & 0 & 0 \end{array}\right\}^T , \quad \{r_{02}\} = \left\{\begin{array}{ccc} 1 & 2 & 0 \end{array}\right\}^T ,$$
$$\{r_{03}\} = \left\{\begin{array}{ccc} 1 & 2 & 1 \end{array}\right\}^T , \quad \{r_{04}\} = \left\{\begin{array}{ccc} 0 & 2 & 1 \end{array}\right\}^T .$$

$$\{r_1\} = \left\{\begin{array}{ccc} 2 & 3 & 1 \end{array}\right\}^T , \quad \{r_2\} = \left\{\begin{array}{ccc} 0 & 3 & 1 \end{array}\right\}^T ,$$
$$\{r_3\} = \left\{\begin{array}{ccc} 0 & 3 & 2 \end{array}\right\}^T , \quad \{r_4\} = \left\{\begin{array}{ccc} 0 & 2 & 2 \end{array}\right\}^T .$$

the initial and final positions of four points attached to a body. These data refer to the motion shown in Fig. 4.

Since the data are not affected by errors, the matrix $\left[\widehat{R}\right]$ is the identity matrix and directly from [27] follows the transform matrix

$$\left[{}^1_0\widehat{A}\right] = \left[\begin{array}{ccc} -\varepsilon & -1 & 2\varepsilon \\ 1 & -\varepsilon & -2\varepsilon \\ 2\varepsilon & 2\varepsilon & 1 \end{array}\right]$$

The screw motion parameters coincide with those obtained in the previous numerical example.

5. Rigid body motion interpolation techniques

In this section some basic dual quaternions interpolation algorithms are presented.

5.1. Screw linear interpolation

This is a generalization of the well known Spherical Linear Interpolation (SLERP) scheme. Let denote by $\widehat{\mathbf{Q}}_1$ and $\widehat{\mathbf{Q}}_2$ two dual quaternions expressing the initial and final pose of a rigid body, respectively.

The ScLERP function (Screw Linear Interpolation) is defined as follows

$$ScLERP\left(t; \widehat{\mathbf{Q}}_1, \widehat{\mathbf{Q}}_2\right) = \widehat{\mathbf{Q}}_1 * \left(\widehat{\mathbf{Q}}_1^{-1} * \widehat{\mathbf{Q}}_2\right)^t \tag{30}$$

with $t \in [0, 1]$.

A kinematic interpretation of this interpolation scheme is presented in Refs. [11, 12]. Since $\widehat{\mathbf{Q}}_1^{-1} * \widehat{\mathbf{Q}}_2$ represents the finite screw motion between the initial and final pose of the rigid body, the product

$$\left(\widehat{\mathbf{Q}}_1^{-1} * \widehat{\mathbf{Q}}_2\right)^t = \cos\left(t\frac{\widehat{\theta}}{2}\right) + \sin\left(t\frac{\widehat{\theta}}{2}\right)\widehat{\mathbf{u}} \tag{31}$$

defines a screw motion of a dual angle $t\widehat{\theta}$ along the screw axis.

Algorithm 1 sDLB: Simple Dual quaternion Linear Blending

1: **procedure** SDLB($\widehat{\mathbf{Q}}_1, \widehat{\mathbf{Q}}_2, t$)

2: $\qquad SDLB\left(t; \widehat{\mathbf{Q}}_1, \widehat{\mathbf{Q}}_2\right) = \dfrac{(1 - t)\,\widehat{\mathbf{Q}}_1 + t\widehat{\mathbf{Q}}_2}{\left\|(1 - t)\,\widehat{\mathbf{Q}}_1 + t\widehat{\mathbf{Q}}_2\right\|}$

3: **end procedure**

5.2. Dual quaternion linear blending

The dual quaternion linear blending interpolation scheme (DLB) is defined as follows

$$DLB\left(t; \widehat{\mathbf{Q}}_1, \widehat{\mathbf{Q}}_2\right) = \frac{(1 - t)\,\widehat{\mathbf{Q}}_1 + t\widehat{\mathbf{Q}}_2}{\left\|(1 - t)\,\widehat{\mathbf{Q}}_1 + t\widehat{\mathbf{Q}}_2\right\|} \tag{32}$$

This can be extended to the interpolation of several poses as follows

$$DLB\left(t; \widehat{\mathbf{Q}}_1, \ldots, \widehat{\mathbf{Q}}_n\right) = \frac{w_1(t)\widehat{\mathbf{Q}}_1 + \ldots + w_n(t)\widehat{\mathbf{Q}}_n}{\left\|w_1(t)\widehat{\mathbf{Q}}_1 + \ldots + w_n(t)\widehat{\mathbf{Q}}_n\right\|} \tag{33}$$

In Refs. [12, 13] the weights are assumed to be convex(*i.e.* $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$).

In this investigation, the weights are made coincident with the coefficients of the classical Lagrange's interpolating polynomials

$$w_j(t) = \prod_{\substack{k = 1 \\ k \neq j}}^{n} \frac{t - t_k}{t_j - t_k} \tag{34}$$

Algorithm 2 DLB: Extension of the Dual quaternion Linear Blending

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1: **procedure** DLB$(\widehat{\mathbf{Q}}_1, \ldots, \widehat{\mathbf{Q}}_n, t)$

2:     Compute weights w_i $(i = 1, \ldots, n)$

3:     $DLB\left(t; \widehat{\mathbf{Q}}_1, \ldots, \widehat{\mathbf{Q}}_n\right) = \dfrac{w_1(t)\widehat{\mathbf{Q}}_1 + \ldots + w_n(t)\widehat{\mathbf{Q}}_n}{\left\| w_1(t)\widehat{\mathbf{Q}}_1 + \ldots + w_n(t)\widehat{\mathbf{Q}}_n \right\|}$

4: **end procedure**

There are different possibilities for the choice of the t value [6]. In this investigation the value of t coincides with the Euclidean distance between the coordinates of origins of the moving Cartesian coordinate system in the different poses.

5.3. Dual quaternion iterative blending

If in the DLB algorithm the lengthy normalization is avoided, then the resulting dual quaternion does not necessarily belong to the hypersphere. A refinement of DLB is the Dual quaternion iterative blending proposed in Ref. [13].

The steps of the improved algorithm are outlined in Algorithm 3

Algorithm 3 DIB: Dual quaternion Iterative Blending

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  procedure DIB$(\widehat{\mathbf{Q}}_1, \ldots, \widehat{\mathbf{Q}}_n, t)$ $\widehat{\mathbf{B}} = DLB(\widehat{\mathbf{Q}}_1, \ldots, \widehat{\mathbf{Q}}_n, t)$

    repeat $\widehat{\mathbf{X}} = \displaystyle\sum_{i=1}^{n} w_i \log\left(\widehat{\mathbf{B}}^{-1} * \widehat{\mathbf{Q}}_i\right)$ $\widehat{\mathbf{B}} = \widehat{\mathbf{B}} * \exp\left(\widehat{\mathbf{X}}\right)$

    until $\left\|\widehat{\mathbf{X}}\right\| \leq h$

  end procedure

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6. Numerical tests on dual quaternion interpolation

In this section we will report the numerical results obtained with the interpolating schemes previously described. In particular, we will prescribe the poses of a rigid body compute the intermediate poses through interpolation.

The numerical results obtained with the algorithms herein considered have been summarized in Tab. 1. In particular, with the bold characters are denoted the numerical data defining the prescribed poses $\widehat{\mathbf{Q}}_1, \ldots, \widehat{\mathbf{Q}}_n$. The coincidence between the numerical results of the ScLERP and DIB algorithms is due to the simmetry of the intermediate pose with respect to the first and last one.

Table 1.

Numerical results

	ScLERP				sDLB		
t	$\vec{u}$	$\vec{d}$	θ (rad)	t	$\vec{u}$	$\vec{d}$	θ (rad)
0	$\{0,0,1\}^T$	$\{0,0,0\}^T$	**0.000000**	**0**	$\{0,0,1\}^T$	$\{0,0,0\}^T$	**0.000000**
0.2	$\{0,0,1\}^T$	$\{0,0,2\}^T$	0.157080	0.2	$\{0,0,1\}^T$	$\{0,0,1.925100\}^T$	0.155128
0.4	$\{0,0,1\}^T$	$\{0,0,4\}^T$	0.314159	0.4	$\{0,0,1\}^T$	$\{0,0,3.962077\}^T$	0.313176
0.6	$\{0,0,1\}^T$	$\{0,0,6\}^T$	0.471239	0.6	$\{0,0,1\}^T$	$\{0,0,6.037923\}^T$	0.472222
0.8	$\{0,0,1\}^T$	$\{0,0,8\}^T$	0.628319	0.8	$\{0,0,1\}^T$	$\{0,0,8.074900\}^T$	0.630270
1.0	$\{0,0,1\}^T$	$\{0,0,10\}^T$	**0.785398**	**1.0**	$\{0,0,1\}^T$	$\{0,0,10\}^T$	**0.785398**

	DLB				DIB		
t	$\vec{u}$	$\vec{d}$	θ (rad)	t	$\vec{u}$	$\vec{d}$	θ (rad)
0	$\{0,0,1\}^T$	$\{0,0,0\}^T$	**0.000000**	**0**	$\{0,0,1\}^T$	$\{0,0,0\}^T$	**0.000000**
0.2	$\{0,0,1\}^T$	$\{0,0,2.036640\}^T$	0.158043	0.2	$\{0,0,1\}^T$	$\{0,0,1.999999\}^T$	0.157080
0.4	$\{0,0,1\}^T$	$\{0,0,4.018432\}^T$	0.314643	0.4	$\{0,0,1\}^T$	$\{0,0,4.000002\}^T$	0.314159
0.5	$\{0,0,1\}^T$	$\{0,0,5\}^T$	**0.392699**	**0.5**	$\{0,0,1\}^T$	$\{0,0,5\}^T$	**0.392699**
0.6	$\{0,0,1\}^T$	$\{0,0,5.981568\}^T$	0.470756	0.6	$\{0,0,1\}^T$	$\{0,0,5.999998\}^T$	0.471239
0.8	$\{0,0,1\}^T$	$\{0,0,7.963360\}^T$	0.627355	0.8	$\{0,0,1\}^T$	$\{0,0,8.000001\}^T$	0.628318
1.0	$\{0,0,1\}^T$	$\{0,0,10\}^T$	**0.785398**	**1.0**	$\{0,0,1\}^T$	$\{0,0,10\}^T$	**0.785398**

7. Biomechanic analysis by means of rigid body motion interpolation

In the current investigation the application of motion interpolation techniques to the mathematical description of human body motion is attempted.

Motion capture systems supply the user with numerical data regarding the cartesian coordinates of markers on a body. Through dual algebra, given these coordinates, one can compute the pose of the body [10, 14, 15].

Our aim is to describe the relative motion between two adjacent anatomical segments through a limited finite numbers of poses of the two bodies. The

missing poses are estimated by means of the dual quaternion interpolation techniques mentioned in the previous section.

7.1. Experimental human motion analysis

The motion capture system OptiTrack has been used for the experimental tracking of markers positions. In our analysis four passive markers for each body segment (arm and forearm) are used.

Since we are interested in the relative motion forearm.vs.arm, the arm is kept still and only the forearm is moved.

OptiTrack allows to associate a set of at least three markers with a rigid body. The pattern of the markers must be different. For each rigid body the software returns:
–   the absolute coordinates of the baricenter of the markers;
–   the Euler parameters of Cartesian system framed to the moving body (see Fig. 9).

The motion of the markers is monitored with eight FLEX:V100 cameras acquiring at 100 frames per second. The positions of the cameras is such that the markers are always visible at least by four cameras.

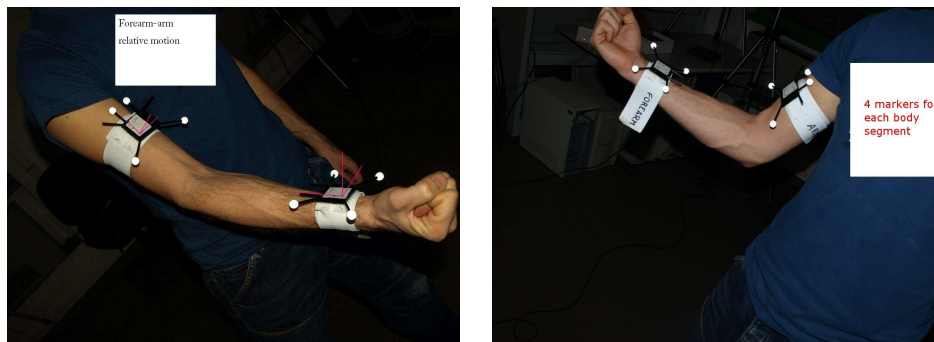The markers positions on the body segments are shown in Fig. 6.

Fig. 6. Positions of the markers

The angle of rotation θ of the forearm with respect to the arm, as obtained from the Euler parameters computed by the Optitrack software is plotted in Fig. 7.

In particular, at each frame, from the values provided by the Optitrack system, the dual quaternions (9), (10) and (11) are readily obtained.

Let $\widehat{\mathbf{Q}}_0$ and $\widehat{\mathbf{Q}}_i$ the dual quaternions at the beginning of the acquisition and at the $i^t h$ frame. One can transform these quaternions into the materices $\left[\widehat{\mathbf{A_0}}\right]$ and $\left[\widehat{\mathbf{A_i}}\right]$, respectively.

The dual matrix describing the relative motion is

$$\left[{}_{0}^{i}\widehat{A} \right] = \left[\widehat{A_0} \right]^{-1} \left[\widehat{A_i} \right] \tag{35}$$

From this matrix the motion parameters can be obtained [10].

7.2. Human motion interpolation by means of dual quaternions

The motion of the forearm has been reproduced through dual quaternion motion interpolation.

In particular three dual quaternions have been interpolated by means of the DLB algorithm. The rotation angle θ of the forearm with respect to the arm can be estimated from the interpolated dual quaternions.
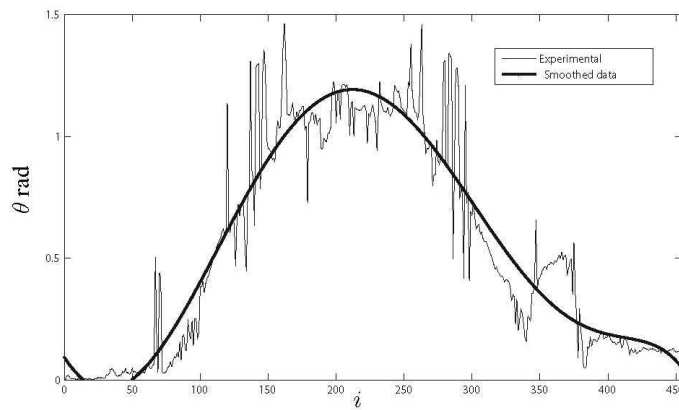

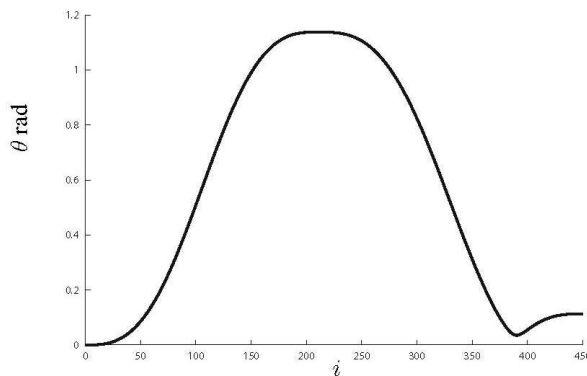Fig. 7. Experimental rotation angle θ forearm w.r.t. arm

Fig. 8. Rotation angle θ forearm w.r.t. arm from dual quaternion interpolation

The quality of interpolation depends strongly on the law of variation of parameter t. A linear variation of t (*i.e.* of θ) has been discarded because would result in a linear increase and decrease of the rotation angle θ. Observing the experimental plot of Fig. 7 one immediately conclude that the linear law is not appropriate for θ.

In our analysis a cycloidal variation of t has been applied. This choice is supported by the experimental evidence. Furthermore in biomechanics several authors hinted that skilled human movements are associated with a decrease in jerk ([4], p.207). In the field of cam design, due to the minimization of jerk, the cycloidal law is often adopted for the cam profiles.

In particular, the following function has been prescribed for the t parameter:

$$t = \begin{cases} \dfrac{1}{\pi}\left[\dfrac{\pi}{i_h}i - \dfrac{1}{2}\sin\left(\dfrac{2\pi}{i_h}i\right)\right] & 0 \le i < i_h \\[3ex] \dfrac{1}{\pi}\left[\dfrac{\pi}{n - i_h}(i - i_h) - \dfrac{1}{2}\sin\dfrac{2\pi}{n - i_h}(i - i_h)\right] & i_h \le i \le n \end{cases} \tag{36}$$

where n is the total number of frames monitored and i_h the frame where the motion inversion begins. In our case $n = 450$ and $i_h = 211$.
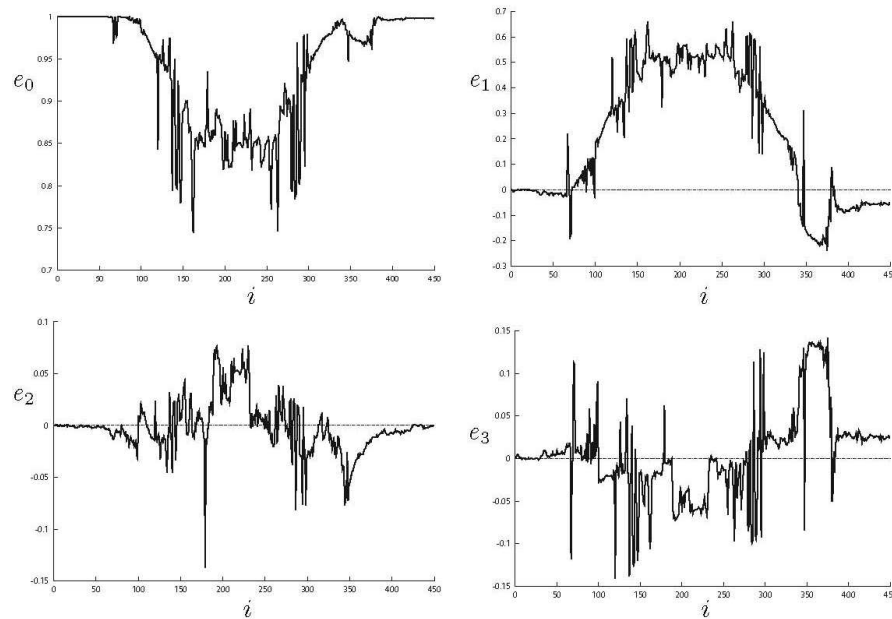

Fig. 9. Euler parameters of the forearm from experimental data

The plot of angle θ obtained from the interpolation and a variation of t according to (36) is shown in Fig. 8. The values of Euler parameters computed from the interpolated dual quaternions are plotted in Fig. 10.
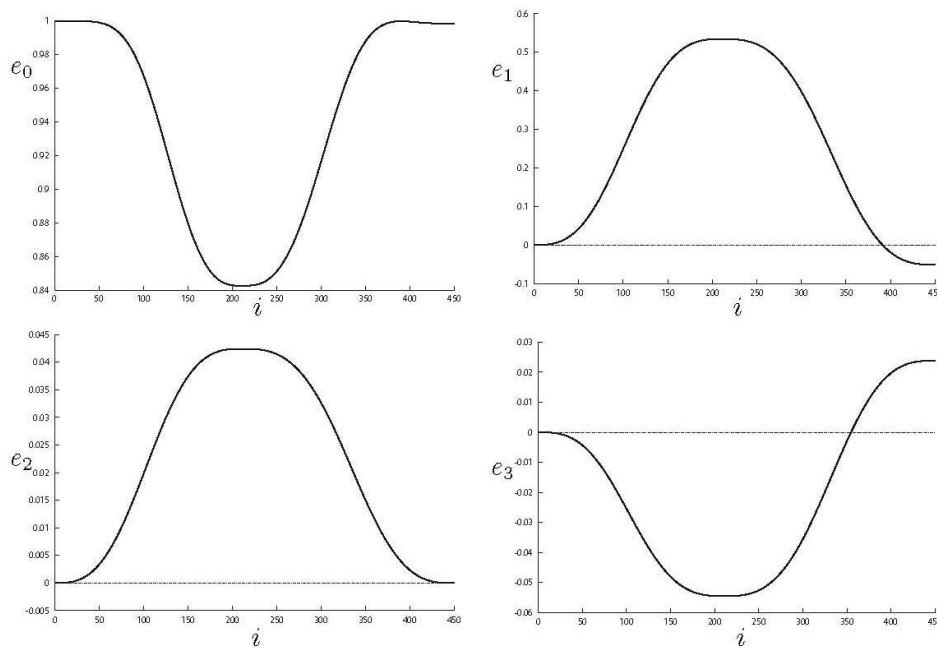

Fig. 10. Euler parameters of the forearm from dual quaternion interpolation

8. Conclusions

In this paper some preliminary results regarding the application of dual quaternion interpolation to biomechanics have been presented.

Although some issues remain open these results are encouraging. The choice of the cycloidal law for the variation of the t parameter seems appropriate, although other choices need to be evaluated. The possibility of using the dual quaternion iterative blending (DIB) interpolation scheme with $n \geq 3$ is currently under test.

Our impression is that the matching between experimental and theoretical analyses could be improved through the filtering of data and by reducing the acquisition frequency of the cameras.

In this field it is very important the availability of a simple analytical criterion for the evaluation of differences between experimental motion and theoretically computed rigid body features. There are some scientific contri-

butions toward the development of such criterion, but it seems that there is not a general consensus.

REFERENCES

[1] M. Hiller, C. Woernle, A unified representation of spatial displacements. Mechanism and Machine Theory, 19, 477-486, 1984.

[2] J.S. Dai, An historical review of the theoretical development of rigid body displacements from Rodrigues parameters to the finite twist. Mechanism and Machine Theory, 41, 41-52, 2006.

[3] L. Chèze, and J. Dimnet, Three-Dimensional Analysis of Human Movement, chapter Modeling Human Body Motions by the Techniques Known to Robotics, 177-200. Human Kinetics, 1995.

[4] V. M. Zatsiorsky, Kinematics of Human Motion, Human Kinetics, 1998.

[5] M. McCarthy, Introduction to theoretical kinematics. The MIT Press, 1990.

[6] B. Jüttler, Visualization of moving objects using dual quaternion curves. Computers & Graphics, 18(3), 315-326, 1994.

[7] A. McAulay, Octonions - A Development of Clifford's Bi-Quaternions. Cambridge University Press, 1898.

[8] A. T. Yang, Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms. PhD thesis, Columbia University, 1963.

[9] E. Pennestrì, R. Stefanelli, Linear algebra and numerical algorithms using dual numbers. Multibody System Dynamics, 18, 323-344, 2007.

[10] E. Pennestrì, and P.P. Valentini, Linear dual algebra algorithms and their Application to Kinematics. In C.L. Bottasso, editor, Multibody Dynamics Computational Methods and Applications, Vol. 12, Springer Verlag, 2008.

[11] K.R. Etzel, and J.M. McCarthy, Spatial motion interpolation in an image space of so(4). In Proceedings of The 1996 ASME Design Engineering Technical Conference and Computers in Engineering Conference, 96-DETC/MECH-1164, 1996.

[12] L. Kavan,and S. Collins, and C. O'Sullivan, and J. Zara, Dual Quaternions for Rigid Transformation Blending. Technical Report TCD-CS-2006-46, The University of Dublin, Trinity College, 2006.

[13] L. Kavan, and S. Collins, and Zara, and C. O'Sullivan, Geometric Skinning with Approximate Dual Quaternion Blending. ACM Transaction on Graphics, 27, 105, 2008.

[14] K.K. Teu, and W. Kim, Estimation of the axis of a screw motion from noisy data—A new method based on Plücker lines. Journal of Biomechanics, 39, 2857-2862, 2006.

[15] A. Page, and V. Mata, and J.V. Hoyos, and R. Porcar, Experimental determination of instantaneous screw axis in human motion. Error analysis.Mechanism and Machine Theory Mechanism and Machine Theory, 42, 429-441, 2007.

[16] H. H. Cheng, Computation of dual numbers in the extended finite dual plane. In Proc. of the 1993 ASME Design Automation Conference, 73-80, Sept. 19-22, 1993.

[17] H. H. Cheng, Programming with dual numbers and its applications in mechanisms design. Engineering with Computers, 10(4), 212-229, 1994.

[18] K. Wohlhart, Motor Tensor Calculus. In J. P. Merlet and B. Ravani, editors, Computational Kinematics 93-102. Kluwer Academic Publishers, 1995.

9. Appendix: Basic dual quaternion operations

For a presentation of algebraic operations on dual numbers and dual vectors the reader may refer to Refs. [16, 17, 18, 9, 10].

A dual quaternion is a vector of the form

$$\widehat{\mathbf{Q}} = \begin{pmatrix} \widehat{d} \\ \widehat{a} \\ \widehat{b} \\ \widehat{c} \end{pmatrix} \tag{37}$$

where the components $\widehat{d}$, $\widehat{a}$, $\widehat{b}$ and $\widehat{c}$ are dual numbers. In order to express in a compact notation the basic algebrai operations of dual quaternions, it is convenient to split the dual quaternion in a dual scalar part $\widehat{d}$ and dual vector part $\widehat{\mathbf{v}} = \widehat{a}\,\vec{i} + \widehat{b}\,\vec{j} + \widehat{c}\,\vec{k}$ as follows

$$\widehat{\mathbf{Q}} = \widehat{d} + \widehat{\mathbf{v}} \tag{38}$$

1. **Sum**

$$\widehat{\mathbf{Q}}_1 + \widehat{\mathbf{Q}}_2 = \widehat{d_1} + \widehat{d_2} + \widehat{\mathbf{v}}_1 + \widehat{\mathbf{v}}_2 \tag{39}$$

2. **Product**

$$\widehat{\mathbf{Q}}_1 * \widehat{\mathbf{Q}}_2 = \begin{pmatrix} \widehat{d_1}\widehat{d_2} - \langle \widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2 \rangle \\ \widehat{d_1}\widehat{\mathbf{v}}_2 + \widehat{d_2}\widehat{\mathbf{v}}_1 + \widehat{\mathbf{v}}_1 \times \widehat{\mathbf{v}}_2 \end{pmatrix} \tag{40}$$

where $\langle \widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2 \rangle$ and $\widehat{\mathbf{v}}_1 \times \widehat{\mathbf{v}}_2$ denote the dot and vector products, respectively.

3. **Conjugation**

$$\widehat{\mathbf{Q}}^* = \widehat{d} - \widehat{\mathbf{v}} \tag{41}$$

4. **Norm**

$$\left\| \widehat{\mathbf{Q}} \right\| = \sqrt{\widehat{\mathbf{Q}} * \widehat{\mathbf{Q}}^*} = \|\mathbf{Q}_O\| + \varepsilon \frac{\langle \mathbf{Q}, \mathbf{Q_O} \rangle}{\|\mathbf{Q_O}\|}$$

$$= \sqrt{\widehat{d}^{\,2} + \widehat{a}^{\,2} + \widehat{b}^{\,2} + \widehat{c}^{\,2}} \tag{42}$$

5. **Inverse**

$$\widehat{\mathbf{Q}}^{-1} = \frac{\widehat{\mathbf{Q}}^*}{\left\| \widehat{\mathbf{Q}} \right\|^2} \tag{43}$$

6. **Exponential**

$$\exp\left(\widehat{\mathbf{Q}} \right) = \exp\left(\widehat{d} \right) \begin{pmatrix} \cos\left(\|\widehat{\mathbf{v}}\| \right) \\ \dfrac{\widehat{\mathbf{v}}}{\|\widehat{\mathbf{v}}\|} \sin\left(\|\widehat{\mathbf{v}}\| \right) \end{pmatrix} \tag{44}$$

The extension of Euler's identity to dual quaternions is expressed by the following unit dual quaternion

$$\exp\left(\widehat{\mathbf{u}} \frac{\widehat{\theta}}{2} \right) = \cos \frac{\widehat{\theta}}{2} + \sin \frac{\widehat{\theta}}{2} \widehat{\mathbf{u}} \tag{45}$$

7. **Logarithm**

$$\ln\left(\widehat{\mathbf{Q}} \right) = \begin{pmatrix} \ln\left(\left| \widehat{\mathbf{Q}} \right| \right) \\ \dfrac{\widehat{\mathbf{v}}}{\|\widehat{\mathbf{v}}\|} \arccos\left(\dfrac{\widehat{d}}{\left\| \widehat{\mathbf{Q}} \right\|} \right) \end{pmatrix} \tag{46}$$

If $\widehat{\mathbf{Q}}$ is a unit dual quaternion expressed by (45), then

$$\ln\left(\widehat{\mathbf{Q}}\right) = \widehat{\mathbf{u}}\frac{\widehat{\theta}}{2} \tag{47}$$

8. **Power**

$$\widehat{\mathbf{Q}}^t = \exp\left(\ln\left(\widehat{\mathbf{Q}}\right)t\right) \tag{48}$$

Kwaterniony dualne jako narzędzie analizy ruchu ciał sztywnych.
Przykład zastosowań w biomechanice

S t r e s z c z e n i e

Kwaterniony dualne i interpolacja z użyciem kwaternionów dualnych stanowią silne narzędzia matematyczne wykorzystywane analizy ruchu przestrzennego ciał sztywnych. W artykule przedstawiono przegląd podstawowych wzorów i wyników, a następnie zaprezentowano próbę użycia tych narzędzi do modelowania kinematyki stawów w ciele człowieka. W szczególności, parametry kinematyczne wyznaczone na podstawie danych eksperymentalnych porównano z wyliczonymi teoretycznie na podstawie interpolacji ruchu ciał sztywnych z użyciem kwaternionów dualnych.