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Discrete-time feedback stabilization

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This paper presents an algorithm for designing dynamic compensator for infinite-dimensional systems with bounded input and bounded output operators using finite dimensional approximation. The proposed method was then implemented in order to find the control function for thin rod heating process. The optimal sampling time was found depending on discrete output measurements.

Key words: stabilization, feedback stabilization, finite dimensional stabilization, infinite dimensional systems, finite dimensional approximations, continuous-discrete system.

1. Introduction

One of the main areas of automatic control is related to stabilization problems. Usually, in real time application, an algorithm consisting of two stages is used: 1. Bring the system to the valid region of linearization. 2. Stabilize the system using linear approximation. This approach is justified by topological similarity of a nonlinear system and its linearization (valid only for hyperbolic systems without purely imaginary eigenvalues).

Feedback design (design of the stabilizing controller) depends on the system form (usually we have either differential equations or transfer function for time invariant systems).

The design of finite dimensional feedback is useful due to multiple reasons: 1. It is possible to use simple, finite-dimensional methods, e.g., Lyapunov functions and in consequence, Lyapunov equations strictly linked with algebraic Riccati equations. 2. Some of the systems have predefined structure, e.g., the hoisting machine (long line is a distributed system, and the drive may be modeled with finite-dimensional system).

The design of finite-dimensional controllers for infinite systems with finite set of unstable modes (or at least weakly damped ones) is widely analyzed in literature. This class of the systems was described by Triggiani (1975) [34], or even earlier by Fattorini (1967) [11]. Using small disturbance methods and building appropriate invariant sets, Schumacher (1981, 1983) [30, 31] proposed finite dimensional stabilizing controllers

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for distributed and delayed systems (see also Kamen (1985) [14]). Similar results were obtained by Curtain (1984) [4] for parabolic systems with infinite input-output operators. Also the works of Curtain and Salomon (1986) [5], and Sakawa (1983, 1984, 1985) [27-29] are worth noticing. Balas (1983) [2] proposed a finite dimensional dynamic compensator for finite dimensional approximations of infinite systems. Similar methods were proposed by Kobayashi (1983) [16]. Gibson (1981) [13] used finite dimensional approximation of algebraic Riccati equation. The detailed description of those works was done, e.g., by Mitkowski (1991) [20] with 229 books and papers analyzed.

The design of stabilizing controllers is still an interesting problem (see, e.g. Przyşuski (2014) [26]), especially as there are more efficient numerical tools. Thanks to computers, nowadays, we can analyze complex mathematical models of distributed parameter systems, e.g. models of non-integer order Obrźczka (2014) [22], Sierociuk (2015) [32], Oprzêdkiewicz (2016) [24] which sometimes better describe real systems.

In this work, we focused on an algorithm of stabilization of linear infinite dimensional system with bounded input and bounded output operators and with finite set of unstable modes (weakly damped) using finite discrete stabilization. As an example, we used diffusion equation which models the heating process of a thin rod.

2. Problem description

Consider a closed-loop system (with continuous time) shown in Fig. 1.

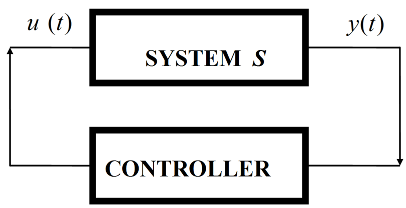

Figure 1: Closed-loop system.

Finite dimensional stabilization problem: for a given infinite system S find a stabilizing controller (finite dimensional) such that the closed-loop system is exponentially stable with predefined damping coefficient.

In digital control, it is necessary to use a discrete system (computer or other device with discrete time). In order to use a discrete stabilizing controller in continuous time system, we need to use the system (see Fig. 2) in the form of a series of pulser, continuous linear system S, and ZOH (Zero Order Hold) with input $u(k)$ and output $y(k)$, $k = 1,2,3,\dots$.

If the pulser and ZOH work synchronously with time step $h > 0$, then the parameters of discrete linear system S^d denoted for simplicity with A, B, C are given by the formulas

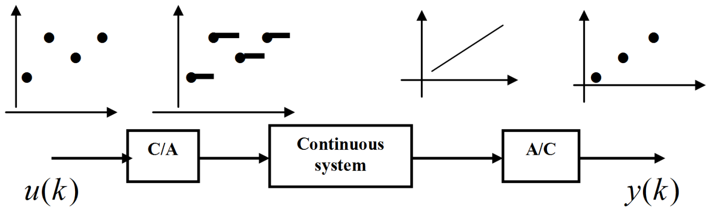

Figure 2: Continuous-discrete system.

calculated on the basis of continuous system:

$$A := e^{Ah}, \quad B := \int_0^h e^{At} B dt, \quad C := C \tag{1}$$

For a valid controller (both continuous and discrete), we need the controllability and observability of continuous system S. The conditions for time step $h > 0$ which guarantee that the discrete system is also controllable and observable are known and may be found, e.g., in Mitkowski (1991, p. 141) [20].

3. The decomposition of the system

There is a group of infinite dimensional systems which can be stabilized using finite dimensional methods. Let us now consider a system (see for example Pazy (1983) [25], Slemrod (1974) [33], Wang (1972) [35], Curtain and Pritchard (1978) [7], Curtain and Zwart (1995) [8])

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), && y(t) = Cx(t) \\ &x(t) \in X, \quad u(t) \in U, && y(t) \in Y \end{aligned} \tag{2}$$

For further use we will denote it as $S(A,B,C)$. Let us now assume that (2) fulfills the following conditions:

- X, Y, U – Hilbert spaces, $\dim U < +\infty$.
- A is an infinitesimal generator C_0 of semi-group $T_A(t)$, for $t \geqslant 0$ in X.
- $B \in L(U,X)$, $C \in L(X,Y)$ are bounded.
- A is a discrete operator with finite number of eigenvalues with $\mathrm{Re}\, s > \beta$, $\beta < +\infty$.

Taking into account the conditions above, we can decompose (2) into (Triggiani (1975) [34]):

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u(t), \qquad y(t) = C_1x_1(t) + C_2x_2(t) + C_3x_3(t), \tag{3}$$

$$x_i(t) \in X_i, i = 1,2,3, X = X_1 + X_2 + X_3, \qquad \dim X_1 < +\infty, \quad \dim X_2 = p < +\infty. \tag{4}$$

The spectrum of A (see (2)) is depicted in the Fig. 3. The operator A_1 is responsible for unstable (or weakly damped) part of the system (3). The operators A_2 and A_3 are exponentially stable.

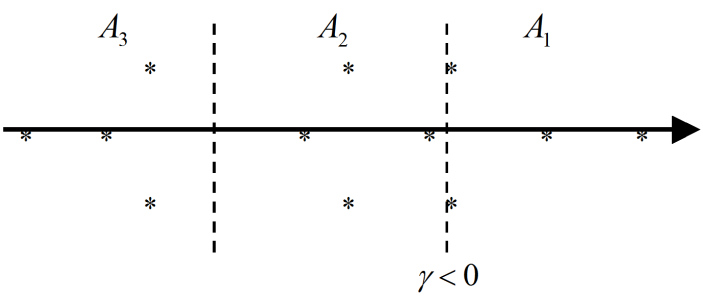

Figure 3: Discrete spectrum of A.

Let us now add the following assumptions:

- $\sup\{\mathrm{Re}\, s : s \in \lambda(A_3) < 0, \sup\{\mathrm{Re}\, s : s \in \lambda(A_2)\} = \gamma < 0.$
- The pair (A_1, B_1) is controllable, the pair (C_1, A_1) is observable.
- $\dim X_2 = p \to +\infty \;\Rightarrow\; \|B_3\| \to 0$ and $\|C_3\| \to 0$.

The last assumption is fulfilled if, e.g., self-adjoint generator A has compact resolvent (the eigenvectors form a basis of the given space).

4. Finite-dimensional stabilizing controller

Let us now consider dynamic feedback Mitkowski (1988 [19, p. 519], 1991, [20, p. 233]) of form:

$$\begin{bmatrix} \dot{w}_1(t) \\ \dot{w}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 - G_1C_1 + B_1K_1 & -G_1C_2 \\ B_2K_1 & A_2 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} y(t), \tag{5}$$
$$u(t) = K_1 w_1(t), \quad w_i(t) \in X_i, \quad i = 1, 2.$$

Let us assume that the conditions mentioned in previous section are fulfilled. There exists a finite dimensional stabilizing controller (5), such that the closed-loop system (2) with (5) is exponentially stable with predefined damping coefficient $\alpha \in (\gamma, 0)$, see Sakawa (1983) [27]. See also Mitkowski (1982, 1986, 1988) [17, 19], Mitkowski (1991, [20, p. 230]) for further details.

The design of feedback (5) may be reduced to finding the matrices K_1 and G_1 which can be done using methods known from finite dimensional system's analysis, e.g., LQ design. The desired damping coefficient $\alpha \in (\gamma, 0)$ can be found by increasing $p = \dim X_2$.

A discrete version of the controller (Mitkowski (1991, [20, p. 236]) can be obtained using formulas (1) and remembering that the system is asymptotically stable if the eigenvalues lie inside the unit circle. The matrices K_1 i G_1 should be found in a way that guarantees that the eigenvalues of matrices $A_1 + B_1K_1$ and $A_1 - G_1C_1$ lie inside the unit circle (for example, we can set them as zeros).

5. Example

Let us now consider the process of heating a thin rod (Oprzedkiewicz (2003, 2016) [23, 24]) depicted in the Fig. 4.

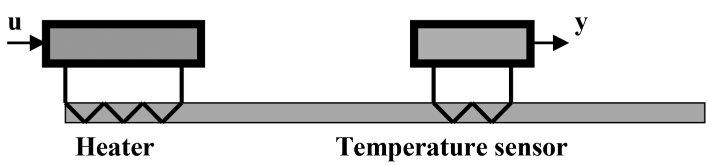

Figure 4: Heating of a thin rod.

A simplified mathematical model of the analyzed process has the form

$$\begin{aligned}
&\frac{\partial x(z,t)}{\partial t} = a\frac{\partial^2 x(z,t)}{\partial z^2} - R_a x(z,t) + b(z)u(t), \quad t \geqslant 0, \quad z \in [0,\,1], \\
&\left.\frac{\partial x(z,t)}{\partial z}\right|_{z=0} = \left.\frac{\partial x(z,t)}{\partial z}\right|_{z=1} = 0, \quad t \geqslant 0, \\
&x(z,0) = 0, \quad z \in (0,\,1), \\
&y(t) = \int_0^1 c(z)x(z,t)dz.
\end{aligned} \tag{6}$$

where

$$b(z) = \begin{cases} 1 & for \quad 0 \leqslant z \leqslant z_0 \\ 0 & for \quad z_0 < z \leqslant 1 \end{cases}$$

$$c(z) = \begin{cases} \bar{c} & for \quad z_1 \leqslant z \leqslant z_2 \\ 0 & for \quad 0 \leqslant z < z_1 \quad and \quad z_2 < z \leqslant 1 \end{cases}$$

$$x(z,t) = \sum_{i=0}^{\infty} x_i(t)h_i(z)$$

After the decomposition, we have system $S(A,B,C,D)$, where

$$\begin{aligned}
&A = diag\,(\lambda_0, \lambda_1, \lambda_2, \ldots\ldots), \quad B = [b_0\; b_1\; b_2\; \ldots\ldots\ldots]^T, \\
&C = [c_0\; c_1\; c_2\; \ldots\ldots\ldots.], \qquad D = 0,
\end{aligned}$$

and

$$\begin{aligned}
&X = L^2(0,\,1;R), \qquad \lambda_i = -i^2\pi^2 a - R_a, \quad i = 0, 1, 2, \ldots \\
&h_i(z) = \begin{cases} 1 & for \quad i = 0 \\ \sqrt{2}\cos(i\pi z) & for \quad i = 1, 2, 3, \ldots.. \end{cases} \\
&b_i = \int_0^1 b(z)h_i(z)dz, \quad c_i = \int_0^1 c(z)h_i(z)dz,
\end{aligned} \tag{7}$$

we have the following parameters for model (6) (verified in a laboratory, Oprzedkiewicz (2003, 2004) [26, 21]):

$$a = 0.000945,\ R_a = 0.0271,\ \bar{c} = 25.7922\ z_0 = 1/13,\ z_1 = 25/52,\ z_2 = 27/52$$

From (7), we have

$$\begin{aligned}
A = diag(&\text{ - }0.0269\text{ - }0.0358\text{ - }0.0624\text{ - }0.1068\text{ - }0.1690\text{ - }0.2490\text{ - }0.3467 \\
&\text{ - }0.4621\text{ - }0.5954\text{ - }0.7464\text{ - }0.9152\text{ - }1.1017\text{ - }1.3060\text{ - }1.5281\text{ - }1.7679 \\
&\text{ - }2.0255\text{ - }2.3009\text{ - }2.5940\text{ - }2.9049\text{ - }3.2335\text{ - }3.5800\text{ - }3.9441\text{ - }4.3261 \\
&\text{ - }4.7258\text{ - }5.1433)
\end{aligned}$$

$B = [0.0769\ 0.1077\ 0.1046\ 0.0995\ 0.0926\ 0.0842\ 0.0745\ 0.0638$
$0.0526\ 0.0412\ 0.0299\ 0.0190\ 0.0090\ \text{-}\ 0.0000\ \text{-}\ 0.0077\ \text{-}\ 0.0139$
$\text{-}\ 0.0187\ \text{-}\ 0.0218\ \text{-}\ 0.0234\ \text{-}\ 0.0235\ \text{-}\ 0.0223\ \text{-}\ 0.0200\ \text{-}\ 0.0168\ \text{-}\ 0.0130$
$\text{-}\ 0.0087]^T$

$C = [1.0171\ 0\ \text{-}\ 1.4348\ \text{-}\ 0.0000\ 1.4244\ \text{-}\ 0.0000\ \text{-}\ 1.4070\ 0.0000$
$1.3830\ \text{-}\ 0.0000\ \text{-}\ 1.3524\ \text{-}\ 0.0000\ 1.3156\ \text{-}\ 0.0000\ \text{-}\ 1.2729\ \text{-}\ 0.0000$
$1.2246\ \text{-}\ 0.0000\ \text{-}\ 1.1711\ \text{-}\ 0.0000\ 1.1130\ \text{-}\ 0.0000\ \text{-}\ 1.0507\ 0.0000\ 0.9848]$

In order to perform the simulation, the heating process was implemented with use of Matlab/Simulink environment (see Fig. 5).

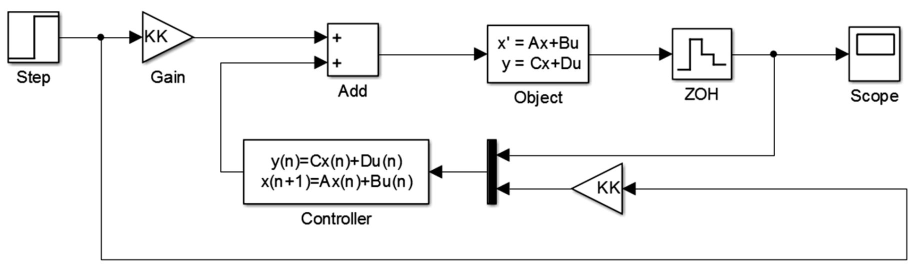

Figure 5: Simulink system.

The zero-order-hold is necessary to simulate a measurement device (e.g. thermometer) with various sampling times. We used the Tustin method (see e.g., Astrom 1990, [1, p. 212]) to discretize the compensator and then find the appropriate sampling frequency. It transforms the continuous system $S(A,B,C,D)$ into a discrete one for a given sampling time h using the formulas

$$\begin{aligned} A^+ &= (I+\frac{h}{2}A)(I-\frac{h}{2}A)^{-1} \\ B^+ &= A^{-1}(A^+ - I)B \\ C^+ &= C \\ D^+ &= D \end{aligned} \tag{8}$$

During the simulation we wanted to find optimal sampling time of the compensator for various sampling frequencies for temperature measurement. We used the performance indicator proposed by Bini and Buttazzo (2014) [3]:

$$J(N) = \frac{1}{N}\int_0^T |\dot{u}(t)|\,dt \tag{9}$$

During the simulations, we set $T = 200$ [s]. For optimization, we used golden search with parabolic interpolation implemented in Matlab Optimization Toolbox. The optimization constraints were chosen as $1 \leqslant N \leqslant 10^6$. The results are gathered in Tab. 1.

Table 1: The results of optimization

Temperature sampling frequency [Hz]	Optimal number of samples N_{opt}	Sampling time $h = \frac{T}{N_{opt}}$ [s]
10	23700	0.0084
1	23896	0.0083
0.1	68957	0.0029
0.03	84140	0.0024
0.02	48284	0.0041
0.01	69997	0.0028

It can be seen that sampling time of the controller increases with increasing sampling frequency. This means that we have a buffer in the controller for doing necessary calculations. The accuracy of temperature measurements and controller performance are depicted in the Figs 6 and 7.

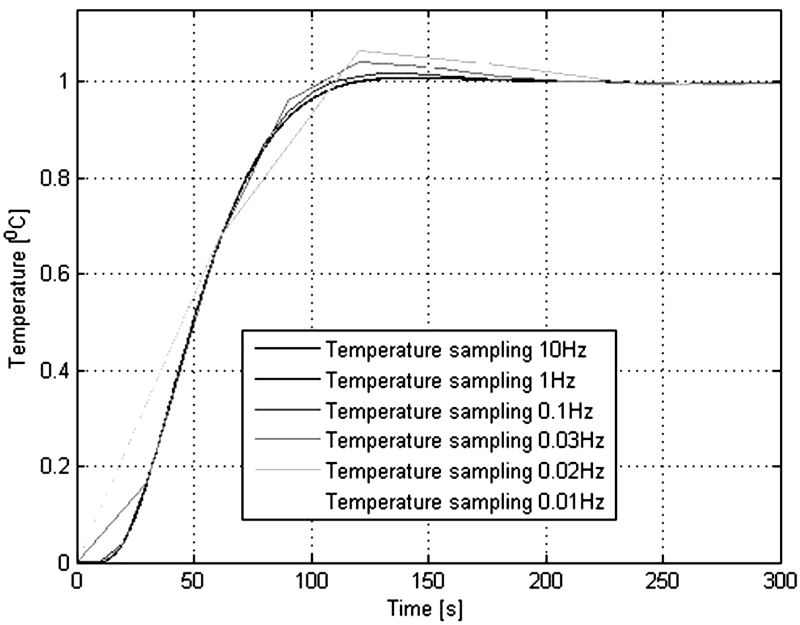

Figure 6: Temperature for various sampling frequencies.

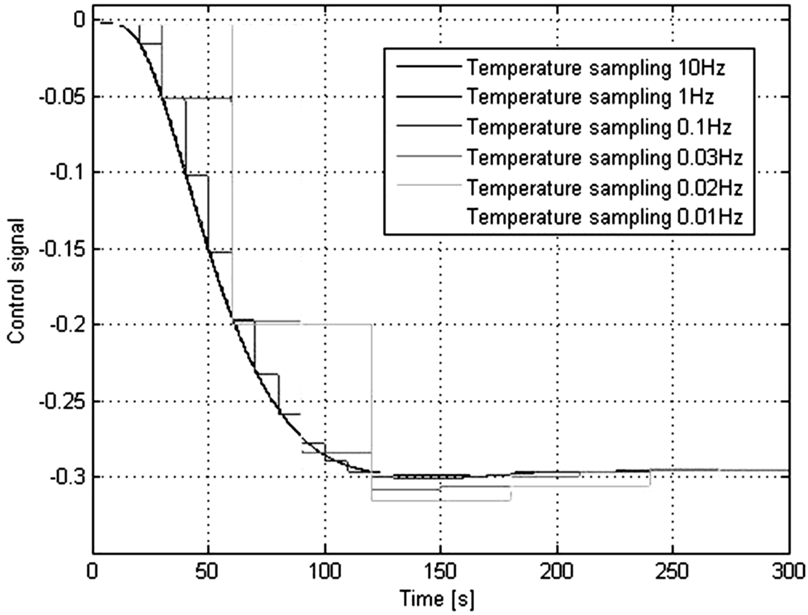

Figure 7: Control signal for various sampling frequencies.

6. Comparison quality index

For all numerical experiments the golden search with parabolic interpolation method has been chosen for tuning N_{opt} parameter. Initial value for all experiments have value 1.

The tests will be conducted for the following quality index:

$$1.\ \frac{1}{N}\int_0^T |\dot{u}(t)|\,dt, \qquad 2.\ \frac{1}{N}\int_0^T u^2 dt \qquad 3.\ \frac{1}{N}\int_0^T |u|dt \qquad 4.\ \frac{1}{N}\int_0^T tudt$$

I can bee seen, that all quality index give the same result for the same sampling frequencies (see Fig. 8, Fig. 9 and Fig. 10). But calculating are the faster for the quality index of form 1 (see Tab. 2) .

7. Conclusion

The main purpose of this work was to present possible way of approximating infinite dimensional systems with finite dimensional ones. The resulting system can then be discretized and implemented in digital controllers. The results were confirmed with simulation as we analyzed the process of thin rod heating. We found optimal sampling time

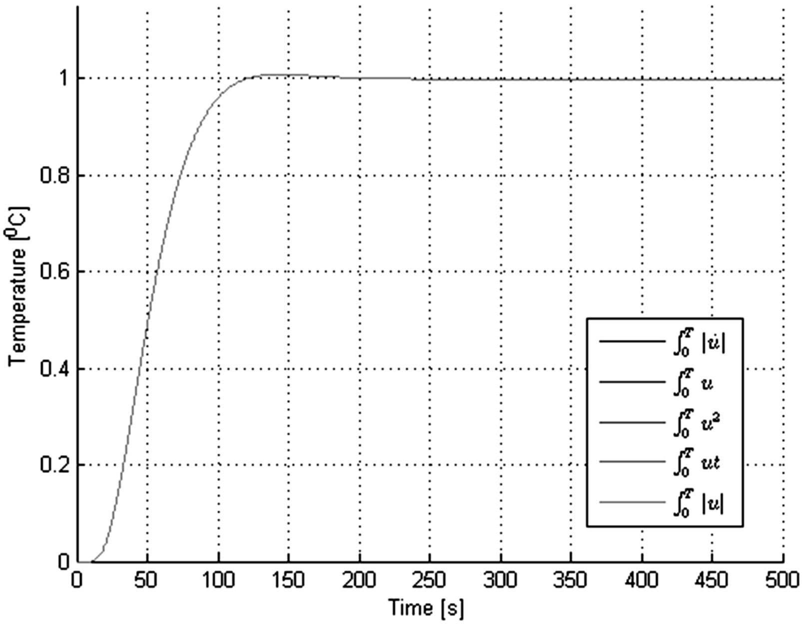

Figure 8: Temperature for various quality index with sampling frequencies 10Hz.

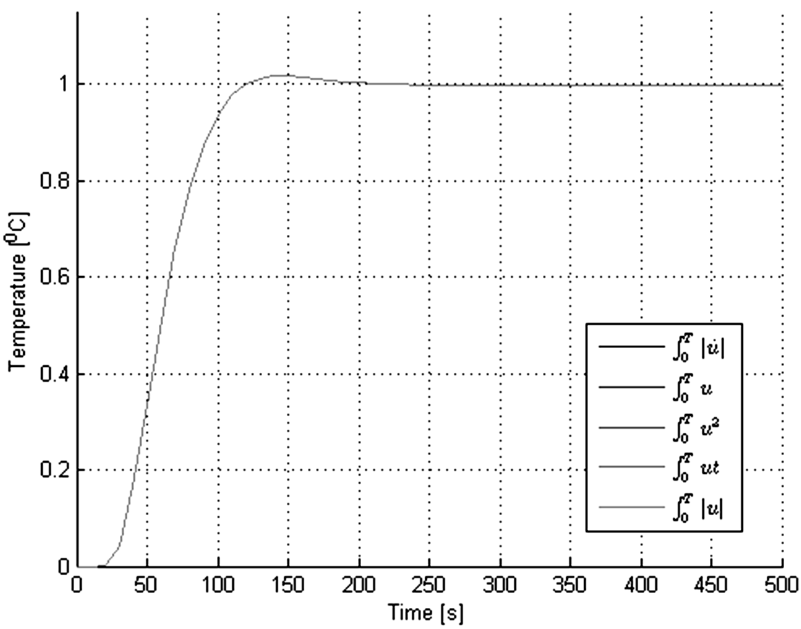

Figure 9: Temperature for various quality index with sampling frequencies 0.01Hz.

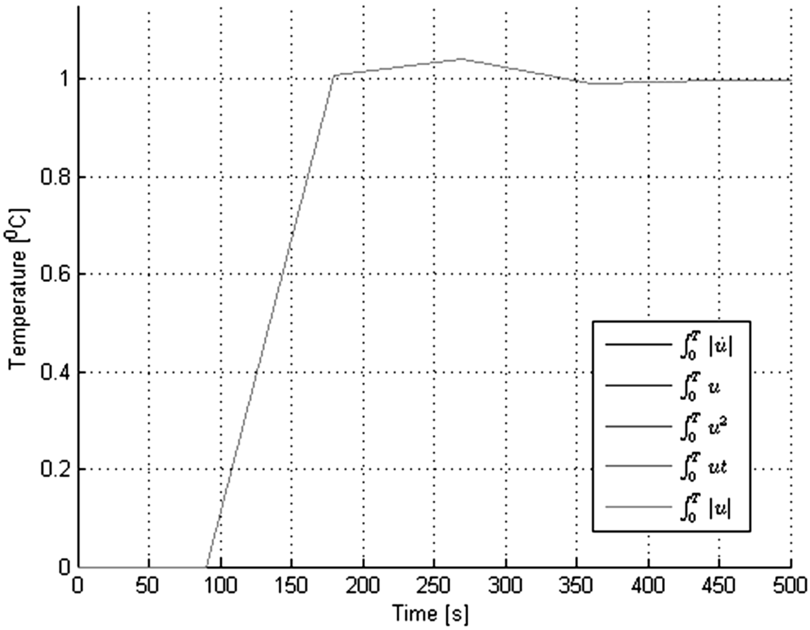

Figure 10: Temperature for various quality index with sampling frequencies 0.1Hz

for the compensator depending on various output sampling frequencies and comparison result for different quality index.

Nevertheless, the proposed algorithm is general and may be used for control of various systems. One of the possible way of applications may be non-integer order diffusion equation Gal and Warma (2016) [12], see also Evans (2007) [10]. However, it will require further analysis and research, as the methods for integer order systems cannot be directly applied to them.

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Table 2: The results of optimization

Quality index number	Temperature sampling frequency [Hz]	Error	Time of calculation [s]
1	10	0.0545	200
	1	0.0548	191
	0.1	0.0586	93
	0.03	0.0687	61
	0.02	0.0853	102
	0.01	0.0889	80
2	10	0.0545	455
	1	0.0548	448
	0.1	0.0586	135
	0.03	0.0687	113
	0.02	0.0853	87
	0.01	0.0889	90
3	10	0.0545	466
	1	0.0548	454
	0.1	0.0586	158
	0.03	0.0687	119
	0.02	0.08537	87
	0.01	0.08897	90
4	10	0.0545	441
	1	0.0548	432
	0.1	0.0586	173
	0.03	0.0687	116
	0.02	0.0853	102
	0.01	0.0889	101

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