

Economic growth in the European Union modelled with fractional derivatives: first results

I. TEJADO¹, E. PÉREZ¹, and D. VALÉRIO^{2*}

¹ Industrial Engineering School, University of Extremadura, Avda. de Elvas, s/n, 06006 Badajoz, Spain

² IDMEC, Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

Abstract. This paper presents models of economic growth for all states of the European Union (EU), since either 1970 or the year of accession to the EU. Both integer and fractional order models are obtained, where the gross domestic product (GDP) is a function of the country's land area, gross capital formation (GCF), exports of goods and services, and average years of school attendance.

Key words: economic growth, Europe, fractional calculus.

1. Introduction

Economic growth is conditioned by many factors, which act over time. This effect can be modelled using fractional derivatives, more accurately than using integer derivatives only [1–5]. In fact, it is reasonable to speak of the diffusion of several of the factors that condition economic growth, and of the diffusion of economic growth itself [6–8]. Diffusion processes in biological systems can often be modelled using fractional derivatives [9], and published results show that this also happens with financial models [10–23]. Fractional derivatives themselves have an economic interpretation [24] and are needed in the formulation of models for economic processes with long memory [25]. Fractional order models have been built for the GDP of several countries at a world level, both for recent years only and for longer time series [26, 27].

In previous papers we developed integer order and fractional order models, with the latter outperforming the former, for the economic growth of four economies in Western Europe, all bordering the Mediterranean [28, 29]: Portugal, Spain, France and Italy. In this paper we develop similar models for all states of the European Union (EU). This choice is motivated by the high degree of integration of the national economies involved, allowing to assume that similar patterns can be found in the resulting models. Continuous series of data are available from 1970 on; models are thus obtained in the 1970–2016 period for the states which in 1970 were members of the European Economic Community (EEC), predecessor of the EU, established in 1993 by the Maastricht Treaty. Models for other states are presented from the year of accession to the EEC or the EU (see Table 1).

The paper presents the methodology followed in Sec. 2 and the results obtained in Sec. 3. A discussion and conclusions are given in Sec. 4. The data employed for the models are tabulated in an Appendix.

*e-mail: duarte.valerio@tecnico.ulisboa.pt

Manuscript submitted 2017-11-16, revised 2018-01-23, initially accepted for publication 2018-01-26, published in August 2018.

2. Methodology

The models considered have the following form for each of the member states of the EU:

$$y(t) = f(x_1, x_4, x_5, x_6). \quad (1)$$

The output model y is the GDP (in 2016 euros). The x_k are the variables on which the output depends:

- x_1 : land area (km²);
- x_4 : school attendance (years);
- x_5 : gross capital formation (GCF) (in 2016 euros);
- x_6 : exports of goods and services (in 2016 euros).

The rationale behind this choice of variables is the following:

- natural resources are represented by x_1 ;
- the quality of human resources is represented by x_4 ;
- the resources manufactured and the impact of investment on the economy are represented by x_5 ;
- external impacts on the economy are represented by x_6 .

The numeration of these variables is not consecutive, because they are a subset of those used in [28, 29], mentioned above in Sec. 1. The variables retained in this paper are those that were shown to be relevant for all the four models developed in those references.

The integer order model considered is

$$y(t) = C_1 x_1(t) + C_4 x_4(t) + C_5 \int_{t_0}^t x_5(t) dt + C_6 x_6(t), \quad (2)$$

where C_k are constant weights for each of the variables, and t_0 is the first year considered. Notice that the accumulated

gross capital formation $\int_{t_0}^t x_5(t) dt$ is used as a measure of manufactured resources.

Its generalization to non-integer orders is as follows:

$$y(t) = \sum_{k=1,4,5,6} C_k D^{\alpha_k} x_k(t), \quad (3)$$

where α_k are the differentiation orders of each variable. The Caputo definition of fractional derivative D^{α_k} was used [30].

Table 1
Year of accession to the EEC (1957–1993) or the EU (1993–present) of all member states

Year	Accession of states
1957	Belgium (BEL), France (FRA), Germany* (DEU), Italy (ITA), Luxembourg (LUX), Netherlands (NDL)
1973	Denmark (DNK), Ireland (IRL), United Kingdom (GBR)
1981	Greece (GRC)
1986	Portugal (PRT), Spain (ESP)
1995	Austria (AUT), Finland (FIN), Sweden (SWE)
2004	Czech Republic (CZE), Cyprus (CYP), Estonia (EST), Hungary (HUN), Latvia (LVA), Lithuania (LTU), Malta (MLT), Poland (POL), Slovakia (SVK), Slovenia (SVN)
2007	Bulgaria (BGR), Romania (ROU)
2013	Croatia (HRV)

* In 1990, the former German Democratic Republic was integrated into the Federal Republic of Germany. There was no increase in the number of member-states, but the EEC territory got larger, and variables for Germany have large variations in that year.

3. Results

This section contains the models for the economies of all states of the EU in the period between 1970 to 2016 (see economic data in Tables 4 and 5 in the Appendix).

The fitting procedure is implemented in MATLAB. Nelder-Mead's simplex search method (implemented in function *fminsearch*) is used to minimize the mean square error (MSE), given by

$$\text{MSE} = \frac{\sum_{j=1}^N (y_j - \hat{y}_j)^2}{N}. \quad (4)$$

Here N is the number of points, and y_j and \hat{y}_j are the real output and the model output, respectively. The MSE alone is not relied upon to evaluate the quality of the fit obtained by the resulting models: other performance indices were calculated as well. These were:

1. The mean absolute deviation (MAD), given by

$$\text{MAD} = \frac{\sum_{j=1}^N |y_j - \hat{y}_j|}{N}. \quad (5)$$

2. The coefficient of determination ($R^2 \in (0, 1)$), given by

$$R^2 = 1 - \frac{\sum_{j=1}^N (y_j - \hat{y}_j)^2}{\sum_{j=1}^N (y_j - \bar{y})^2}. \quad (6)$$

Here \bar{y} is the mean of the GDP.

3. The t -values and p -values for each variable.

These are calculated with MATLAB command *regstats*.

As will be seen below, not all four variables x_1 , x_4 , x_5 and x_6 turned out to be necessary for every single model. This could be evaluated from the t - and p -values for each variable, by checking whether or not the performance indexes MAD and R^2 deteriorate significantly when removing one or more variables from the model, and also using the Akaike Information Criterion (AIC):

$$\text{AIC} = N \log \frac{\sum_{j=1}^N (y_j - \hat{y}_j)^2}{N} + 2K + \frac{2K(K+1)}{N-K-1}. \quad (7)$$

Here K is the number of parameters of the model. The value of the AIC does not give information about the quality of a model. However, comparing the AIC values of different models, it can be seen which ones are more likely to be a good model for the data, as a lower value indicates a more likely model. Furthermore, if there are M models, the Akaike weight, given by

$$w_i = \frac{\exp\left(-\frac{\text{AIC}_i - \min \text{AIC}}{2}\right)}{\sum_{j=1}^M \exp\left(-\frac{\text{AIC}_j - \min \text{AIC}}{2}\right)}, \quad (8)$$

provides the probability of model i being the best of all the M models.

The results of the models for the several EU member-states are shown in Fig. 1 and 2, with performance indices in Table 2. In that table, t -values that correspond to variables necessary for the model, assuming a 5% significance level, are given in bold. The values of the orders α and the coefficients C are given in Table 3 (notice that such orders for integer model (2) are $\alpha_1 = 0$, $\alpha_4 = 0$, $\alpha_5 = -1$ and $\alpha_6 = 0$).

4. Discussion and conclusions

As can be seen, the first conclusion to be drawn is that fractional order models are better in terms of the indices considered and the Akaike weight w calculated, although this may be considered unsurprising because they have more parameters, i.e., more flexibility for fitting. For this reason, the analysis of the results given in the previous section will be performed adopting different points of view: 1) the significance of the four variables of the model (x_1 , x_4 , x_5 and x_6) after fitting, 2) the values of the order for each variable in the fractional model, and 3) the values of coefficients C . For a better interpretation of the results given in Table 3, the orders α of the model variables (except for x_1) and the coefficients C of fractional model (3) are plotted on a EU map in Figs. 3 and 4, respectively. Although included in plots and tables, the results for HRV are omitted from the discussion for obvious reasons (only data for four years is available).

Economic growth in the European Union modelled with fractional derivatives: first results

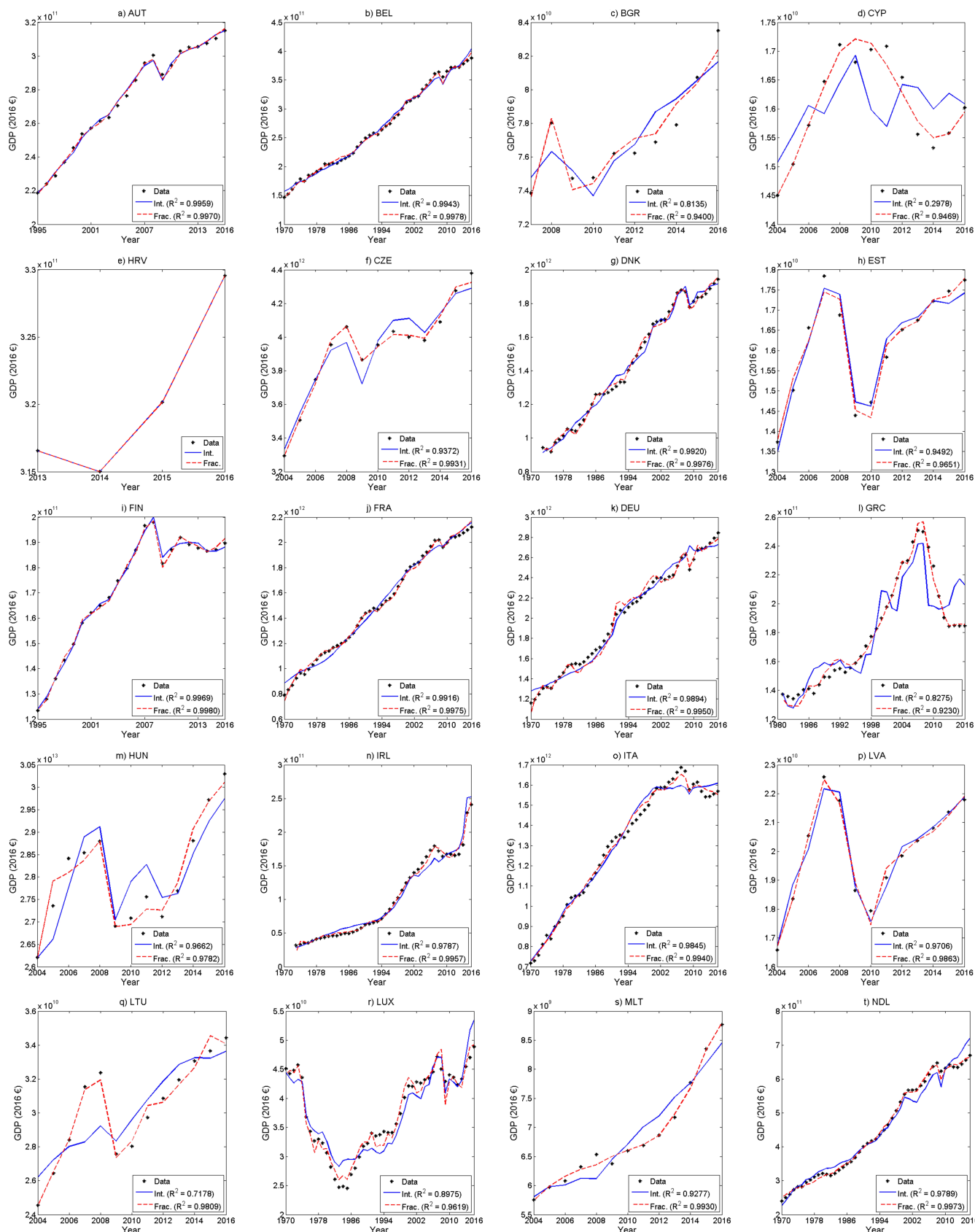


Fig. 1. Fitting results for integer model (2) and fractional model (3) for EU states: a) Austria, b) Belgium, c) Bulgaria, d) Cyprus, e) Croatia, f) Czech Republic, g) Denmark, h) Estonia, i) Finland, j) France, k) Germany, l) Greece, m) Hungary, n) Ireland, o) Italy, p) Latvia, q) Lithuania, r) Luxembourg, s) Malta, t) Netherlands (for illustration purposes, notice that the scale of y - and x -axis is not the same for all states)

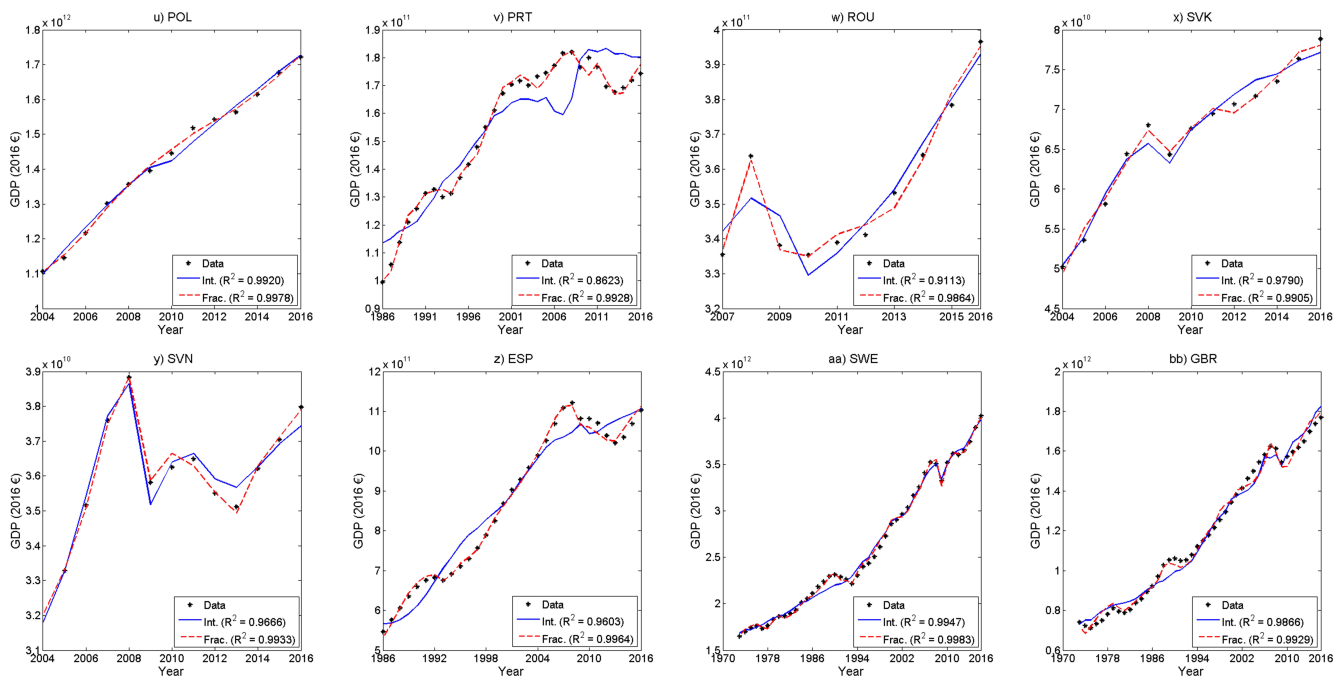


Fig. 2. Fitting results for integer model (2) and fractional model (3) for EU states (cont.): u) Poland, v) Portugal, w) Romania, x) Slovak Republic, y) Slovenia, z) Spain, aa) Sweden, bb) United Kingdom (for illustration purposes, notice that the scale of y - and x -axis is not the same for all states)

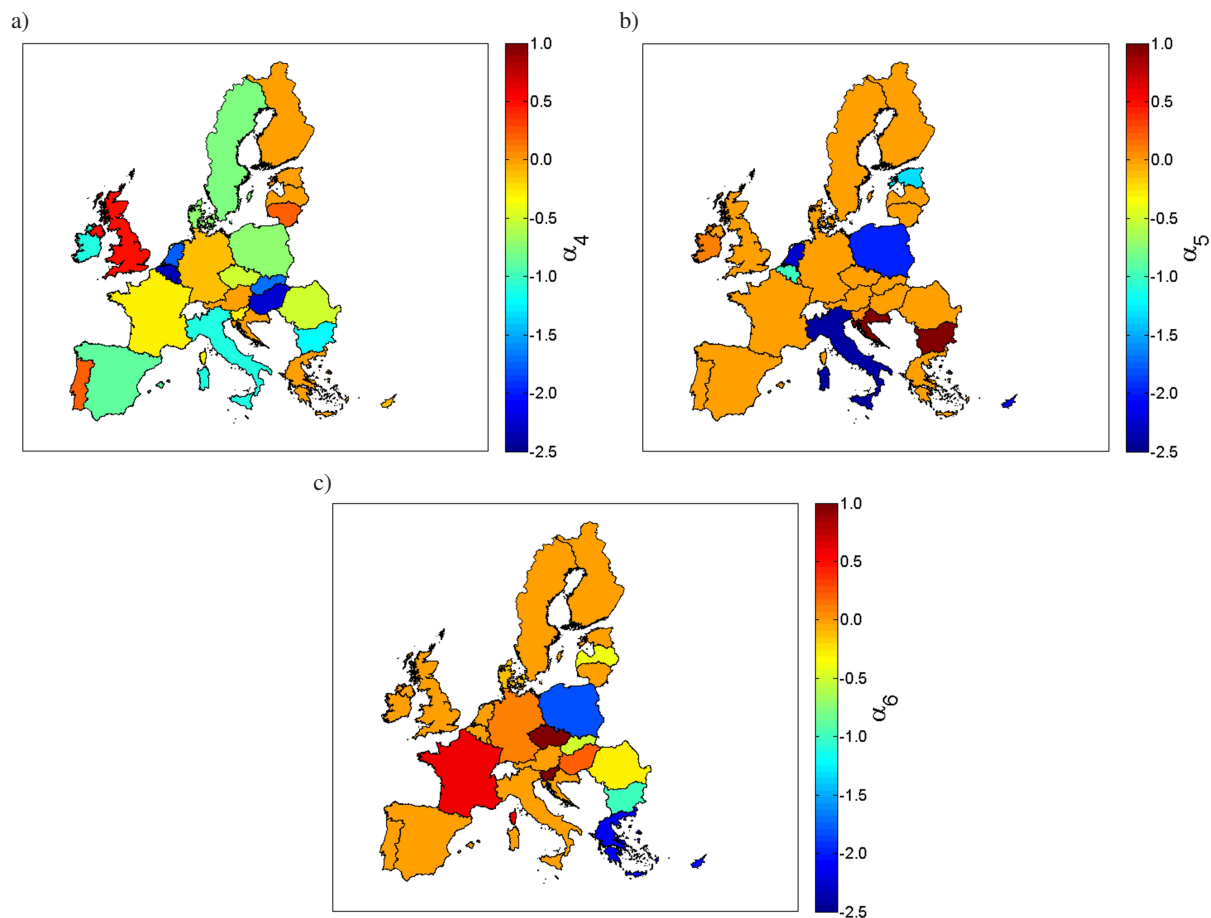


Fig. 3. Results for the orders of the variables of the fractional model (3) on an EU map: a) order α_4 , b) order α_5 , c) order α_6

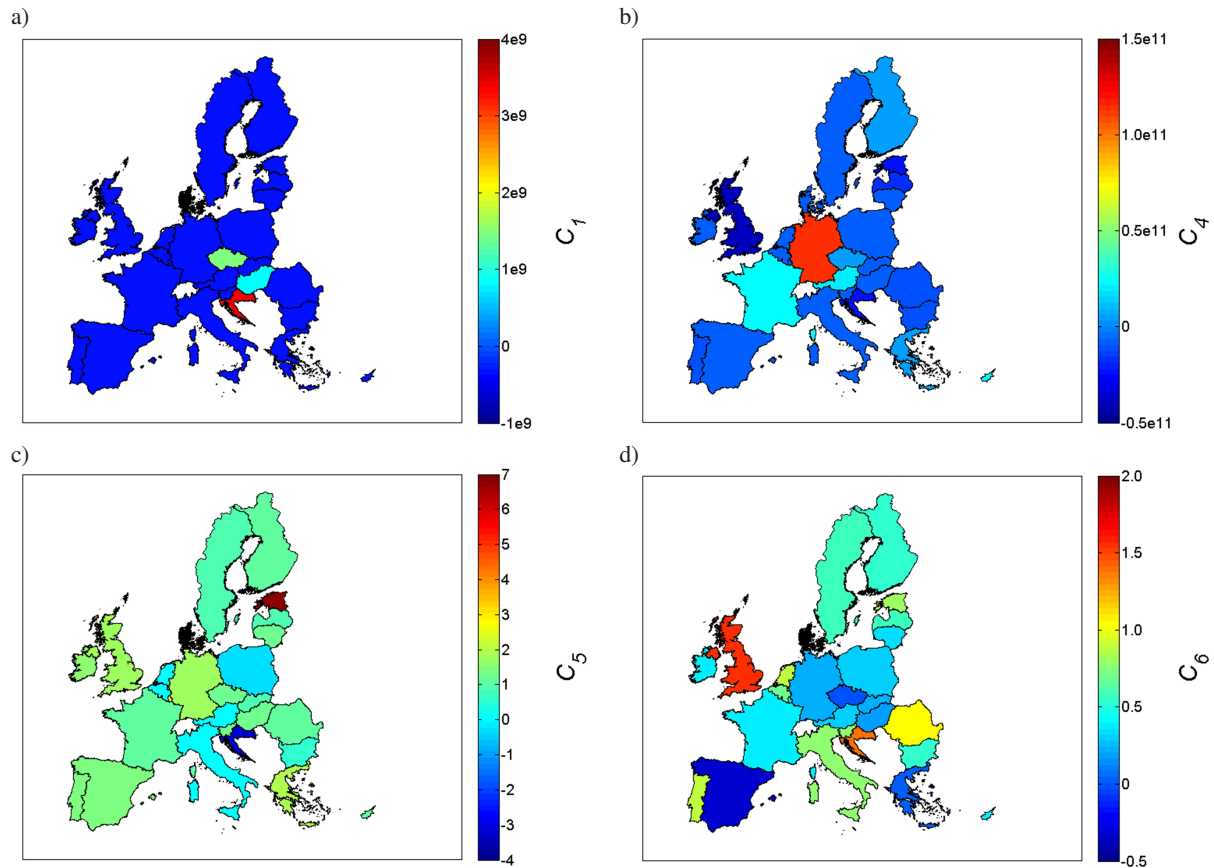


Fig. 4. Results for the coefficients of the fractional model (3) on an EU map: a) coefficient C_1 , b) coefficient C_4 , c) coefficient C_5 , d) coefficient C_6 (Denmark is colored black whenever her value is outside the range considered.)

As far as the significance of the variables of the model is concerned, it is observed that:

- For integer models of the form of (2), not all the variables are significant. This is true for all states, except for the group of five states formed by EST, FIN, HUN, LVA and SVN. Furthermore, for these five countries all the variables are also required for the fractional model.
- For fractional models of the form of (3), all the variables are significant. This is true for all states, except for the group of seven states which are (giving the variable, or variables, with no significance in brackets): AUT (x_5), CYP (x_4), CZE (x_6), FRA (x_6), GRC (x_1, x_4, x_6), POL (x_4), and ESP (x_6).

Taking into account the orders of the fractional models (see Fig. 3):

- As might be expected, the order of variable x_1 is always 0, with the exception of GRC, although even in this case the value of α_1 can be negligible. (It should be noticed that variable x_1 is constant for all countries, except for DEU.)
- The models for AUT and FIN have all the orders equal to 0, i.e., the models for these states are of integer order instead. One reason for this result may be due to the fact that the integer order models can fit the data meaningfully well

(the value of the index R^2 is higher than 0.99, the highest in the table). This circumstance makes the optimization process in MATLAB more difficult: more iterations, and consequently more time, are required to find the minimum of the MSE. However, the cause for the change of the order in x_5 cannot be ascertained with ease.

- From Fig. 3b, it can be observed that the variable x_5 has influence on GDP of three different forms depending on the value of its corresponding order as follows:
 1. when $\alpha_5 = -1$ (as for the integer model (2)), it is a measure of manufactured resources. This is the case for BEL. In the case of EST, $\alpha_5 = -1.32$. Furthermore, there is a group of countries with $\alpha_5 = -2$, or closer, which are ITA, MLT, NLD, and POL.
 2. when $\alpha_5 = 1$, the effect of x_5 is a measure of the impact of the variation of investment in the economy. This is the case of BGR.
 3. when $\alpha_5 = 0$, which is the most common case obtained for the EU states (the remaining ones), x_5 measures the impact of investment in the economy. In this group, it should be mentioned that α_5 is not exactly equal to 0 for GRC, IRL, PRT and SVN, but small and positive.

Table 2

Performance indices for integer model (2) and fractional model (3) for EU states (note for HRV: it is not possible to obtain t - and p -values since the matrix has more predictor variables than observations. These indices are marked as *

Index	Variable	AUT		BEL		BGR		CYP		HRV		CZE		DNK	
		Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)
MSE		3.630×10^{18}	2.688×10^{18}	3.066×10^{19}	1.208×10^{19}	1.464×10^{18}	4.706×10^{17}	4.692×10^{17}	3.548×10^{16}	1.918×10^{-6}	0	4.852×10^{21}	5.338×10^{20}	8.989×10^{20}	2.643×10^{20}
R ²		0.9959	0.9970	0.9943	0.9978	0.8135	0.9400	0.9278	0.9469	1	0.9372	0.9931	0.9920	0.9976	
MAD		1.521×10^9	1.397×10^9	4.261×10^9	2.527×10^9	1.048×10^9	5.729×10^8	5.814×10^8	1.392×10^8	1.129×10^{-3}	0	5.755×10^{10}	1.833×10^{10}	2.498×10^{10}	1.398×10^{10}
t -values	x_1	-1.589	-2.954	-3.704	34.464	-1.678	27.880	2.708	75.264	*	*	2.901	22.574	-1.600	57.835
	x_4	4.444	5.868	11.169	-19.634	1.990	-4.863	-1.383	0.904	*	*	2.518	21.339	-12.923	18.992
	x_5	-0.842	2.696	-0.273	16.505	-1.647	4.943	1.840	-6.705	*	*	-1.209	13.507	-3.186	18.350
	x_6	6.154	2.914	5.385	11.059	1.347	5.052	-1.356	6.456	*	*	4.431	-1.508	7.690	-3.867
	x_6	1.295×10^{-1}	8.488×10^{-3}	6.011×10^{-4}	5.899×10^{-33}	1.443×10^{-1}	1.409×10^{-7}	2.406×10^{-2}	6.518×10^{-14}	*	*	1.756×10^{-2}	3.113×10^{-9}	1.175×10^{-1}	3.556×10^{-40}
p -values	x_1	3.136×10^{-4}	1.479×10^{-5}	2.709×10^{-14}	4.336×10^{-23}	9.371×10^{-2}	2.813×10^{-3}	2.000×10^{-1}	3.897×10^{-1}	*	*	3.287×10^{-2}	5.123×10^{-9}	7.309×10^{-16}	1.273×10^{-21}
	x_4	4.107×10^{-1}	1.479×10^{-2}	7.861×10^{-1}	3.209×10^{-20}	1.507×10^{-1}	2.596×10^{-3}	9.890×10^{-2}	8.800×10^{-5}	*	*	2.575×10^{-1}	2.793×10^{-7}	2.792×10^{-3}	4.384×10^{-21}
	x_5	8.238×10^{-6}	9.257×10^{-3}	2.847×10^{-6}	3.723×10^{-14}	2.265×10^{-1}	2.327×10^{-3}	2.080×10^{-1}	1.173×10^{-4}	*	*	1.644×10^{-3}	1.659×10^{-1}	2.088×10^{-9}	3.961×10^{-4}
	AIC	950.5	943.9	2117.8	2074.0	434.3	422.9	542.0	508.4	*	*	662.1	633.44	2131.9	2078.1
	w (%)	3.5	96.4	0	100	0.3	99.7	0	100	*	*	0	100	0	100
AIC without one variable	x_1	950.4	949.6	2128.4	2229.3	432.1	465.6	545.4	587.8	*	*	666.4	681.81	2132.2	2270.9
	x_4	963.8	964.4	2179.4	2180.0	433.3	432.9	540.1	504.8	*	*	664.7	680.4	2201.8	2177.0
	x_5	948.4	948.4	2115.5	2165.3	432.0	433.1	541.8	527.3	*	*	659.8	668.8	2139.4	2174.3
	x_6	972.4	949.4	2139.6	2134.9	430.9	433	540.0	526.5	*	*	672.8	632.0	2169.4	2089.6
w found from AIC	x_1	30	25.3	0	0	23	0	3	0	*	*	3.3	0	97.4	0
	x_4	0	0	0	0	12	38.2	39	100	*	*	7.4	0	0	0
	x_5	70	46.9	100	0	24	33.6	17	0	*	*	89.2	0	2.6	0
	x_6	0	27.8	0	100	41	28.2	41	0	*	*	0.1	100	0	100
Index	Variable	EST		FIN		FRA		DEU		GRC		HUN		IRL	
		Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)
MSE		8.534×10^{16}	5.858×10^{16}	1.512×10^{18}	9.720×10^{17}	1.409×10^{21}	4.168×10^{20}	4.677×10^{21}	2.555×10^{21}	2.247×10^{20}	1.734×10^{19}	2.468×10^{23}	5.365×10^{22}	7.494×10^{19}	1.515×10^{19}
R ²		0.9492	0.9969	0.9969	0.9975	0.9916	0.9975	0.9894	0.9950	0.8275	0.9230	0.9662	0.9782	0.9957	
MAD		2.515×10^8	1.889×10^8	9.985×10^8	8.529×10^8	3.055×10^{10}	1.618×10^{10}	5.402×10^{10}	4.113×10^{10}	1.169×10^{10}	3.214×10^9	4.274×10^{11}	1.782×10^{11}	6.465×10^9	2.953×10^9
t -values	x_1	8.535	9.862	-5.529	-5.400	1.781	24.167	6.015	9.922	-1.901	0.631	-5.984	17.575	-2.696	12.169
	x_4	-8.102	-9.477	7.958	9.480	9.948	30.136	1.934	15.680	2.978	1.440	6.498	10.546	3.577	9.210
	x_5	4.421	5.708	-6.332	8.508	1.405	9.253	10.644	13.606	-3.726	8.266	-10.236	9.270	-1.895	11.604
	x_6	7.395	10.544	12.239	5.832	1.907	0.883	-6.546	4.532	5.607	1.696	9.380	6.718	7.585	13.179
	x_6	1.314×10^{-5}	4.016×10^{-6}	3.000×10^{-5}	3.935×10^{-5}	8.200×10^{-2}	1.189×10^{-26}	3.474×10^{-7}	1.094×10^{-12}	6.633×10^{-2}	5.322×10^{-1}	2.065×10^{-3}	2.829×10^{-8}	1.022×10^{-2}	5.022×10^{-15}
p -values	x_1	2.000×10^{-5}	5.583×10^{-6}	2.641×10^{-7}	2.018×10^{-8}	1.012×10^{-12}	1.509×10^{-30}	5.967×10^{-2}	2.140×10^{-19}	5.496×10^{-3}	1.117×10^{-3}	2.295×10^{-6}	9.274×10^{-6}	1.963×10^{-11}	
	x_4	1.668×10^{-3}	2.912×10^{-4}	5.750×10^{-6}	1.009×10^{-7}	1.673×10^{-1}	8.622×10^{-12}	1.256×10^{-13}	3.503×10^{-17}	7.517×10^{-4}	1.921×10^{-9}	2.946×10^{-6}	6.700×10^{-2}	6.528×10^{-2}	
	x_5	4.123×10^{-5}	2.298×10^{-6}	3.670×10^{-10}	1.593×10^{-5}	6.321×10^{-2}	3.820×10^{-1}	5.862×10^{-8}	4.613×10^{-5}	3.396×10^{-6}	9.949×10^{-2}	6.081×10^{-6}	8.676×10^{-5}	2.906×10^{-9}	
	AIC	519.8	514.9	931.3	921.5	2297.7	2240.5	2354.1	2325.7	1696.3	1604.1	713.2	693.4	2022.6	1952.3
	w (%)	8	92	0.8	99.2	0	100	0	100	0	100	0	100	0	100
AIC without one variable	x_1	544.2	542.7	950.1	939.7	2298.7	2364.0	2362.6	2346.5	1697.6	1665.1	707.1	7353.8	2027.5	2018.0
	x_4	543.0	541.7	961.4	957.9	2351.5	2383.6	2354.8	2376.7	1702.6	1725.8	708.9	7227.4	2032.4	2000.0
	x_5	530.5	530.5	954.0	954.0	2297.4	2289.5	2388.5	2360.6	1706.7	1720.5	719.2	7196.7	2024.0	2014.7
	x_6	540.9	544.3	977.4	941.9	2299.1	2238.9	2384.8	2330.2	1718.4	1604.4	724.7	7123.7	2059.4	2032.6
w found from AIC	x_1	0.1	0.2	87.5	74.5	27.4	0	2	0	91.4	0	70.3	0	14.2	0
	x_4	0.2	0.4	0.3	0	0	0	98	0	7.7	0	29.5	0.5	1.3	99.9
	x_5	99.2	99.3	12.2	0.1	50.9	0	0	0	0.9	0	0.2	2.5	84.5	0.1
	x_6	0.5	0.1	0	25.4	21.7	100	0	100	0	100	0	97	0	0
Index	Variable	ITA		LVA		LTU		LUX		MLT		NLD		POL	
		Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)	Int. (2)	Frac. (3)
MSE		1.357×10^{21}	5.234×10^{20}	8.403×10^{16}	3.923×10^{16}	2.371×10^{18}	1.601×10^{17}	5.063×10^{18}	1.880×10^{18}	5.634×10^{16}	5.438×10^{15}	4.127×10^{20}	5.308×10^{19}	2.901×10^{20}	8.156×10^{19}
R ²		0.9845	0.9940	0.9706	0.9863	0.9718	0.9809	0.8975	0.9619	0.9277	0.9930	0.9789	0.9973	0.9920	0.9978
MAD		3.010×10^{10}	1.998×10^{10}	2.432×10^8	1.468×10^8	1.225×10^9	3.176×10^8	1.896×10^{10}	1.093×10^9	1.918×10^8	5.219×10^7	1.548×10^{10}	5.901×10^9	1.394×10^{10}	7.590×10^9
t -values	x_1	-10.502	50.626	11.998	5.289	0.131	23.507	0.890	13.121	-0.282	7.154	-2.871	36.489	-3.041	120.108
	x_4	15.350	18.160	-11.692	-4.820	6.493×10^{-2}	-3.214	-8.233×10^{-2}	-7.463	0.610	-6.226	4.267	14.807	3.927	0.838
	x_5	-11.023	-25.825	7.020	12.932	-1.247×10^{-2}	8.194	-1.384	8.194	0.507	-5.153	-1.761	-25.323	3.697	-3.630
	x_6	1.875	4.882	8.067	8.055	1.058	17.500	11.677	7.454	-0.721	5.580	3.984	11.808	-0.778	3.637
	x_6	1.912×10^{-13}	5.784×10^{-40}	7.712×10^{-7}	5.010×10^{-4}	8.989×10^{-1}	2.174×10^{-9}	3.784×10^{-1}	1.242×10^{-16}	7.841×10^{-1}	5.342×10^{-5}	6.327×10^{-3}	5.486×10^{-34}	1.398×10^{-2}	9.764×10^{-16}
p -values	x_1	4.667×10^{-19}	8.740×10^{-22}	9.603×10^{-7}	9.464×10^{-4}	9.496×10^{-1}	1.059×10^{-2}	9.348×10^{-1}	2.763×10^{-9}	5.570×10^{-1}	$1.539 \$				

Economic growth in the European Union modelled with fractional derivatives: first results

Table 3

Fitting results for integer model (2) and fractional model (3) for EU states: coefficients and orders of the fractional operator (notice that the orders α_k for integer model are $\alpha_1 = 0, \alpha_4 = 0, \alpha_5 = -1$ and $\alpha_6 = 0$)

AUT		x_1	x_4	x_5	x_6	BEL		x_1	x_4	x_5	x_6
Int. (2)	C_k	-1.122×10^6	3.082×10^{10}	-4.117×10^{-3}	5.350×10^{-1}	Int. (2)	C_k	-2.672×10^6	2.665×10^{10}	-3.469×10^{-3}	5.165×10^{-1}
Frac. (3)	C_k	-2.107×10^6	3.667×10^{10}	7.373×10^{-2}	3.065×10^{-1}	Frac. (3)	C_k	3.706×10^6	-9.664×10^6	1.047×10^{-2}	6.761×10^{-1}
	α_k	0	0	0	0		α_k	0	-2.31	-1	0
BGR		x_1	x_4	x_5	x_6	CYP		x_1	x_4	x_5	x_6
Int. (2)	C_k	-3.112×10^6	3.754×10^{10}	-8.978×10^{-2}	2.743×10^{-1}	Int. (2)	C_k	4.559×10^6	-1.502×10^9	1.706×10^{-1}	-1.482×10^{-2}
Frac. (3)	C_k	5.674×10^5	-1.202×10^9	4.636×10^{-1}	5.279×10^{-1}	Frac. (3)	C_k	8.945×10^5	4.230×10^{10}	1.146	3.607×10^{-1}
Orders	α_k	0	-1.25	1	-1	Orders	α_k	0	-0.10	-2.20	-2.03
HRV		x_1	x_4	x_5	x_6	CZE		x_1	x_4	x_5	x_6
Int. (2)	C_k	4.028×10^8	-1.845×10^{12}	2.542	6.607×10^{-1}	Int. (2)	C_k	2.149×10^9	-1.142×10^{12}	3.282×10^{-2}	8.532×10^{-1}
Frac. (3)	C_k	6.164×10^8	-1.941×10^{10}	-3.296	1.416	Frac. (3)	C_k	2.874×10^8	1.580×10^{10}	1.204	-3.528×10^{-2}
	α_k	0	2.50×10^{-4}	1	0		α_k	0	-0.52	0	1
DNK		x_1	x_4	x_5	x_6	EST		x_1	x_4	x_5	x_6
Int. (2)	C_k	-2.833×10^6	1.086×10^{11}	-3.894×10^{-2}	1.104	Int. (2)	C_k	4.701×10^6	-1.595×10^{10}	11.231	7.549×10^{-1}
Frac. (3)	C_k	3.938×10^9	1.008×10^9	9.430×10^{10}	-5.001	Frac. (3)	C_k	5.602×10^6	-1.911×10^{10}	6.785	8.292×10^{-1}
	α_k	0	-0.52	0	1		α_k	0	0	-1.32	0
FIN		x_1	x_4	x_5	x_6	FRA		x_1	x_4	x_5	x_6
Int. (2)	C_k	-5.971×10^5	2.963×10^{10}	-2.556×10^{-2}	9.395×10^{-1}	Int. (2)	C_k	2.424×10^5	13.160×10^{10}	1.286×10^{-2}	4.038×10^{-1}
Frac. (3)	C_k	-3.153×10^5	1.886×10^{10}	1.020	5.326×10^{-1}	Frac. (3)	C_k	8.945×10^5	4.230×10^{10}	1.146	3.607×10^{-1}
	α_k	0	0	0	0		α_k	0	-0.3	0	0.63
DEU		x_1	x_4	x_5	x_6	GRC		x_1	x_4	x_5	x_6
Int. (2)	C_k	9.862×10^5	1.520×10^{11}	5.359×10^{-2}	-4.703×10^{-1}	Int. (2)	C_k	-1.448×10^6	3.571×10^{10}	-1.677×10^{-1}	3.364
Frac. (3)	C_k	1.763×10^6	1.568×10^{11}	1.847	2.014×10^{-1}	Frac. (3)	C_k	-4.889×10^5	1.742×10^{10}	1.915	2.676×10^{-4}
	α_k	0	-0.1	0	0.1		α_k	1.25×10^{-5}	0	2.80×10^{-17}	-2.10
HUN		x_1	x_4	x_5	x_6	IRL		x_1	x_4	x_5	x_6
Int. (2)	C_k	2.070×10^8	0	-9.426×10^{-2}	6.105×10^{-1}	Int. (2)	C_k	-1.676×10^6	1.648×10^{10}	-7.688×10^{-2}	7.877×10^{-1}
Frac. (3)	C_k	1.921×10^8	2.362×10^9	1.262	1.732×10^{-1}	Frac. (3)	C_k	2.858×10^5	9.213×10^7	1.536	3.594×10^{-1}
	α_k	0	-2.29	0	0.25		α_k	0	-1.1	0.11	0
ITA		x_1	x_4	x_5	x_6	LVA		x_1	x_4	x_5	x_6
Int. (2)	C_k	-5.227×10^6	3.824×10^{11}	-1.116×10^{-1}	5.085×10^{-1}	Int. (2)	C_k	4.971×10^6	-2.871×10^{10}	2.564×10^{-1}	1.405×10^{-1}
Frac. (3)	C_k	2.257×10^6	3.260×10^9	-1.800×10^{-3}	7.875×10^{-1}	Frac. (3)	C_k	1.907×10^6	-1.015×10^{10}	9.610×10^{-1}	5.549×10^{-1}
	α_k	0	-1.12	-2.42	0		α_k	0	0	0	-0.40
LTU		x_1	x_4	x_5	x_6	LUX		x_1	x_4	x_5	x_6
Int. (2)	C_k	2.306×10^5	6.599×10^8	-2.440×10^{-3}	4.206×10^{-1}	Int. (2)	C_k	1.045×10^7	-3.287×10^8	-9.418×10^{-2}	5.632×10^{-1}
Frac. (3)	C_k	2.911×10^5	-2.339×10^8	1.230	3.510×10^{-1}	Frac. (3)	C_k	4.585×10^6	-1.378×10^6	2.623	3.902×10^{-1}
	α_k	0	0.22	0	1.31×10^{-3}		α_k	0	-2.20	0	0
MLT		x_1	x_4	x_5	x_6	NLD		x_1	x_4	x_5	x_6
Int. (2)	C_k	-1.987×10^7	1.251×10^8	1.424×10^{-1}	-2.038×10^{-1}	Int. (2)	C_k	-1.502×10^7	7.902×10^{-10}	-7.496×10^{-2}	1.090
Frac. (3)	C_k	1.395×10^8	-3.685×10^9	-5.846×10^{-1}	1.557×10^{-1}	Frac. (3)	C_k	5.840×10^6	1.191×10^8	-3.795×10^{-3}	8.798×10^{-1}
	α_k	0	0	-2.00	-1.73		α_k	0	-1.80	-2.30	0
POL		x_1	x_4	x_5	x_6	PRT		x_1	x_4	x_5	x_6
Int. (2)	C_k	-1.220×10^7	4.350×10^{11}	1.239×10^{-1}	-2.221×10^{-1}	Int. (2)	C_k	1.884×10^5	1.879×10^{10}	8.558×10^{-2}	-1.318
Frac. (3)	C_k	3.605×10^6	1.729×10^9	-3.696×10^{-1}	2.853×10^{-1}	Frac. (3)	C_k	9.288×10^5	-3.980×10^9	1.487	8.998×10^{-1}
	α_k	0	-0.70	-2.00	-1.82		α_k	0	0.22	1.25×10^{-3}	0
ROU		x_1	x_4	x_5	x_6	SVK		x_1	x_4	x_5	x_6
Int. (2)	C_k	1.322×10^7	-2.535×10^{11}	2.389×10^{-1}	-6.986×10^{-2}	Int. (2)	C_k	-5.295×10^6	2.413×10^{10}	-9.739×10^{-2}	4.468×10^{-1}
Frac. (3)	C_k	9.832×10^5	-4.304×10^9	9.883×10^{-1}	1.056	Frac. (3)	C_k	7.653×10^5	-3.000×10^7	9.159×10^{-1}	2.184×10^{-1}
	α_k	0	-0.52	0	-0.30		α_k	0	-1.72	0	-0.52
SVN		x_1	x_4	x_5	x_6	ESP		x_1	x_4	x_5	x_6
Int. (2)	C_k	1.681×10^7	-2.720×10^{10}	1.210×10^{-1}	7.472×10^{-1}	Int. (2)	C_k	-7.682×10^6	1.373×10^{11}	2.265×10^{-2}	-6.055×10^{-1}
Frac. (3)	C_k	9.491×10^5	6.290×10^8	1.159	7.646×10^{-1}	Frac. (3)	C_k	8.059×10^5	2.515×10^9	1.433	-3.610×10^{-1}
	α_k	0	-0.28	0.04	1		α_k	0	-0.88	0	0
SWE		x_1	x_4	x_5	x_6	GBR		x_1	x_4	x_5	x_6
Int. (2)	C_k	2.640×10^6	3.673×10^{10}	3.397×10^{-2}	7.826×10^{-1}	Int. (2)	C_k	4.238×10^6	-5.823×10^{10}	7.558×10^{-2}	1.684
Frac. (3)	C_k	2.654×10^6	3.401×10^9	9.186×10^{-1}	5.846×10^{-1}	Frac. (3)	C_k	1.910×10^6	-3.845×10^{10}	1.829	1.581
	α_k	0	-0.8	0	0		α_k	0	0.5	0	9.04×10^{-5}

Table 4 Economic data for all states of the EU in the period 1970–2016

Table with 47 columns (years 1970-2016) and 47 rows (countries: AUT, BEL, BGR, CYP, DEU, DNK, ESP, EST, FIN, FRA, GBR, GRC, HRV). Each row contains economic data points for each year, including GDP, unemployment, and other indicators.

Economic growth in the European Union modelled with fractional derivatives: first results

Table 5 Economic data for all states of the EU in the period 1970–2016 (cont)

	1970	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016			
BEU	3.18	3.31	3.50	3.55	3.84	4.11	4.24	4.37	4.52	4.62	4.61	4.81	4.96	4.94	5.17	5.43	5.75	6.24	6.57	7.14	7.82	8.35	9.45	10.23	11.34	12.47	13.22	13.96	14.47	15.44	16.33	17.29	17.85	17.16	16.38	16.71	16.71	16.32	16.70	16.80	16.89	16.89	16.82	22.88	24.07					
DEU	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89	6.89				
FRU	8.47	8.63	8.79	8.96	9.13	9.33	9.51	9.69	9.86	10.01	10.15	10.27	10.37	10.45	10.53	10.56	10.60	10.64	10.68	10.71	10.74	10.77	10.80	10.82	10.85	10.89	11.03	11.15	11.29	11.45	11.62	11.80	11.99	12.13	12.27	12.38	12.45	12.64	12.74	12.82	12.85	12.93	13.05	13.18	13.31	13.44	13.58	13.72	13.86	
GRU	6.60	6.64	7.15	7.73	8.81	9.90	10.54	11.21	11.42	12.06	13.32	15.33	16.65	17.05	18.36	21.11	23.29	25.32	26.76	30.48	33.44	38.50	46.18	53.91	61.02	68.65	76.65	85.06	93.86	103.06	112.66	122.66	133.19	144.29	156.00	168.37	181.50	195.40	210.00	225.30	241.30	258.00	275.30	293.20	311.70	330.80	350.50	370.80		
IRL	7.17	7.30	7.57	8.11	8.56	8.98	9.21	9.51	10.07	10.42	10.51	11.02	11.33	11.65	12.02	12.33	12.95	13.21	13.41	13.52	13.41	13.70	14.09	15.37	15.89	16.15	16.30	16.33	16.39	16.45	16.49	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59	16.59
ITA	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	2.94	
NET	5.97	5.98	6.09	6.20	6.45	6.58	6.71	6.84	6.97	7.09	7.20	7.31	7.41	7.52	7.63	7.76	7.89	8.03	8.17	8.32	8.46	8.61	8.75	8.89	9.04	9.19	9.33	9.47	9.61	9.73	9.84	10.03	10.12	10.20	10.38	10.48	10.59	10.71	10.79	10.86	10.94	11.01	11.08	11.16	11.24	11.32	11.40	11.48	11.56	
NLD	1.69	1.58	1.64	1.91	2.09	1.63	1.91	1.80	1.79	1.96	2.18	1.99	1.96	1.88	2.07	2.14	2.22	2.38	2.50	2.61	2.73	2.78	2.73	2.74	2.85	2.76	2.75	2.82	2.92	3.06	3.24	3.31	3.46	3.47	3.54	3.73	3.69	3.12	3.30	3.26	2.78	2.62	2.65	2.72	2.72	2.72	2.72	2.72	2.72	
POL	7.45	8.00	8.08	9.20	9.85	10.02	11.22	12.41	13.76	14.83	13.59	14.52	14.33	14.88	16.02	16.60	16.87	17.57	18.48	20.06	21.30	20.88	22.41	24.42	26.77	30.13	30.63	32.18	33.06	32.74	36.05	36.57	36.10	38.32	39.62	42.88	45.52	44.12	36.15	40.41	42.51	43.50	43.80	44.99	46.95	48.10	49.30	50.50		
PTU	4.51	4.43	4.48	4.36	3.68	3.43	3.27	3.30	3.23	3.06	2.82	2.61	2.47	2.48	2.45	2.69	2.80	2.99	3.18	3.23	3.40	3.36	3.38	3.43	3.41	3.41	3.56	3.74	4.01	4.21	4.21	4.28	4.26	4.32	4.35	4.45	4.72	4.50	4.29	4.40	4.36	4.24	4.33	4.54	4.70	4.89	5.09			
ROU	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59	2.59			
SWE	7.27	7.49	7.70	7.90	8.07	8.21	8.32	8.40	8.46	8.51	8.55	8.60	8.64	8.70	8.76	8.83	8.91	8.99	9.07	9.16	9.24	9.32	9.39	9.45	9.51	9.57	9.63	9.69	9.75	9.83	9.92	10.03	10.15	10.29	10.44	10.59	10.75	10.90	11.05	11.20	11.33	11.39	11.44	11.49	11.55	11.61	11.67	11.73	11.79	11.85
SVK	7.95	8.32	8.14	8.24	6.65	5.41	5.00	3.83	4.99	4.00	4.45	4.10	3.85	3.57	3.37	3.01	3.88	4.52	5.11	5.33	5.41	6.03	5.11	5.40	5.30	5.83	6.61	6.11	6.08	7.84	7.82	7.43	6.88	7.31	7.92	7.52	8.32	8.84	7.67	7.87	8.64	8.49	8.14	8.76	9.02	8.92				
TUR	3.65	3.62	3.62	3.69	3.93	3.99	3.75	2.69	2.68	2.41	2.61	2.50	2.69	2.61	2.27	2.37	2.45	2.52	2.73	3.00	3.05	3.23	3.22	3.20	3.33	3.76	3.85	4.21	4.70	5.22	5.84	6.02	6.08	6.10	6.61	6.81	6.99	7.12	7.68	7.73	7.73	7.68	8.06	8.76	9.59					
UKU	6.22	6.48	7.57	8.45	8.69	8.43	9.21	9.08	9.46	10.20	10.46	10.76	10.65	10.89	11.73	12.26	12.51	12.98	14.14	15.35	16.22	17.29	17.80	18.31	20.11	21.09	22.89	25.11	26.80	29.17	32.89	33.36	34.08	37.11	39.20	41.95	44.31	45.13	41.11	45.44	49.24	50.29	52.43	53.13	53.15	53.15	53.15	53.15		
YUN	2.40	2.50	2.59	2.73	2.83	2.95	3.03	3.11	3.17	3.21	3.19	3.15	3.21	3.31	3.40	3.49	3.56	3.68	3.84	4.01	4.10	4.17	4.23	4.35	4.49	4.65	4.85	5.07	5.32	5.55	5.67	5.69	5.69	5.80	5.93	6.14	6.36	6.47	6.23	6.32	6.42	6.35	6.34	6.43	6.56	6.70	6.83			
FIN	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38		
ESP	8.48	8.65	8.82	8.98	9.13	9.28	9.42	9.55	9.67	9.77	9.87	9.95	10.02	10.09	10.15	10.22	10.29	10.37	10.45	10.52	10.59	10.65	10.71	10.76	10.81	10.86	10.91	11.01	11.14	11.15	11.14	11.15	11.14	11.14	11.15	11.18	11.25	11.35	11.50	11.71	11.78	11.82	11.86	11.86	11.90	11.95	12.00	12.05		
CYU	6.82	6.70	6.31	6.80	6.82	6.23	6.50	6.83	6.89	6.67	6.86	5.96	6.08	6.53	6.90	7.44	7.34	8.00	8.63	8.84	8.88	9.11	8.44	8.93	9.43	10.18	10.97	11.65	12.50	12.66	12.90	12.86	11.89	12.05	12.50	13.35	14.32	14.69	13.10	12.90	13.34	12.52	12.03	12.41	13.18	13.81				
CZE	6.22	6.48	7.57	8.45	8.69	8.43	9.21	9.08	9.46	10.20	10.46	10.76	10.65	10.89	11.73	12.26	12.51	12.98	14.14	15.35	16.22	17.29	17.80	18.31	20.11	21.09	22.89	25.11	26.80	29.17	32.89	33.36	34.08	37.11	39.20	41.95	44.31	45.13	41.11	45.44	49.24	50.29	52.43	53.13	53.15	53.15	53.15			
HRU	2.40	2.50	2.59	2.73	2.83	2.95	3.03	3.11	3.17	3.21	3.19	3.15	3.21	3.31	3.40	3.49	3.56	3.68	3.84	4.01	4.10	4.17	4.23	4.35	4.49	4.65	4.85	5.07	5.32	5.55	5.67	5.69	5.69	5.80	5.93	6.14	6.36	6.47	6.23	6.32	6.42	6.35	6.34	6.43	6.56	6.70	6.83			
EST	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38	3.38			
LTU	6.82	6.70	6.31	6.80	6.82	6.23	6.50	6.83	6.89	6.67	6.86	5.96	6.08	6.53	6.90	7.44	7.34	8.00	8.63	8.84	8.88	9.11	8.44	8.93	9.43	10.18	10.97	11.65	12.50	12.66	12.90	12.86	11.89	12.05	12.50	13.35	14.32	14.69	13.10	12.90	13.34	12.52	12.03	12.41	13.18	13.81				
UKR	6.22	6.48</																																																

With respect the coefficients C (see Fig. 4), the following can be concluded:

- There is some uniformity in coefficient C_1 (Fig. 4a), except for DNK (it is represented in black because is out of range), CZE, and HUN. This may mean that the land area has a similar influence on GDP for each country.
- For C_4 (Fig. 4b), there is less uniformity than in the previous case, but still some, except mainly in the case of DEU, and less of FRA and AUT.
- Although with some exceptions for center and Eastern states, the values of coefficient C_5 seem to be regular (Fig. 4c).
- There is no apparent pattern for C_6 (Fig. 4d) for the whole EU, or even for regions.

To sum up, we consider that an in-depth study is needed in order to obtain more conclusive results concerning economic growth modelling of all states of the EU, since either 1970 or the year of accession to the EU to 2016. The consideration of other variables, such as all those considered in [28, 29], may be required for a better description of GDP.

A. Appendix: Data

The economic data used in this work can be found in Tables 4 and 5. Sources: data for x_1 , x_5 and x_6 can be found in World Bank database [31], whereas data for variable x_4 was obtained for most of EU countries from Lee-Lee dataset [32], except for Croatia, Estonia, Latvia, Lithuania, Slovenia and Slovak Republic; for those six countries, Barro-Lee database [33] was used instead. Since data for x_4 are included in both databases, it should be mentioned that: 1) it was presented every 5 years, so a third-order spline interpolation was used to obtain the data for every year; and 2) it was available for the period 1970–2010, so was extended to 2015 using the Wittgenstein projection from [34] (the last available value of [32] or [33] was extrapolated using the increase rate of the spline interpolation of [34]).

REFERENCES

- [1] Y. Hu and B. Oksendal, “Fractional white noise calculus and applications to finance”, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* 6(1), 2003.
- [2] I. Petrás and I. Podlubny, “State space description of national economies: The V4 countries”, *Computational Statistics & Data Analysis* 52(2), 1223–1233 (2007).
- [3] T. Skovranek, I. Podlubny, and I. Petrás, “Modeling of the national economies in state-space: A fractional calculus approach”, *Economic Modelling* 29(4), 1322–1327 (2012).
- [4] Y. Xu and Z. He, “Synchronization of variable-order fractional financial system via active control method”, *Central European Journal of Physics* 11(6), 824–835 (2013).
- [5] Y. Yue, L. He, and G. Liu, “Modeling and application of a new nonlinear fractional financial model”, *Journal of Applied Mathematics* 2013, 1–9 (2013).
- [6] D.E. Bloom, D. Canning, and J. Sevilla, *Technological diffusion, conditional convergence, and economic growth*, 2002.
- [7] S. Sasia and M. Goaid, “Financial development, ICT diffusion and economic growth: Lessons from MENA region”, *Telecommunications Policy* 37(4–5), 252–261 (2013).
- [8] A. Seck, “International technology diffusion and economic growth: Explaining the spillover benefits to developing countries”, *Structural Change and Economic Dynamics* 23(4), 437–451 (2012).
- [9] R.L. Magin, *Fractional Calculus in Bioengineering*, Begell House, 2004.
- [10] B. Baeumer and M.M. Meerschaert, “Fractional diffusion with two time scales”, *Physica A: Statistical Mechanics and its Applications* 373, 237–251 (2007).
- [11] J. Blackledge, “Application of the fractal market hypothesis for modelling macroeconomic time series”, *ISAST Transactions on Electronics and Signal Processing* 2(1), 89–110 (2008).
- [12] J. Blackledge, “Application of the fractional diffusion equation for predicting market behaviour”, *International Journal of Applied Mathematics* 40(3), 130–158 (2010).
- [13] M. Boleantu, “Fractional dynamical systems and applications in economy”, *Differential Geometry – Dynamical Systems* 10, 62–70 (2008).
- [14] Á. Cartea and D. del Castillo-Negrete, “Fractional diffusion models of option prices in markets with jumps”, *Physica A: Statistical Mechanics and its Applications* 374(2), 749–763 (2007).
- [15] S.A. David, J.A.T. Machado, D.D. Quintino, and J.M. Baltazar, “Partial chaos suppression in a fractional order macroeconomic model”, *Mathematics and Computers in Simulation* 122, 55–68 (2016).
- [16] R. Gorenflo, F. Mainardi, E. Scalas, and M. Raberto, *Mathematical Finance Trends in Mathematics*, chapter Fractional Calculus and Continuous-Time Finance III: the Diffusion Limit, pages 171–180, Birkhäuser Basel, 2001.
- [17] N. Laskin, “Fractional market dynamics”, *Physica A: Statistical Mechanics and its Applications* 287, 482–492 (2000).
- [18] F. Mainardi, M. Raberto, R. Gorenflo, and E. Scalas, “Fractional calculus and continuous-time finance II: The waiting-time distribution”, *Physica A: Statistical Mechanics and its Applications* 287, 468–481 (2000).
- [19] O. Marom and E. Momoniat, “A comparison of numerical solutions of fractional diffusion models in finance”, *Nonlinear Analysis: Real World Application* 10, 3435–3442 (2009).
- [20] M.M. Meerschaert and E. Scalas, “Coupled continuous time random walks in finance”, *Physica A: Statistical Mechanics and its Applications* 370, 114–118 (2006).
- [21] M.M. Meerschaert and A. Sikorski, *Stochastic Models for Fractional Calculus*, volume 43 of *Studies in Mathematics*, Walter de Gruyter & Co, 2012.
- [22] E. Scalas, “The application of continuous-time random walks in finance and economics”, *Physica A: Statistical Mechanics and its Applications* 362, 225–239 (2006).
- [23] E. Scalas, R. Gorenflo, and F. Mainardi, “Fractional calculus and continuous-time finance”, *Physica A: Statistical Mechanics and its Applications* 284(1–4), 376–384 (2000).
- [24] V.V. Tarasova and V.E. Tarasov, “Economic interpretation of fractional derivatives”, *Progress in Fractional Differentiation and Applications* 1, 1–6 (2017).
- [25] V.E. Tarasov, “Long and short memory in economics: Fractional-order difference and differentiation”, *IRA – International Journal of Management and Social Sciences* 5(2), 327–334 (2016).

Economic growth in the European Union modelled with fractional derivatives: first results

- [26] J.A.T. Machado and M.E. Mata, “Pseudo phase plane and fractional calculus modeling of western global economic downturn”, *Communications in Nonlinear Science and Numerical Simulation* 22(1–3), 396–406 (2015).
- [27] J.A. Tenreiro Machado, M.E. Mata, and A.M. Lopes, “Fractional state space analysis of economic systems”, *Entropy* 17, 5402–5421 (2015).
- [28] I. Tejado, D. Valério, E. Pérez, and N. Valério, “Fractional calculus in economic growth modelling: The economies of France and Italy”, in *International Conference on Fractional Differentiation and its Applications*, 2016.
- [29] I. Tejado, D. Valério, E. Pérez, and N. Valério, “Fractional calculus in economic growth modelling. The Spanish and Portuguese cases”, *International Journal of Dynamics and Control* 5(1), 208–222 (2017).
- [30] D. Valério and J. Sá da Costa, *An Introduction to Fractional Control*, IET, Stevenage, 2013, ISBN 978-1-84919-545-4.
- [31] World Bank, World development indicators, 2017, Last Updated: 07/20/2017. Access Date: 07/25/2017.
- [32] J.-W. Lee and H. Lee, “Lee and Lee long-run education dataset”, 2016, Last Updated: 01/01/2016. Access Date: 06/26/2017.
- [33] World Bank, Education statistics – All indicators, 2017, Last Updated: 05/25/2017. Access date: 06/29/2017.
- [34] Wittgenstein Centre for Demography and Global Human Capital, Wittgenstein centre data explorer version 1.2, 2015, Access Date: 07/05/2017.