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An attempt to determine the effect of increase of observation correlations on detectability and identifiability of a single gross error

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Abstract: The paper presents the results of investigating the effect of increase of observation correlations on detectability and identifiability of a single gross error, the outlier test sensitivity and also the response-based measures of internal reliability of networks. To reduce in a research a practically incomputable number of possible test options when considering all the non-diagonal elements of the correlation matrix as variables, its simplest representation was used being a matrix with all non-diagonal elements of equal values, termed uniform correlation. By raising the common correlation value incrementally, a sequence of matrix configurations could be obtained corresponding to the increasing level of observation correlations. For each of the measures characterizing the above mentioned features of network reliability the effect is presented in a diagram form as a function of the increasing level of observation correlations. The influence of observation correlations on sensitivity of the *w*-test for correlated observations (Förstner 1983,Teunissen 2006) is investigated in comparison with the original Baarda's *w*-test designated for uncorrelated observations, to determine the character of expected sensitivity degradation of the latter when used for correlated observations. The correlation effects obtained for different reliability measures exhibit mutual consistency in a satisfactory extent.
As a by-product of the analyses, a simple formula valid for any arbitrary correlation matrix is proposed for transforming the Baarda's *w*-test statistics into the *w*-test statistics for correlated observations.

Keywords: single outlier case, correlated observations, detectability, identifiability, outlier-test sensitivity, mean-shift model

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1. Introduction

In general, every measurement process generates correlations between the individual observations. The correlations contain stochastic information about the measurement process and thus, should be taken into account in constructing the models for the problems to be solved. It is well known that observation correlations may obscure the location of a gross error, thus making the outlier identification less effective or even unfeasible. In specific GMM models where the transfer of the effect of gross error from one part of a network to another is not possible due to network's structure, the observation correlations may cause outlier hiding effects (Prószyński, 2000).

It is often not possible to possess fairly reliable non-diagonal elements of the covariance matrix for observations. Towards solving of this problem the research is carried out to work out methods of estimating the covariance matrix for a given measurement process (e.g. Ananga et al., 1994; Leandro et al., 2005).

The disturbing role of observation correlations upon outlier identification is analyzed in literature in terms of the resulting correlations between the outlier test statistics. As it follows from (Förstner, 1983), the smaller the correlations between the outlier test statistics the more effective is identification of the contaminated observation. This statement was fully confirmed by the in-depth analyses in many publications, especially in the area of satellite based navigation systems (e.g. Wang and Knight, 2012; Wang et al., 2012). However, the question arises, how are the correlations between the outlier test statistics as well as the measures of internal reliability of networks influenced by observation correlations themselves.

The objective of the present paper is to investigate the effect of increase of observation correlations on detectability and identifiability of a single outlier, on outlier test sensitivity as well as on the response-based reliability measures. That might provide some indications as regards the magnitudes of correlation effects. The use of several reliability measures, which are in the nature of things interrelated, was meant to check correctness of research results.

Due to complexity of the above task it turned out necessary to work out an appropriate research method. To reach this goal the following questions had to be answered first:

– how to represent the correlation matrix in order to reduce the computations in numerical tests to an acceptable size,
– how to obtain a sequence of configurations of the correlation matrix that are ordered according to the increasing level of observation correlations,
– how to investigate the correlation effect so as to find out its properties that are common (or highly similar) for different types of networks.

In addition, a comparison is carried out of the effects of observation correlations on sensitivity of the *w*-test for correlated observations (Forstner, 1983; Teunissen, 2006) and the original Baarda's *w*-test (Baarda, 1968) designated only for uncorrelated observations. The comparison allows to determine the character of expected sensitivity degradation of Baarda's *w*-test due to its use for correlated observations. The former *w*-test has already been investigated on a theoretical basis (Teunissen, 2006). The present analysis will make it possible to empirically verify the proved properties and hopefully, may provide some complementary detailed properties not specified in the proofs.

Confining the research to a single outlier case was due to the fact that even in this elementary case the scope of study is considerably wide.

2. Notation and basic formulas

Instead of the original form of Gauss-Markov model (GMM), being

$$\mathrm{E}(\mathbf{y})=\mathbf{A}\mathbf{x} \qquad \mathrm{D}(\mathbf{y})=\mathbf{C}_{\mathbf{y}} \tag{1}$$

where: $\mathbf{x}(u\times 1)$, $\mathbf{y}(n\times 1)$, $\mathbf{A}(u\times n)$, rank $\mathbf{A}\leq u$, $\mathbf{C}_{\mathrm{y}}(n\times n)$ (positive definite)

the modified (i.e. standardized) form exposing the correlation matrix (Prószyński 2010) will be used

$$\mathrm{E}(\mathbf{y}_{\mathrm{s}})=\mathbf{A}_{\mathrm{s}}\mathbf{x} \qquad \mathrm{D}(\mathbf{y}_{\mathrm{s}})=\mathbf{C}_{\mathbf{y},\mathrm{s}} \tag{2}$$

where $\mathbf{C}_{\mathrm{y,s}}$ is a correlation matrix (positive definite), further denoted as $\mathbf{C}_{\mathrm{s}}$.
The redundancy of the model (1) or (2) will be denoted by f, where $f = n$ – rank $\mathbf{A}$.
As we concentrate on the aspects of a priori analysis we do not single out in $\mathrm{D}(\mathbf{y})$ the variance factor.
The vector of standardized least squares (LS) residuals and its covariance matrix will be denoted by

$$\mathbf{v}_{\mathrm{s}}=-\mathbf{H}\mathbf{y}_{\mathrm{s}} \qquad \mathbf{C}_{\mathbf{v}_{\mathrm{s}}}=\mathbf{H}\mathbf{C}_{\mathrm{s}}\mathbf{H}^{\mathrm{T}}$$

where $\mathbf{H}=\mathbf{I}-\mathbf{A}_{\mathrm{s}}(\mathbf{A}_{\mathrm{s}}^{\mathrm{T}}\mathbf{C}_{\mathrm{s}}^{-1}\mathbf{A}_{\mathrm{s}})^{+}\mathbf{A}_{\mathrm{s}}^{\mathrm{T}}\mathbf{C}_{\mathrm{s}}^{-1}$ is the operator of oblique projection, the symbol "+" denotes the pseudo-inverse.
Assuming a single outlier case, we shall write

$$\mathbf{y}_{\mathrm{s}}=\mathbf{y}_{\mathrm{s}}^{\mathrm{true}}+\mathbf{e}_{\mathrm{s}}+\Delta\mathbf{y}_{\mathrm{s}(i)} \tag{3}$$

where
$\mathbf{e}_{\mathrm{s}}$– the $n\times 1$ vector of random observation errors, such that due to $\mathrm{E}(\mathbf{e}) = \mathbf{0}$, we have also $\mathrm{E}(\mathbf{e}_{\mathrm{s}}) = \mathbf{0}$; $\Delta\mathbf{y}_{\mathrm{s(i)}}$ – the $n\times 1$ vector with the i-th non-zero element being the standardized gross error $\Delta y_{\mathrm{s,i}}$ in the i-th observation.

With (3), we get

$$\mathbf{v}_{\mathrm{s}(i)}=-\mathbf{H}\mathbf{e}_{\mathrm{s}}-\mathbf{H}\cdot\Delta\mathbf{y}_{\mathrm{s}(i)} \qquad \mathbf{C}_{\mathbf{v}_{\mathrm{s}(i)}}=\mathbf{C}_{\mathbf{v}_{\mathrm{s}}} \tag{4}$$

For the j-th observation ($j = 1,\ldots,n$), we get

$$v_{s,j(i)} = -\{\mathbf{H}\}_{j\bullet}\mathbf{e}_s - \{\mathbf{H}\}_{ji} \cdot \Delta y_{s,i} \qquad \sigma_{v_{s,j(i)}} = \sqrt{\{\mathbf{H}\mathbf{C}_s\mathbf{H}^T\}_{jj}} \tag{5}$$

where $\{\mathbf{H}\}_{j\bullet}$ denotes the j-th row of $\mathbf{H}$.
We shall also consider the mean-shift model (MSM) (see e.g. Cook and Weisberg 1982, Kok 1984), written in the notation of the model (2)

$$\mathrm{E}(\mathbf{y}_s) = \mathbf{A}_s\mathbf{x} + \boldsymbol{\theta}_j z_{j(i)} \qquad \mathrm{D}(\mathbf{y}_s) = \mathbf{C}_{\mathbf{y},s} \tag{6}$$

where $\boldsymbol{\theta}_j$ – the $n\times 1$ vector with 1 in the j-th element and 0 in the remaining elements; $z_{j(i)}$ – the parameter placed in observation equation for the j-th observation, reflecting the system response to a single (standardized) gross error in the i-th observation, j can be chosen within the interval $[1, n]$.
Following the LS solution for $z_{j(i)}$ given in (Teunissen 2006, Knight et al. 2010) and taking into account (3), we obtain

$$\hat{z}_{j(i)} = \{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\}_{jj}^{-1} \cdot \left[\{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\}_{j\bullet}\mathbf{e}_s + \{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\}_{ji}\Delta y_{s,i}\right]; \quad \sigma_{\hat{z}_{j(i)}} = \sqrt{\{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\}_{jj}^{-1}} \tag{7}$$

With $j = 1,\ldots,n$ the formula (7) will correspond to n MSM models.

3. The assumed research method

Investigating the effect of increase of observation correlations on internal reliability of networks is a complex task and needs specially devised research method. In this Section we present the main features of the assumed method.

3.1. Representation of correlation matrices

For the purpose of research we introduce an auxiliary term *configuration* of the correlation matrix.

A *configuration* of a positive definite correlation matrix $\mathbf{C}_s(n\times n)$, $n\geq 2$, will be denoted by

$$\mathbf{C}_s(\rho_{1,2},\ldots,\rho_{n-1,n})$$

where due to symmetry of $\mathbf{C}_s$ only the elements over the main diagonal are shown.

Considering the correlations $\rho_{1,2},\ldots,\rho_{n-1,n}$ as continuous variables bound by the condition of positive definiteness of $\mathbf{C}_s$, we would obtain an infinite set of configurations of the correlation matrix. Among the elements of this set there would certainly be the so called realistic configurations, i.e. those that reflect the stochastic features of measurement technologies being in practical use.

Even assuming for each of the above variables a discrete set of values we would get, especially for greater n, a large number of matrix configurations.

In addition to the problem of computation size, it would not be possible to rank the configurations so as to obtain a sequence of increasing level of observation correlations. Such a requirement is indispensable for investigating the effect of increase of observation correlations.

After analyzing several options, the simplest possible representation of the correlation matrix $\mathbf{C}_s$ was finally applied, termed *uniform correlation*. It allows one to reduce the computations to an acceptable size and ensures the possibility of ranking the test configurations. It is as follows

$$\mathbf{C}_s\,(a, \ldots, a),\; q < a < 1$$

where q is such a negative value of a for which det $\mathbf{C}_s = 0$, determined from the formula (Dufresne 2005, Prószyński et al., 2011)

$$q = -\frac{1}{n-1} \qquad n \geq 2$$

e.g. $n = 2$, $q = -1$; $n = 3$, $q = -0.5$; … ; $n = 10$, $q = -0.1$.

One cannot exclude the cases that uniform correlation may be a realistic stochastic model for some particular measurement processes.

Assuming several values for a ranked in a growing order, we obtain a discrete set of configurations of the matrix $\mathbf{C}_s$, forming a sequence of increasing level of observation correlations. Further in the text instead of $\mathbf{C}_s(a, \ldots, a)$ an abbreviated form $\mathbf{C}_s(a)$ will be used.

3.2. The attempt to extract the effect of increasing correlations

In general each reliability feature x can be expressed as a function of the form x($\mathbf{A}$, $\boldsymbol{\sigma}$, $\mathbf{C}_s$), where the matrix $\mathbf{A}$ represents the network structure, the column vector $\boldsymbol{\sigma}$ represents observation accuracies and the matrix $\mathbf{C}_s$ – the observation correlations. Hence, to investigate the effect of observation correlations we need to consider a particular network of specified observation accuracies and correlations. We cannot extract the "pure effect", i.e. x($\mathbf{C}_s$), by neglecting $\mathbf{A}$ and $\boldsymbol{\sigma}$. Obviously, this problem remains also in case of reducing the number of possible test options by assuming the uniform correlation and unitary accuracies of the observations, i.e. x[$\mathbf{A}$, $\boldsymbol{\sigma} = \mathbf{1}$, $\mathbf{C}_s(a)$], where $\mathbf{1}$ denotes a vector of ones.

Therefore, we may only seek the properties of x[$\mathbf{A}$, $\mathbf{1}$, $\mathbf{C}_s(a)$] that are similar, to a certain degree, for different network structures.

When approaching the value a =1 where det $\mathbf{C}_s = 0$, the computed quantities x as above, converge asymptotically to a certain value (0, 1 or some other). This will not be signalized in the diagrams for x[**A**, **1**, $\mathbf{C}_s(a)$] but will only be mentioned in the text. For x[**A**, **1**, $\mathbf{C}_s(a)$] the abbreviated form x(a) will be used.

The essential elements of the research task are the analyses of correlation effect carried out on basis of the existing or the derived formulas and the diagrams prepared on basis of numerical tests.

3.3. Test networks

The test networks, treated as free networks, are shown in Figure 1. Their brief characteristics shown above each sketch contain the following features:
– type of network: V – leveling, H – horizontal;
– the range of internal reliability indices for $\mathbf{C}_s = \mathbf{I}$, e.g. [0.65, 0.75];
– number of observations n, redundancy (i.e. number of degrees of freedom) f.

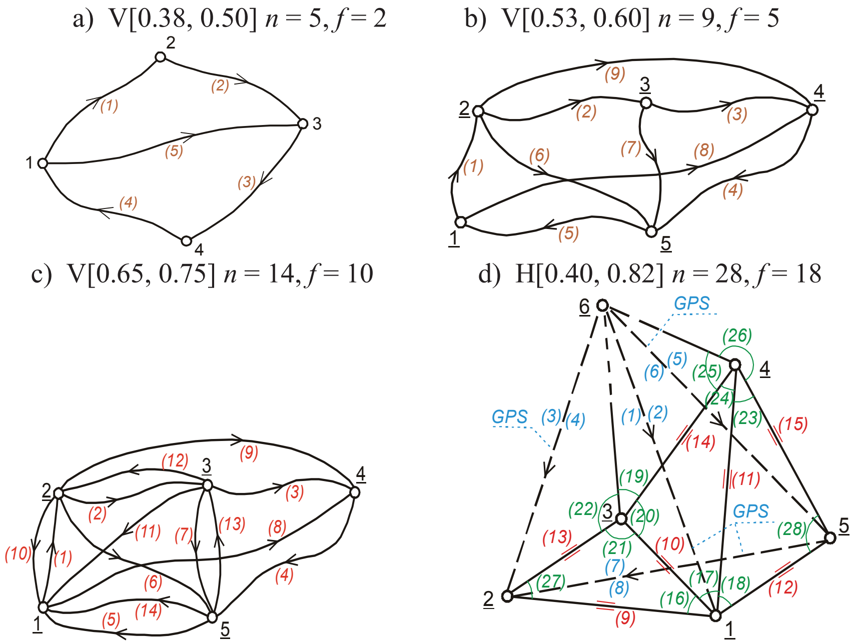


Fig. 1. Test networks

The arrows for leveling lines and GPS vectors in Figure 1 indicate the orientation of differences in height and coordinate differences assumed in the GMM models. The angles have a commonly used clock-wise orientation. It is well known that introducing the changes of orientation for some or all the observations would imply the necessity for a due modification of the stochastic model. To allow for reproducing the test computations for a horizontal network as in Figure 1d, the X,Y coordinates are provided in Table 1.

Table 1. Point coordinates for a horizontal network

Point No.	$X_{[m]}$	$Y_{[m]}$
1	150	650
2	200	100
3	400	400
4	800	700
5	350	950
6	950	350

To reduce the number of possible test options to be analyzed, the case of unitary accuracies is applied as a basic case, i.e. $\sigma_1=\sigma_2=...=\sigma_n=1$; so $\mathbf{A}_s=\mathbf{A}$. Only additionally, several options with varying accuracies are considered, i.e. $\sigma_1\neq\sigma_2\neq...\neq\sigma_n$; $\mathbf{A}_s\neq\mathbf{A}$.

4. Effect of increase of observation correlations on detectability of a single gross error in a GMM

Let us recall a well known formula for minimal detectable bias (MDB) (Wang and Chen, 1994; Teunissen, 1990; 2006), here presented in a standardized form with the use of notation of Sect. 2

$$\mathrm{MDB}_{s,i}=\frac{\sqrt{\lambda_{f,\alpha,\beta}}}{\sqrt{\left\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_{s}^{-1}\mathbf{H}\right\}_{ii}}} \tag{8}$$

where

$\lambda_{f,\alpha,\beta}$ – non-centrality parameter in a global model test, dependent on type I error α, type II error β and redundancy f, further denoted as λ.

$\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_s^{-1}\mathbf{H}\}_{ii}$ is termed a *reliability number* (r_i^*) for the i-th observation. With $\mathbf{C}_s=\mathbf{I}$, r_i^* becomes a *reliability measure* (r_i).

In the computations the values $\alpha=0.05$ and $\beta=0.20$ were used. The resulting values for λ were as follows: $f=2$, $\lambda=9.6$; $f=5$, $\lambda=12.8$; $f=10$, $\lambda=16.2$; $f=18$, $\lambda=20.0$. The diagrams obtained for the relationship x(a), where x = $\mathrm{MDB}_{s,i}$ $i=1,\ldots,n$, for the test networks as in Figure 1 are shown in Figures 2a,b,c,d.

In all the four cases the initial ascent of the MDB_s curves, less noticeable in the fourth case, is followed by the monotonic fall, with 0 being an asymptotic value. This is consistent with the upper bound for r_i^* being infinity (Schaffrin, 1997) and hence, 0 for MDB_s. The spacing of curves at $a=0$ depends on the range of internal reliability measures in a particular network. We can observe that with the increase of redundancy in a network the shape of individual curves becomes more and more consistent and they start monotonic fall at smaller and smaller values of a. However, even for f =18 we do not observe monotony of the curves over the whole interval of the values of a. We can see it more distinctly for the observation 25 in Figure 7b (Appendix A).

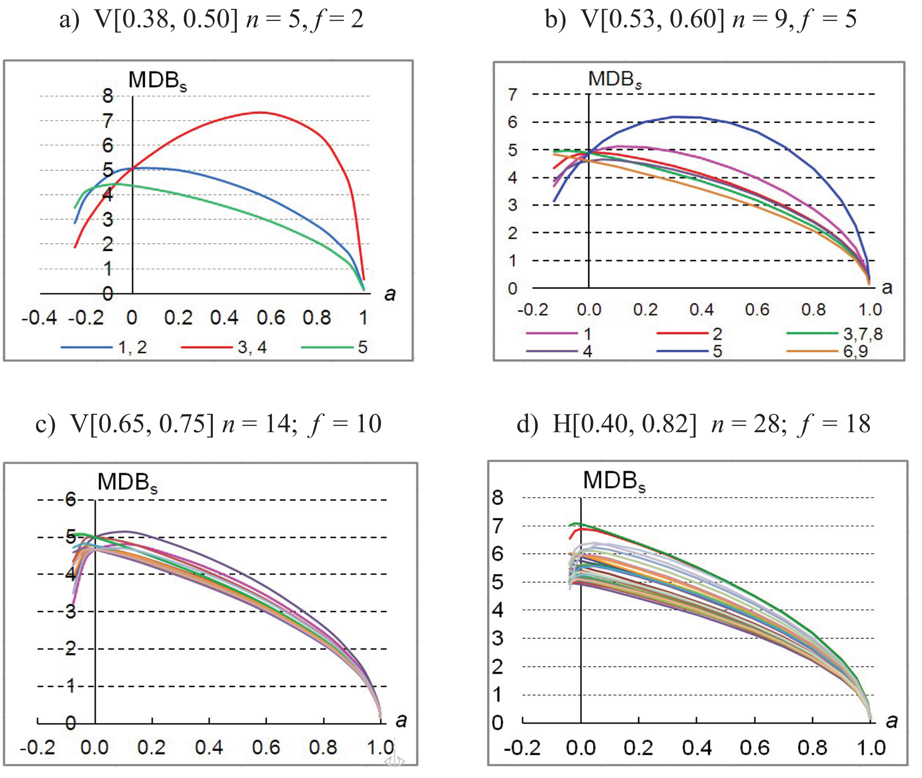


Fig. 2. Diagrams for $\mathrm{MDB}_{s,i}(a)$; $\boldsymbol{\sigma} = \mathbf{1}$

It should be noted that in a network a) the curve that departs mostly from the remaining curves represents the observation with the lowest reliability measure, i.e. 0.38. Although to a lesser degree, but we can see a similar effect in a network b).

The computations not presented here show that for varying accuracies in a network a) we have bigger spacing of the $\mathrm{MDB}_{s,i}$ values at a = 0 and interchanging between some of the curves. The influence of varying accuracies is considerably smaller in networks c) and d).

It seems that the above empirical material might be useful for discussion whether or to what extent the $\mathrm{MDB}_{s,i}$ as in formula (8) can be a measure of outlier detectability in networks with correlated observations. A requirement for $\mathrm{MDB}_{s,i}$ of being uniquely interpretable in terms of the level of observation correlations would be a basic issue in that discussion.

5. Effect of increase of observation correlations on sensitivity to a single gross error of the *w*-test for correlated observations and Baarda's *w*-test

It is only for research purposes that we analyze the response to increasing observation correlations both for a classical *w*-test for correlated observations and Baarda's *w*-test that only applies for uncorrelated observations. For simplicity of notation, we shall refer to the former as a "w^*-test" and to the latter as a "*w*-test".

Let us consider the test statistics for two characteristic types of observations, i.e. $i(i)$ – the observation in which a gross error resides and $j(i)$, $j \neq i$ – any other observation not contaminated by a gross error. The corresponding test statistics will be the following:

$$w\text{-test} \qquad w_{j(i)} = \frac{v_{s,j(i)}}{\sigma_{v_{s,j(i)}}} \qquad w_{i(i)} = \frac{v_{s,i(i)}}{\sigma_{v_{s,i(i)}}} \tag{9}$$

$$w^*\text{-test} \qquad w^*_{j(i)} = \frac{\hat{z}_{j(i)}}{\sigma_{\hat{z}_{j(i)}}} \qquad w^*_{i(i)} = \frac{\hat{z}_{i(i)}}{\sigma_{\hat{z}_{i(i)}}} \tag{10}$$

Using the formulas (9), (10) we can determine the parameters for the above *w*-variables and w^*-variables

$$w_{i(i)} \sim \mathrm{N}(\mu_i, 1) \qquad \mu_i = -\frac{\{\mathbf{H}\}_{ii}}{\sqrt{\{\mathbf{H}\mathbf{C}_s\mathbf{H}^{\mathrm{T}}\}_{ii}}} \cdot \Delta y_{s,i} \tag{11}$$

$$w_{j(i)} \sim \mathrm{N}(\mu_j, 1) \qquad \mu_j = -\frac{\{\mathbf{H}\}_{ji}}{\sqrt{\{\mathbf{H}\mathbf{C}_s\mathbf{H}^{\mathrm{T}}\}_{jj}}} \cdot \Delta y_{s,i} \tag{12}$$

$$\rho_{ij} = \mathrm{cor}\left(w_{i(i)}, w_{j(i)}\right) = \frac{\{\mathbf{H}\mathbf{C}_s\mathbf{H}^{\mathrm{T}}\}_{ij}}{\sqrt{\{\mathbf{H}\mathbf{C}_s\mathbf{H}^{\mathrm{T}}\}_{ii}}\sqrt{\{\mathbf{H}\mathbf{C}_s\mathbf{H}^{\mathrm{T}}\}_{jj}}} \tag{13}$$

For w^*-variables

$$w^*_{i(i)} \sim \mathrm{N}(\mu^*_i, 1) \qquad \mu^*_i = \sqrt{\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_s^{-1}\mathbf{H}\}_{ii}}\,\Delta y_{s,i} \tag{14}$$

$$w^*_{j(i)} \sim \mathrm{N}(\mu^*_j, 1) \qquad \mu^*_j = \frac{\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_s^{-1}\mathbf{H}\}_{ji}}{\sqrt{\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_s^{-1}\mathbf{H}\}_{jj}}}\,\Delta y_{s,i} \tag{15}$$

$$\rho^*_{ij} = \mathrm{cor}\left(w^*_{i(i)}, w^*_{j(i)}\right) = \frac{\left\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathrm{s}}^{-1}\mathbf{H}\right\}_{ij}}{\sqrt{\left\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathrm{s}}^{-1}\mathbf{H}\right\}_{ii}}\sqrt{\left\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathrm{s}}^{-1}\mathbf{H}\right\}_{jj}}} \tag{16}$$

Derivation of the formulas for correlations ρ_{ij} and ρ^*_{ij} is shown in a condensed form in Appendix C.
Substituting $\Delta y_{s,i} = \mathrm{MDB}_{s,i}$ as in (8), into (11) and (12), we obtain

$$\mu_i = -\frac{\{\mathbf{H}\}_{ii}}{\sqrt{\left\{\mathbf{H}\mathbf{C}_{\mathrm{s}}\mathbf{H}^{\mathrm{T}}\right\}_{ii}}\sqrt{\left\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathrm{s}}^{-1}\mathbf{H}\right\}_{ii}}} \cdot \sqrt{\lambda}$$

$$\mu_j = -\frac{\{\mathbf{H}\}_{ji}}{\sqrt{\left\{\mathbf{H}\mathbf{C}_{\mathrm{s}}\mathbf{H}^{\mathrm{T}}\right\}_{jj}}\sqrt{\left\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathrm{s}}^{-1}\mathbf{H}\right\}_{ii}}} \cdot \sqrt{\lambda} \;; \qquad \rho_{ij}$$

Introducing the coefficient $k_{ji} = \dfrac{\{\mathbf{H}\}_{ji} \cdot \sqrt{\left\{\mathbf{H}\mathbf{C}_{\mathrm{s}}\mathbf{H}^{\mathrm{T}}\right\}_{ii}}}{\{\mathbf{H}\}_{ii} \cdot \sqrt{\left\{\mathbf{H}\mathbf{C}_{\mathrm{s}}\mathbf{H}^{\mathrm{T}}\right\}_{jj}}}$

we get $\mu_j = k_{ji} \cdot \mu_i \qquad (k_{ji} \in R)$

$$\mu^*_i = \sqrt{\lambda} \;; \qquad \mu^*_j = \rho^*_{ij} \cdot \sqrt{\lambda} \;; \qquad \rho^*_{ij} \qquad \text{where} \quad \mu^*_j = \rho^*_{ij} \cdot \mu^*_i \qquad \left|\rho^*_{ij}\right| \le 1 \tag{17}$$

We shall also take into account the relationship between μ_i and μ^*_i defined by

$$\mu_i = -k_i \cdot \mu^*_i \qquad \text{where} \quad k_i = \frac{\{\mathbf{H}\}_{ii}}{\sqrt{\left\{\mathbf{H}\mathbf{C}_{\mathrm{s}}\mathbf{H}^{\mathrm{T}}\right\}_{ii}}\sqrt{\left\{\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathrm{s}}^{-1}\mathbf{H}\right\}_{ii}}} \tag{18}$$

The above formulas allow one to conclude the following properties of the considered test statistics:

i. for w-variables

– the noncentralities μ_i are all of one sign but varying magnitudes; the magnitudes being dependent on observation correlations; the noncentralities μ_j can be of opposite sign to that of μ_i;

ii. for w^*-variables

– the noncentralities μ^*_i are all of one sign and equal magnitudes; the magnitudes being independent of observation correlations; the noncentralities μ^*_j can be of opposite sign to that of μ^*_i. Since $\left|\rho^*_{ij}\right| \le 1$, we shall always have $\mu^*_i \ge \left|\mu^*_j\right|$.

Due to varying magnitudes of noncentralities μ_i the w-test with a constant type I error α for individual contaminated observation will have varying type II errors β and thus, cannot be coordinated with a global model test. The w^*-test, according to its theoretical properties (Teunissen 2006), keeps type II errors (and hence, the power) at a constant level.

For further properties of the above test statistics we resort to numerical tests assuming uniform correlation. Figure 3 and 4 show variability of ρ_{ij} and ρ^*_{ij} and also that of k_i for different values of correlation coefficient a, i.e. ρ_{ij} (a), ρ^*_{ij} (a), k_i (a).

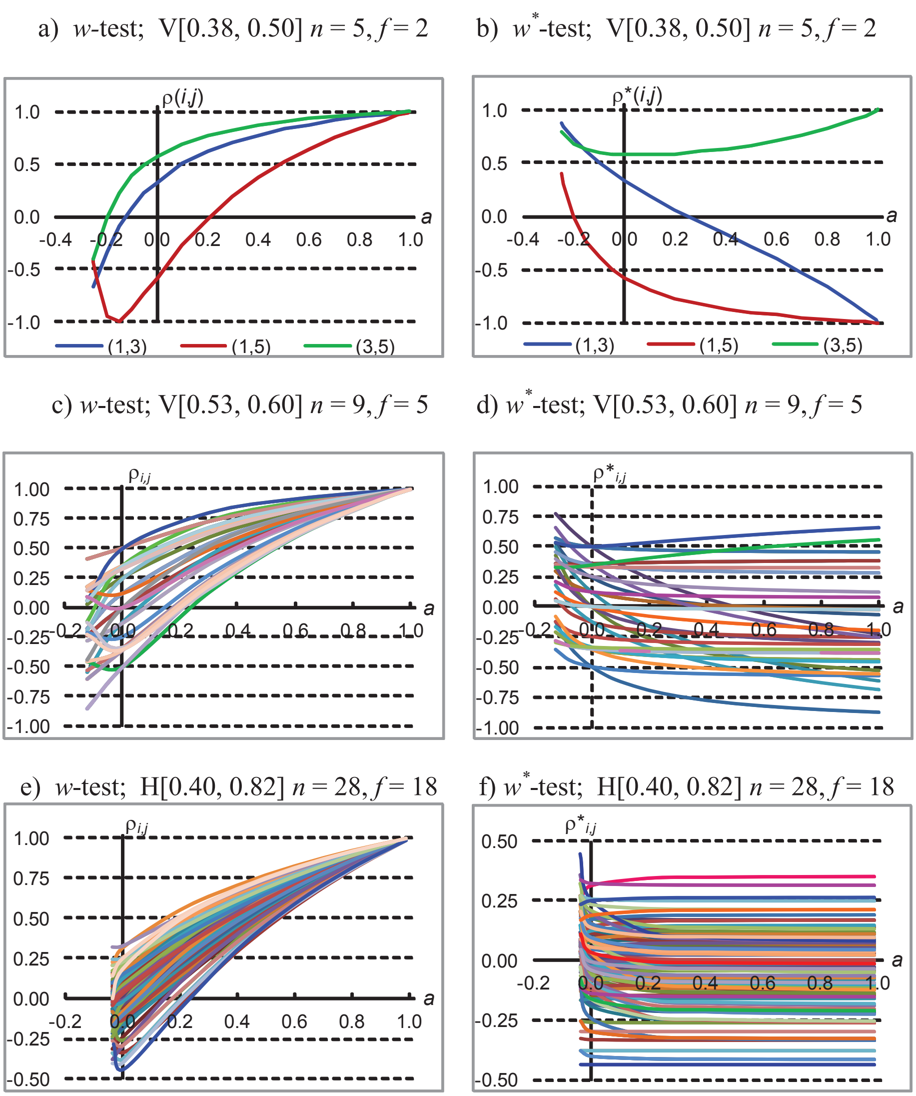


Fig. 3. Diagrams for ρ_{ij} (a) and ρ^*_{ij} (a); $\boldsymbol{\sigma} = \mathbf{1}$

Like in the case of MDB, with the increase of redundancy of a network the shapes of curves for each of the tests become more and more consistent. For a w-test all the curves converge asymptotically to $\rho_{ij}= 1$. For the w^*-test for $f=2$ one curve converges to $\rho^*_{ij}=1$ and the other two to $\rho^*_{ij}=-1$, and for $f=5$ and $f= 18$ the curves converge to the values of ρ^*_{ij} within the range [–0.87, 0.65] and [–0.45, 0.33] respectively. In a w^*-test, for $a > 0.2$ the curves for $f= 18$ keep almost constant level. The diagrams for a w-test show that with the increase of observation correlations the chance for identifying a contaminated observation by separating it from the remaining error-free observations is systematically decreasing. For the w^*-test such a chance is much bigger and for some observations may even improve slightly at higher redundancy. These results are the illustration of the advantageous properties of the w^*-test, proved in (Teunissen 2006).

a) V[0.38, 0.50] $n = 5, f= 2$ b) V[0.53, 0.60] $n = 9, f= 5$ c) H[0.40, 0.82] $n = 28, f= 18$

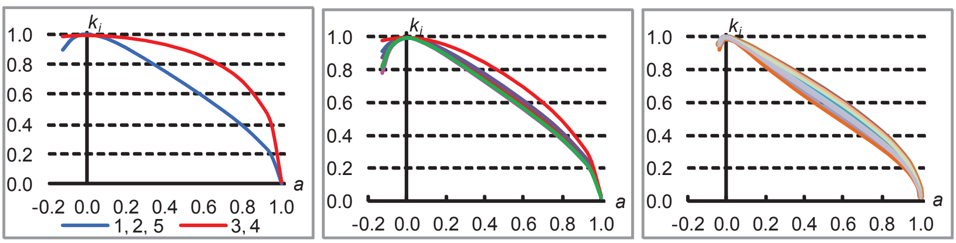


Fig. 4. Variability of k_i with respect to observation correlations; **σ = 1**

We can see in Figures 4a,b,c that for positive values of a the curves $k_i(a)$ for all the observations fall monotonously to zero as an asymptotic value. This indicates that with the increase of observation correlations the noncentralities μ_i in a w-test decrease in comparison to those (i.e. μ^*_i) in the w^*-test.

The numerical tests not included in the present paper show that the values of $k_{ji}(a)$ being the ratio of noncentralities μ_i, μ_j for a w-test (see derivation of formulas in this Section), may for greater correlations exceed 1, i.e. $\mu_j > \mu_i$.

All the above diagrams and results of numerical tests show the rate of sensitivity degradation of the w-test.

A by-product of the analyses is a formula linking w-test and w^*-test, offering time savings in computing the latter test statistics (see Appendix B).

6. Effect of increase of observation correlations on identifiability of a single gross error in a GMM

The identifiability index for the i-th observation contaminated with a gross error (Prószyński 2015) is defined as

$$\mathrm{ID}_i = \mathrm{P}\left(\mathrm{Z}_{1(i)} < 1 \cap \ldots \cap \mathrm{Z}_{j(i)} < 1 \cap \ldots \cap \mathrm{Z}_{n\text{-}1(i)} < 1 \;\middle|\; \overline{\mathrm{R}}\right) \qquad j \neq i \tag{19}$$

where

$$\mathrm{Z}_{j(i)} = \frac{\left|w^*_{j(i)}\right|}{\left|w^*_{i(i)}\right|} < 1 \;\; ; \qquad j \neq i,\; j = 1, \ldots, n$$

$$\mathrm{R} = \left|w^*_{1(i)}\right| < c \cup \ldots \cup \left|w^*_{i(i)}\right| < c \cup \ldots \cup \left|w^*_{n(i)}\right| < c$$

$Z_{j(i)}$ – a ratio of two folded normal variables (Kim 2006, 2014), the variables being correlated; $\overline{\mathrm{R}}$ – an event opposite to R; $w^*_{i(i)}$, $w^*_{j(i)}$ – values of the w^*-test statistics as in Sect.3; c – critical value determined as $\sqrt{w^2_{\mathrm{crit}}}$, where w^2_{crit} is a critical value in a global test (Knight et al., 2010).

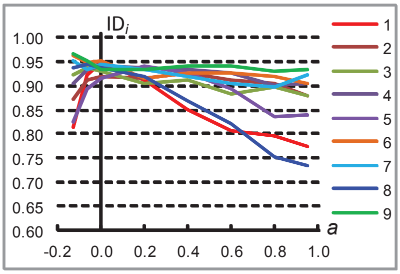
The ID index given by the formula (19) is a probability that the w^*-test statistic for a contaminated observation will be dominating in a set of the w^*-test statistics that exceed the critical value in a global model test. This is based on coordination between the outlier test and a global model test as in (Knight et al., 2010).

Due to a high complexity of the definition (19), an empirical method based on numerical simulation of random observation errors is applied. The simulation procedure is carried out so that the resulting random errors are correlated exactly according to a given matrix $\mathbf{C}_s(a)$.

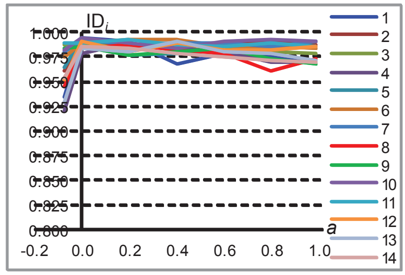
The diagrams presenting $\mathrm{ID}_i(a)$ for three test networks (Figures 5a,b,c) confirm the advantageous properties of a w^*-test discussed in Section 5. In each of the cases the decrease of identifiability is very small. For the networks b) and c) of higher redundancy the ID_i fluctuations, even for a approaching 1, are of very small magnitudes. This means that each considered observation when contaminated with the smaller and smaller gross error (see diagrams for $\mathrm{MDB}_{s,i}(a)$ in Figures 2b,c) will still be clearly identifiable. This is mainly due to an advantageous property of the w^*-test but also to a fact that in the present research the observations are simulated so that their random errors are correlated exactly according to $\mathbf{C}_s(a)$. In other words, in each test option the correlation matrix assigned to a system agrees exactly with the correlations of simulated observation errors.

It should be noted that in Figure 5a the curves falling down most significantly correspond to observations (8,1,5) with lower reliability indices. Similar effect, although to a much lesser extent, takes place in Figure 5c for observations (25, 26, 28) with the lowest reliability indices. Due to high values of IDs there was no need to determine the type III errors (Hawkins, 1980; Förstner, 1983).

a) V[0.53, 0.60] $n = 9, f = 5$



b) V[0.65, 0.75] $n = 14, f = 10$



c) H[0.40, 0.82] $n = 28, f = 18$

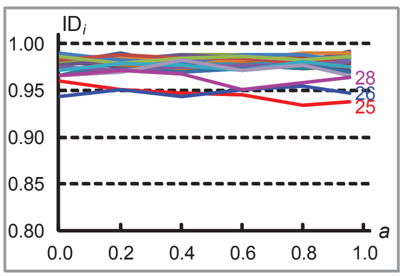


Fig. 5. Effect of increasing observation correlations on identifiability indices; **σ = 1**

The computations not included in the present paper indicate that the effect of varying accuracies, especially for networks b) and c), is not significant.

It is obvious that with the increase of the magnitude of gross error over its MDB value we would get higher and higher values of ID indices. For a network V[0.53, 0.60] $n = 9$ $f = 5$ and 2 × MDB magnitude of gross error the ID indices are practically reaching 1. This property is covered in the concept of Minimal Separable Bias (Wang and Knight 2012).

7. Effect of increase of observation correlations on response-based measures of GMM internal reliability

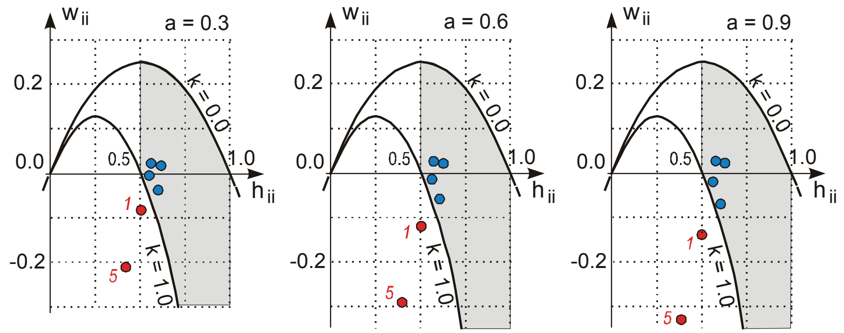
We shall use reliability indices (h,w) as in (Prószyński 2010), having clear interpretation in terms of network responses to a single gross error. Below, we recall the formulas for the i-th observation:

$$h_i = \{\mathbf{H}\}_{ii} \qquad w_i = \{\mathbf{H}\}_{ii} - \{\mathbf{H}^{\mathrm{T}}\mathbf{H}\}_{ii} \tag{20}$$

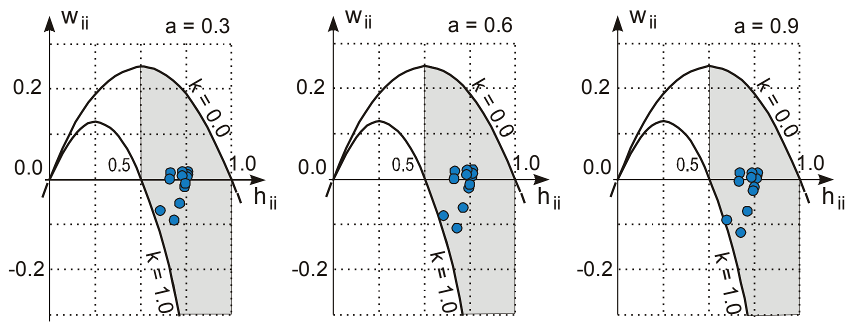
where

h_i – a local response, w_i – coefficient of asymmetry of projection operator **H**.

a) V[0.53, 0.60] $n = 9, f = 5$



b) V[0.65, 0.75] $n = 14, f = 10$



c) H[0.40, 0.82] $n = 28, f = 18$

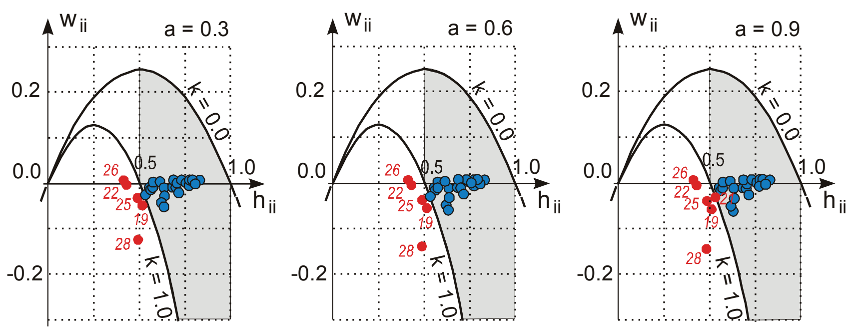


Fig. 6. The effect of observation correlations on response-based reliability measures (h,w); the shaded zone denotes *an outlier exposing area*; $\boldsymbol{\sigma} = \mathbf{1}$

The (h,w) area where the responses satisfy the reliability criteria, called in the above publication *an outlier exposing area* and delimited in terms of h and an auxiliary parameter k, will be shown in Figures 6a,b.c. The parameter k is a ratio of the squared quasi-global response to the squared local response, i.e.

$$k_i = \frac{Q_{(i)}^2}{L_{(i)}^2} = \frac{h_i - h_i^2 - w_i}{h_i^2} \qquad \text{for} \quad h_i \neq 0$$

For reference with the diagrams for relationship $\mathrm{MDB}_{s,i}(a)$ as shown in Figures 2b,c,d we present in Figures 6a,b,c the location of pairs h_i, w_i computed for certain chosen values of the parameter a for three test networks.

It should be emphasized that the response-based measures have been derived directly on basis of the elements of oblique projection operator **H** and they are not associated by an explicit relation with the probabilities of identifying a single outlier. Without going into a detailed analysis of the values of h and w, we shall only analyze the changes in location of the pairs (h, w) with the increase of observation correlations. It should be noted that the pairs (h, w) lying outside the *outlier exposing area* (Figures 6a,c), correspond to observations with reliability indices located at or close to a lower limit of the range.

In Figure 6a we can see a slight decrease of response-based reliability, especially for two observations of (h, w) pairs being outside the *outlier exposing area*. In Figures 6b and 6c we observe a hardly noticeable decrease of response-based reliability. The character of changes in location of the pairs (h, w) for all the three networks is largely in line with the course of curves ID(a) for these networks (Figure 5). This shows that the effect of increasing observation correlations on ID indices and on (h, w) indices is reflected in a substantially similar way.

8. Summing up the studies

The uniform correlation of observations assumed in the present research is a specific option among a variety of possible configurations of the correlation matrix. Therefore, the conclusions formulated on the basis of numerical tests and computations are valid only within the limits of the above assumption.

As it follows from the presented studies, with the increasing redundancy of networks the effect of increase of observation correlations for a particular network becomes more and more consistent between individual observations. Some minor inconsistencies in the shape of graphs for individual observations can be observed at small correlations. Taking all the test networks under consideration one can see high similarities in general shape of each analyzed relationship.

Satisfactory compliance was achieved for the effects of increase of observation correlations on identifiability indices and response-based reliability measures, despite their different theoretical bases. Furthermore, it could be observed that with the increase of global model test sensitivity to presence of a gross error in a network (reflected in the decrease of MDB_s), also the chance to identify this error in the outlier test increases.

As might be expected, for a network of high internal reliability the w^*-test proved to ensure high outlier identifiability over the whole domain of parameter a (including

$a \approx 1$ where the MDB_s values are infinitely small). It should be emphasized that the correlation matrix assumed in a model corresponded exactly with that used for generating random observation errors in a simulation process. However, this empirically obtained possibility of identifying among highly (or even extremely highly) correlated observations the one contaminated by an exceptionally small error, cannot be encountered in practice. The reasons are the following:

– measurement processes that generate highly correlated observations depart too much from the requirement functioning in metrology (and hence in geodesy), of striving for mutual independence of the generated observations. So, such processes are not allowed to practical use.
– it is obvious that high correlation of observations can be caused by the influence of external factors even for an appropriately structured measurement process;
– the absolute correspondence between the correlation matrix assumed in the model ($\mathbf{C}_s(a)$ in the tests) and the actual observation correlations cannot be ensured in practice. It is because we do not have a sufficient knowledge about the actual correlation of observations generated by the measurement process in use.

With the increase of observation correlations, an increasing degradation of the sensitivity of Baarda's *w*-test to a single gross error could be observed. The power of the test was systematically decreasing. Although one might expect it from the very beginning, the tests showed the rate of this degradation.

It seems that the results of the study can be used as an auxiliary information in pre-analyses of network reliability when there is no knowledge about the possible correlation of observations.

The presented approach is an initial attempt needing further development based on a wider set of observation systems. More attention should then be paid to options of varying accuracies and highly diversified indices of internal reliability.

For comparison, it would be interesting to investigate the effect of increase of observation correlations on reliability characteristics in robust estimation. However, also in such investigations the problem of suitable representation of the correlation matrix will remain an important issue.

Acknowledgments

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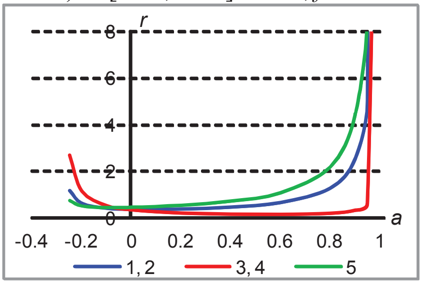
Appendix A

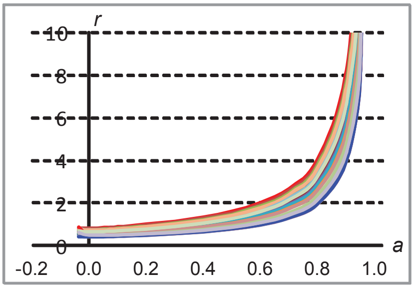
The theoretical and the test-based properties of expressions $\mathbf{HC}_\mathrm{s}\mathbf{H}^\mathrm{T}$ and $\mathbf{H}^\mathrm{T}\mathbf{C}_\mathrm{s}^{-1}\mathbf{H}$ (excluding $\mathbf{H} = 0$)

1. $\mathbf{HC}_\mathrm{s}\mathbf{H}^\mathrm{T} = \mathbf{HC}_\mathrm{s}$; $\mathbf{H}^\mathrm{T}\mathbf{C}_\mathrm{s}^{-1}\mathbf{H} = \mathbf{C}_\mathrm{s}^{-1}\mathbf{H}$
2. $\mathbf{C}_\mathrm{s}^{-1}\cdot\mathbf{HC}_\mathrm{s}\mathbf{H}^\mathrm{T}\cdot\mathbf{C}_\mathrm{s}^{-1} = \mathbf{H}^\mathrm{T}\mathbf{C}_\mathrm{s}^{-1}\mathbf{H}$
3. $\mathbf{HC}_\mathrm{s}\mathbf{H}^\mathrm{T}\cdot\mathbf{H}^\mathrm{T}\mathbf{C}_\mathrm{s}^{-1}\mathbf{H} = \mathbf{H}$
4. For $\mathbf{C}_\mathrm{s} = \mathbf{I}$, $\mathbf{HC}_\mathrm{s}\mathbf{H}^\mathrm{T} = \mathbf{H}^\mathrm{T}\mathbf{C}_\mathrm{s}^{-1}\mathbf{H}$
5. $0 < \left\{\mathbf{HC}_\mathrm{s}\mathbf{H}^\mathrm{T}\right\}_{ii} < 1$
   $\mathbf{HC}_\mathrm{s}\mathbf{H}^\mathrm{T} = \mathbf{HC}_\mathrm{s} = \mathbf{I} - \mathbf{A}_\mathrm{s}(\mathbf{A}_\mathrm{s}^\mathrm{T}\mathbf{C}_\mathrm{s}^{-1}\mathbf{A}_\mathrm{s})^{+}\mathbf{A}_\mathrm{s}^\mathrm{T} = \mathbf{I} - \mathbf{C}_{\hat{\mathbf{y}},\mathrm{s}}$; hence,
   $\left\{\mathbf{HC}_\mathrm{s}\mathbf{H}^\mathrm{T}\right\}_{ii} = 1 - \sigma^2_{\hat{y}_{\mathrm{s},i}}$
6. $0 < \left\{\mathbf{H}^\mathrm{T}\mathbf{C}_\mathrm{s}^{-1}\mathbf{H}\right\}_{ii} < \infty$ (Schaffrin 1997)

Figures 7a,b show the numerically obtained diagrams for the functions $r_i(a)$, $i = 1,\ldots,n$, where $r_i = \left\{\mathbf{H}^\mathrm{T}\mathbf{C}_\mathrm{s}^{-1}\mathbf{H}\right\}_{ii}$ (to simplify notation the asterisk above r_i is omitted) for the networks as in Figures 1a,d. The shape of characteristic fragment of the curve $r_i(a)$ for observation 25 is explained in Figure 7c with the aid of the corresponding curve for $\mathrm{MDB}_s(a)$.

Except for correlations up to 0.1 the function $r_i(a)$ is monotonously increasing for all the observations. Most dynamic increase takes place for correlations greater than 0.80. In the numerical tests carried out for networks with minimal (f =1) or small (f =2, 3) redundancies, the function $r_i(a)$ exhibits significant differentiation of courses for individual observations and becomes monotonously increasing at higher correlations.





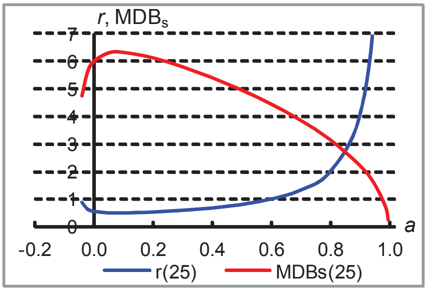


Fig. 7. a),b) Variability of the reliability number with respect to observation correlations; c) indicating a disturbed monotony for obs. 25 with the aid of $MDB_s(a)$; $\boldsymbol{\sigma} = \mathbf{1}$

Appendix B

Formula linking w-test and w^*-test
We introduce the following new variable

$$\mathbf{u}_s = \mathbf{C}_s^{-1}\mathbf{v}_s \text{ with the covariance matrix } \mathbf{C}_{\mathbf{u}_s} = \mathbf{C}_s^{-1}\mathbf{C}_{\mathbf{v}_s}\mathbf{C}_s^{-1} \tag{21}$$

which, after simple manipulations based on properties of the expressions $\mathbf{HC}_s\mathbf{H}^T$ and $\mathbf{HC}_s\mathbf{H}^T$ (see Appendix A), takes the form

$$\mathbf{u}_s = \mathbf{C}_s^{-1}\mathbf{v}_s = -\mathbf{C}_s^{-1}\mathbf{H}\mathbf{y}_s = -\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\mathbf{y}_s, \mathbf{C}_{\mathbf{u}_s} = \mathbf{C}_s^{-1}\mathbf{HC}_s\mathbf{H}^T\mathbf{C}_s^{-1} = \mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H} \tag{22}$$

Maintaining the structure of $\mathbf{y}_s$ as in (3), we get for the j-th element of $\mathbf{u}_s$

$$u_{s,j(i)} = -\{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\}_{j*}\mathbf{e}_s - \{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\}_{ji}\cdot\Delta y_{s,i} \qquad \sigma_{u_{s,j(i)}} = \sqrt{\{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\}_{jj}}$$

Hence, $-\dfrac{u_{s,j(i)}}{\sigma_{u_{s,j(i)}}} = \dfrac{\hat{z}_{s,j(i)}}{\sigma_{\hat{z}_{s,j(i)}}}$, and we get $w^*_{j(i)} = -\dfrac{u_{s,j(i)}}{\sigma_{u_{s,j(i)}}}$ $j = 1,\ldots, n$

where $w^*_{j(i)}$ as in (10).

Appendix C

Derivation of the formulas for correlations between the outlier test statistics w and between test statistics w^*

Since $w_{i(i)}$, $w_{j(i)}$ as well as $w^*_{i(i)}$, $w^*_{j(i)}$ are pairs of standardized variables, their covariances are correlation coefficients. For each pair we only show the elements that are necessary to apply the law of covariance propagation, i.e.

$$\begin{bmatrix} w_{i(i)} \\ w_{j(i)} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{\left\{\mathbf{H}\mathbf{C}_s\mathbf{H}^T\right\}_{ii}}} & 0 \\ 0 & \dfrac{1}{\sqrt{\left\{\mathbf{H}\mathbf{C}_s\mathbf{H}^T\right\}_{jj}}} \end{bmatrix} \begin{bmatrix} v_{s,i(i)} \\ v_{s,j(i)} \end{bmatrix} \tag{23}$$

$$\mathbf{C}\begin{bmatrix} v_{s,i(i)} \\ v_{s,j(i)} \end{bmatrix} = \begin{bmatrix} \left\{\mathbf{H}\mathbf{C}_s\mathbf{H}^T\right\}_{ii} & \left\{\mathbf{H}\mathbf{C}_s\mathbf{H}^T\right\}_{ij} \\ \left\{\mathbf{H}\mathbf{C}_s\mathbf{H}^T\right\}_{ji} & \left\{\mathbf{H}\mathbf{C}_s\mathbf{H}^T\right\}_{jj} \end{bmatrix}$$

$$\begin{bmatrix} w^*_{i(i)} \\ w^*_{j(i)} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{\left\{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\right\}_{ii}}} & 0 \\ 0 & \dfrac{1}{\sqrt{\left\{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\right\}_{jj}}} \end{bmatrix} \begin{bmatrix} \hat{z}_{i(i)} \\ \hat{z}_{j(i)} \end{bmatrix} \tag{24}$$

$$\mathbf{C}\begin{bmatrix} \hat{z}_{i(i)} \\ \hat{z}_{j(i)} \end{bmatrix} = \begin{bmatrix} \left\{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\right\}_{ii} & \left\{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\right\}_{ij} \\ \left\{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\right\}_{ji} & \left\{\mathbf{H}^T\mathbf{C}_s^{-1}\mathbf{H}\right\}_{jj} \end{bmatrix}$$

assuming $\mathbf{C}\begin{bmatrix} \hat{z}_{i(i)} \\ \hat{z}_{j(i)} \end{bmatrix} = \mathbf{C}\begin{bmatrix} u_{s,i(i)} \\ u_{s,j(i)} \end{bmatrix}$

On applying the law of covariance propagation to relationship (23) and then to (24) we obtain the formulas as in (13) and in (16) respectively.