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# Measuring the Natural Rates of Interest in Germany and Italy

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**Abstract**

In this paper a semi-structural econometric model is implemented in order to estimate the natural rates of interest in two large economies of the Euro Area: Germany an Italy. The estimates suggest that after the financial crisis of 2007–2008 a decrease of the growth rate of potential output and the corresponding natural rate of interest was greater in Italy than in Germany which could have had important implications for the effectiveness of a common monetary policy. Unlike in other studies, it is found that the monetary policy stance was less expansionary in Italy as compared to Germany for the whole after-crisis period.


**Keywords:** natural rate of interest, potential output, Euro Area, state-space model, Kalman filter

**JEL Classification**: C32, C51, E43, E52

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# 1 Introduction

The Great Recession and the accompanying decline of interest rates have brought the concept of the natural (neutral, equilibrium) rate of interest to the core of macroeconomic policy debate. The natural rate of interest can be defined as a real rate of interest consistent with the growth rate of real output equal to its potential growth rate in the absence of transitory shocks to demand (see, e.g., Williams 2003). Policy makers at both sides of the Atlantic Ocean use this concept to explicate their decisions (see Yellen 2015 and Constancio 2016).

Laubach and Williams (2003, 2016) proposed a semi-structural model and a procedure for joint estimation of the natural interest rate and the growth rate of potential output in the United States. Their estimates are often used in the discussion of the monetary policy, and some modifications of their methodology are applied to measure the natural rate of interest in the Euro Area (Mesonnier and Renne 2007, Garnier and Wihlhelmsen 2009).

In a recent paper, Holston, Laubach and Williams (2017a) apply the Laubach and Williams (2003) methodology to measure the natural rates of interest and output growth in four advanced economies: the United States, Canada, the United Kingdom and the Euro Area. They use the aggregated Euro Area data from 1972 to 2016 and construct a single natural rate of interest for the conglomerate of heterogeneous economies which may have different natural rates of interest and growth rates of potential output.

In this paper, the differences in the natural rates of interest are evaluated for two large economies of the Euro Area: Germany and Italy. The former is considered as an example of the economy which has returned to a sustainable growth path after the recession of 2008-2009 and the latter as an example of the economy which has undergone an extended period of stagnation. The differences are evaluated using the Holston, Laubach and Williams (2017a) model with three major modifications.

Firstly, the procedure developed in Andrews and Chen (1994) is applied to estimate the persistence of the process describing discrepancies between the natural rate of interest and the growth rate of potential output, which is modeled as a random walk in Holston, Laubach and Williams (2017a). The random walk assumption allows for arbitrarily large differences between these variables. However, the very definition of the natural rate of interest and the growth rate of potential output requires them to be in a stable long-term relation. It is a necessary condition for identification of these unobservable variables measured only indirectly, in terms of other (observable) variables. Although the approximately median-unbiased estimator of Andrews and Chen (1994) includes the unit root in the parameter space, the obtained estimates of the persistence parameter are smaller than one implying stationarity of the discrepancy process, which is consistent with a stable relation between the natural rate of interest and the growth rate of potential output.

Secondly, a diffuse initialization of non-stationary unobservable variables is implemented in estimation and filtering. Holston, Laubach and Williams (2017a) use in-sample information in order to initialize non-stationary unobservable variables in their model. This approach violates the assumptions of the Kalman filter (see Anderson and Moore 1979 and Hamilton 1994). Additionally, in a non-stationary environment the effect of specific initial conditions is not forgotten: it is transferred to the end of the series causing biased estimation of parameters and state variables (inter alia, the natural rate of interest and potential output). In this paper, a variant of the exact diffuse Kalman filter, which is described in Durbin and Koopman (2012), is implemented and the parameter estimation is based on the marginal likelihood function described in Francke, Koopman and de Vos (2010). Conditioning on specific values of non-stationary states is avoided, although a few initial observations which are necessary for the filter convergence, are lost.

Thirdly, in order to control for international demand shocks, the IS equation of the model is augmented by a measure of foreign output gap which is computed using a weighted sum of outputs in major trading partners of the Euro Area.

These methodological innovations allow modeling the growth rate of potential output and the natural rate of interest as non-stationary variables for which a stable long-run equilibrium relation exists. The estimated model makes it possible to explain persistent changes in the output growth and the natural rate of interest and demonstrate differences between their dynamics in Germany and Italy: both the estimated growth rate of potential output and the estimated natural rate of interest have decreased by a larger amount in Italy compared to Germany in the aftermath of the financial crisis of 2007-2008. Conditionally on these estimates, it can be suggested that an effective monetary policy stance was less expansionary in Italy as compared to Germany, which could have contributed to a prolonged stagnation in Italy.

The paper is organized as follows. The next section provides a literature review. A detailed description of the methodology and data is given in Section 3. The results are described in Section 4 and the conclusions are provided in Section 5.

## 2 Literature Review

Theoretical concept of the natural rate of interest which originated with Knut Wicksell (1898), has been recently popularized in empirical research following the publication of Laubach and Williams (2003) in which a semi-structural model was applied to measure the natural rate of interest in the US. Some modifications of this model were estimated for the Euro Area (Mèsonnier and Renne 2007, Garnier and Wilhelmsen 2009).

The Great Recession and the decline of interest rates have brought this model to the attention of policy makers with representatives of the Federal Reserve (Yellen 2015) and the ECB (Constancio 2016) directly citing Laubach and Williams (2003, 2016) in their public speeches. In these speeches, persistent discrepancies between the estimates of potential output growth and the natural interest rate, labeled as “headwinds” by Yellen (2015), are explained by various structural economic factors which have emerged in the aftermath of the financial crisis of 2007-2008.

The concept of the natural rate of interest has also been in the center of the debate concerning the possibility for advanced industrial economies of entering secular stagnation, a prolonged period of low growth caused by the failure of the capital market to achieve equilibrium given the low natural rates of interest. While proponents of this hypothesis (Summers 2014) use the estimates of Laubach and Williams (2003, 2016) to support it, its opponents argue that the estimates of natural rate are very uncertain (see Hamilton et al 2015, Taylor and Wieland 2016, Beyer and Wieland 2017).

Some critics contend that the results obtained by Laubach and Williams (2016) and Holston, Laubach and Williams (2017) may be explained by the choice of a specific model and estimation procedure. The potential output and the natural rate of interest are unobservable variables identified jointly. Their identification depends on a priori assumptions concerning structural relations between these variables. Lewis and Vazquez-Grande (2017) consider alternative specifications of the Laubach and Williams (2003) model and demonstrate that the results obtained by Holston, Laubach and Williams (2017a) for the US depend on the assumption that the non-growth component of the natural rate of interest ("headwinds") is modeled as a random walk.

In Holston, Laubach and Williams (2017a) the filtered natural rate of interest and the growth rate of potential output coincide in the beginning of the sample as a result of tight initial conditions based on in-sample information and diverge at the end of the sample because the "headwinds" process is modeled as a random walk allowing for arbitrary discrepancies between the two rates. The combination of tight initial conditions and persistent "headwinds" produces an estimate of potential growth rate equal to about one percent with near zero output gap and the negative natural rate in 2016 (see Figure 3 in Holston, Laubach and Williams 2017a). It would imply that by the beginning of 2016 the economy of the Euro Area has returned to a sustainable growth path with nearly full employment albeit with a negative level of the natural rate of interest. These estimates are in odds with the data concerning a few vulnerable Euro-Area economies characterized by a near-zero growth and endemic unemployment.

Fries et al (2018) estimate the national natural rates of interest for the four largest economies of the Euro Area (Germany, France, Italy and Spain) using monthly series over 1999-2016. They use a joint model of the four economies and their interactions. Their approach produces estimates of the natural interest rates and real interest rate gaps which diverge across the economies over the years 2010-2013. This is interpreted as a failure for the common monetary policy to have the same effects across the Euro Area economies. In particular, Fries et al (2018) argue that the single monetary policy, being neutral in Germany and France, was strongly dis-inflationary in Italy and Spain in 2011-2012. However, they find that the natural rates and interest rate gaps converged over 2014-2016. Their results may depend on the assumption that for each economy the dynamics of both natural rate of interest and growth rate of potential output is driven by the same stationary (albeit persistent) process. The "headwinds" are modeled as a white noise. These settings exclude the possibility that the Great Recession could have produced a permanent effect onto the potential output (no possibility of secular stagnation): persistent changes in the output growth rate are attributed to demand shocks.

Belke and Klose (2017) use the Laubach and Williams (2003) model in order to estimate the natural rates of interest in 12 economies of the Euro Area and find no evidence of secular stagnation in any economy besides Greece. They model the "headwinds" as a random walk. However, the estimated variance of this component collapses to nearly zero in their study (see Table A1 in Belke and Klose 2017), which implies that the natural rate of interest is proportional to the estimated growth rate of potential output.

In this paper, quarterly time series over 1978-2017 are used in order to estimate the natural interest rates and the growth rates of potential output in Germany and Italy. The assumption that these rates have a random walk component (with a small disturbance variance) is maintained: it is necessary in order to explain long-run changes in the growth rates of output. However, the persistence of the non-growth component of the natural interest rate ("headwinds") is estimated using the approximately median-unbiased estimator of Andrews and Chen (1994). The obtained estimates are smaller than unity which is consistent with "headwinds" being a stationary process. In these settings, the natural rate of interest is anchored to the growth rate of potential output: these rates are cointegrated. Nevertheless, the model encompasses a possibility of secular stagnation given the non-stationarity of the natural interest rate and the growth rate of potential output.

# 3 Methodology and Data

## 3.1 Model Specification

A modification of the semi-structural econometric model considered in Holston, Laubach and Williams (2017a) is given by the system of equations:

Measured Output: $$y_t = y_t^* + \tilde{y}_t, \tag{1}$$

Potential Output: $$y_t^* = y_{t-1}^* + g_t + \varepsilon_{y^*,t},\ \varepsilon_{y^*,t} \sim N\left(0, \sigma_{y^*}^2\right), \tag{2}$$

IS Equation: $$\begin{aligned}\tilde{y}_t \ &= \sum_{k=1}^{2} a_{y,k}\tilde{y}_{t-k} + \sum_{i=1}^{2} a_{x,i}\tilde{x}_{t-i} + \frac{a_r}{2}\sum_{j=1}^{2}\left(r_{t-j} - r_{t-j}^*\right) + \\ &+\varepsilon_{\tilde{y},t},\ \ \varepsilon_{\tilde{y},t} \sim N(0, \sigma_{\tilde{y}}^2),\end{aligned} \tag{3}$$

Phillips Curve: $$\begin{aligned}\pi_t &= b_\pi \pi_{t-1} + (1-b_\pi)\bar{\pi}_{t-2,4} + b_y \tilde{y}_{t-1} + \varepsilon_{\pi,t}, \\ &\varepsilon_{\pi,t} \sim N\left(0, \sigma_\pi^2\right),\end{aligned} \tag{4}$$

Potential Growth: $$g_t = g_{t-1} + \varepsilon_{g,t},\ \varepsilon_{g,t} \sim N\left(0, \lambda_g^2 \sigma_{y^*}^2\right), \tag{5}$$

"Headwinds": $$z_t = \rho_z z_{t-1} + \varepsilon_{z,t},\ \varepsilon_{z,t} \sim N\left(0, 2\lambda_z^2 \sigma_{\tilde{y}}^2 / a_r^2\right), \tag{6}$$

Natural Rate: $$r_t^* = g_t^{(a)} + z_t, \tag{7}$$

where $y_t$ is the logarithm of measured quarterly output in period $t$, $y_t = 100\log(Y_t)$; $y_t^*$ and $\tilde{y}_t$ are (unobservable) domestic potential output and output gap; $\tilde{x}_t$ is the foreign output gap; $g_t$ is the quarterly growth rate of potential output; $g_t^{(a)}$ is the annualized growth rate of potential output, $g_t^{(a)} = 4g_t$; $\pi_t$ is the annualized inflation rate in period $t$ and $\bar{\pi}_{t-2,4}$ is the average annualized inflation rate over periods $t-2$, $t-3$, and $t-4$ ($\bar{\pi}_{t-2,4} = \frac{\pi_{t-2}+\pi_{t-3}+\pi_{t-4}}{3}$); $r_t$ is the measured three-month real rate of interest, $r_t^*$ is the unobservable natural rate of interest and $(r_t - r_t^*)$ is the real rate gap; $z_t$ is the non-growth component of the natural rate of interest ("headwinds"); $\varepsilon_{y^*,t}$, $\varepsilon_{g,t}$, $\varepsilon_{\pi,t}$, $\varepsilon_{\tilde{y},t}$, and $\varepsilon_{z,t}$ are shocks which are assumed to be uncorrelated over time and across variables.

In Holston, Laubach and Williams (2017a) the persistence parameter $\rho_z$ in equation (6) is equal to one which means that the process $\{z_t\}$ is a random walk and as follows from equation (7) the the natural rate of interest $r_t^*$ and the growth rate of potential output $g_t^{(a)}$ can diverge. For values of $\rho_z$ smaller than one, the process $\{z_t\}$ is stationary and the natural rate of interest $r_t^*$ is cointegrated with the growth rate of potential output $g_t$. In this paper, the parameter $\rho_z$ is estimated using the approximate median-unbiased estimator of Andrews and Chen (1994) with the parameter space given by the interval $(-1, 1]$.

### 3.2 Three-Stage Estimation

Holston, Laubach and Williams (2017a) use a three-stage estimation procedure: the first two stages serve to estimate parameters $\sigma_g^2 = Var(\varepsilon_{g,t})$, $\sigma_z^2 = Var(\varepsilon_{z,t})$ specifying unobservable components $g_t$ and $z_t$. These parameters cannot be identified in the complete model (1)-(7), which causes the "pileup problem" discussed in Stock (1994). For this reason, partial models are estimated at the first two stages and the median-unbiased estimation is applied in the partial models to obtain estimates of $\sigma_g^2$ and $\sigma_z^2$. Holston, Laubach and Williams (2017a) apply the procedure developed in Stock and Watson (1998) in order to estimate ratios $\lambda_g = \frac{\sigma_g}{\sigma_{y^*}}$ and $\lambda_z = \frac{a_r \sigma_z}{\sqrt{2}\sigma_{\tilde{y}}}$.

A modified three-stage estimation is implemented in this paper. Following Holston, Laubach and Williams (2017a), the ratios $\lambda_g$ and $\lambda_z$ are estimated using the Stock and Watson (1998) procedure. Additionally, the parameter $\rho_z$ is estimated using the procedure developed in Andrews and Chen (1994). The complete model (1)-(7) is conditioned on these estimates.

At the first stage, interest rates are excluded from the model and the growth rate of potential output is constant: $g_t = g$. The following subsystem is estimated by maximum likelihood:

$$y_t = y_t^* + \tilde{y}_t, \tag{1'}$$

$$y_t^* = y_{t-1}^* + g + \varepsilon_{y^*,t},\ \varepsilon_{y^*,t} \sim N(0, \sigma_{y^*}^2), \tag{2'}$$

$$\tilde{y}_t = \sum_{k=1}^{2} a_{y,k}\tilde{y}_{t-k} + \sum_{i=1}^{2} a_{x,i}\tilde{x}_{t-i} + \varepsilon_{\tilde{y},t},\ \varepsilon_{\tilde{y},t} \sim N(0, \sigma_{\tilde{y}}^2), \tag{3'}$$

$$\pi_t = b_\pi \pi_{t-1} + (1 - b_\pi)\bar{\pi}_{t-2,4} + b_y \tilde{y}_{t-1} + \varepsilon_{\pi,t},\ \varepsilon_{\pi,t} \sim N(0, \sigma_\pi^2). \tag{4'}$$

An estimate of the ratio $\lambda_g = \sigma_g/\sigma_{y^*}$ is obtained by applying the median-unbiased estimator in the equation $\Delta y_{t|T}^* = g_t + \varepsilon_{y^*,t}$ where $\Delta y_{t|T}^*$ is the smoothed growth rate of potential output. The ratio $\lambda_g$ is used at the second stage where the following model is estimated:

$$y_t = y_t^* + \tilde{y}_t, \tag{1''}$$

$$y_t^* = y_{t-1}^* + g_t + \varepsilon_{y^*,t},\ \varepsilon_{y^*,t} \sim N(0, \sigma_{y^*}^2), \tag{2''}$$

$$\begin{aligned}\tilde{y}_t \ &= a_0 + \textstyle\sum_{k=1}^{2} a_{y,k}\tilde{y}_{t-k} + \sum_{i=1}^{2} a_{x,i}\tilde{x}_{t-i} + \frac{a_r}{2}\sum_{j=1}^{2}(r_{t-j} - g_{t-j}) + \varepsilon_{\tilde{y},t},\\ &\varepsilon_{\tilde{y},t} \sim N(0, \sigma_{\tilde{y}}^2),\end{aligned} \tag{3''}$$

$$\pi_t = b_\pi \pi_{t-1} + (1 - b_\pi)\bar{\pi}_{t-2,4} + b_y \tilde{y}_{t-1} + \varepsilon_{\pi,t},\ \varepsilon_{\pi,t} \sim i.i.d.N(0, \sigma_\pi^2), \tag{4''}$$

$$g_t = g_{t-1} + \varepsilon_{g,t},\ \varepsilon_{g,t} \sim N(0, \lambda_g^2\sigma_{y^*}^2). \tag{5''}$$

The measured real rate of interest is included in equation (3″) describing the output gap. However, the "headwinds" process and the natural rate of interest are excluded from the model.

For the smoothed output gap series $\{\tilde{y}_{t|T}\}_{t=1}^{T}$ obtained at the second stage, a sequence of equations is estimated by ordinary least squares:

$$\tilde{y}_{t|T} = a_0^{(s)} + a_{y,1}^{(s)}\tilde{y}_{t-1|T} + a_{y,2}^{(s)}\tilde{y}_{t-2|T} + \frac{a_r^{(s)}}{2}\sum_{j=1}^{2}(r_{t-j} - g_{t-j|T}) + \zeta_s \mathbb{1}_{\{t>s\}} + \varepsilon_{\tilde{y},t}^{(s)}, \tag{8}$$

where $s = \tau + 1, \ldots, T - \tau$, $\mathbb{1}_{\{t>s\}}$ is an indicator function which is equal to zero for $t \leq s$ and one for $t > s$. The estimated series $\{\hat{\zeta}_t\}_{t=\tau+1}^{T-\tau}$ is used to calibrate the parameters of the unobservable "headwinds" process $\{z_t\}$. Following Holston, Laubach and Williams (2017a), the asymptotically median-unbiased estimation of Stock and Watson (1998) is implemented to obtain an estimate of the ratio $\lambda_z = a_r\sigma_z/\sqrt{2}\sigma_{\tilde{y}}$. The approximately median-unbiased estimation of Andrews and Chen (1994) is implemented in order to obtain an estimate of $\rho_z$ in equation

$\hat{\zeta}_t = \rho_z \hat{\zeta}_{t-1} + \sum_{i=1}^{m} \varphi_i \Delta \hat{\zeta}_{t-i} + \hat{\varepsilon}_{\zeta,t}$, where $m$ is selected using the Akaike Information Criterion. The procedure developed in Andrews and Chen (1994) produces an approximately median-unbiased estimator of the persistence parameter with the value of estimate in the interval $(-1, 1]$.

At the third stage, the complete model (1)-(7) is estimated by maximum likelihood.

### 3.3 State-Space Form and Diffuse Initialization

At each stage of the estimation, a state-space representation of the corresponding model is used:

$$\mathbf{y}_t = A'\mathbf{x}_t + H'\xi_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim i.i.d.N(0, R), \tag{9}$$

$$\xi_{t+1} = F\xi_t + \mathbf{v}_{t+1}, \ \mathbf{v}_{t+1} \sim i.i.d.N(0, Q), \tag{10}$$

where $\mathbf{y}_t$ is an $n \times 1$ vector of observable endogenous variables, $\mathbf{x}_t$ is an $k \times 1$ vector of observable predetermined variables, $\xi_t$ is an $r \times 1$ vector of non-observable state variables, $\mathbf{w}_t$ is a vector of measurement errors, and $\mathbf{v}_{t+1}$ is a vector of state disturbances. (For definitions of state-space matrices at each stage see the Appendix). The Kalman filter is applied to compute values of the likelihood function and unobserved states. The implementation of the Kalman filter requires a specification of the initial states $\hat{\xi}_{1|0} = E_0(\xi_1)$ and their covariance matrix $P_{1|0} = Var_0(\xi_1)$. For a stationary system the initial conditions can be computed using unconditional moments of state processes. However, the model of natural interest rate is non-stationary: state variables do not have proper unconditional distributions.

Holston, Laubach and Williams (2017a, 2017b) use stochastic initial conditions and estimate the expected value and the covariance matrix of initial states using in-sample information, which means that both initial states and their covariance matrix are functions of the data. However, the Kalman filter is based on the assumption that initial conditions do not depend on the data (see Anderson and Moore 1979 and Hamilton 1994).

For a non-stationary system the initial states $\xi_{1|0}$ can be estimated using the same data, which are used in the Kalman filtering, if the initial states are treated as fixed parameters ($P_{1|0} = 0$). If the initial states are treated as stochastic, then the choice of the mean and variance of initial states should be based on prior information, or the model can be estimated using the diffuse Kalman filter described in Durbin and Koopman (2012).

For the state-space model (9)-(10) the initial conditions in Durbin and Koopman (2012) can be rewritten as

$$\xi_{1|0} = \alpha + B\beta + C\varepsilon, \ \varepsilon \sim N(0, R_0) \tag{11}$$

where $\alpha$ is an $r \times 1$ fixed (known) vector, $\beta$ is a $q \times 1$ vector of unknown quantities, the $r \times q$ matrix $B$ and the $r \times (r - q)$ matrix $C$ are selection matrices, that is, they consist of columns of the identity matrix $I_r$ and $B'C = 0$. The matrix $R_0$ is assumed to be positive definite and known.

The vector $\beta$ can be treated as a fixed vector of unknown parameters or as a vector of random variables with infinite variances. In the first case it can be estimated by maximum likelihood. In the second case it can be assumed that $\beta \sim N(0, \kappa I_q)$, where $\kappa \to \infty$, and the Kalman filter can be initialized by setting

$$\xi_{1|0} = \alpha \text{ and } P_{1|0} = \kappa P_\infty + P_*, \tag{12}$$

where $P_\infty = BB'$ and $P_* = R_0$. Since $B$ consists of columns of $I_r$, it follows that $P_\infty$ is an $r \times r$ diagonal matrix with $q$ diagonal elements equal to one and other elements equal to zero. Also, Durbin and Koopman (2012) argue that, without loss of generality, when a diagonal element of $P_\infty$ is non-zero the corresponding element of $\alpha$ can be set to zero.

In this paper, the diffuse Kalman filter described in Durbin and Koopman (2012), is implemented to initialize non-stationary state variables at each stage of estimation. This technique is based on the expansion of matrix products as power series in $\kappa^{-1}$ and letting $\kappa \to \infty$ to obtain the dominant term. At each step of the filter recursions, the covariance matrix $P_{t|t-1}$ can be decomposed as

$$P_{t|t-1} = \kappa P_{\infty,t|t-1} + P_{*,t|t-1} + O(\kappa^{-1}), \quad t = 2, \ldots, T, \tag{13}$$

where $P_{\infty,t|t-1}$ and $P_{*,t|t-1}$ do not depend on $\kappa$. Durbin and Koopman (2012) show that after a few initial recursions the diffuse component collapses to zero: $P_{\infty,t|t-1} = 0$ for any $t > d$ where $d$ is a positive integer which is small relative to $T$. The consequence is that the usual Kalman filter applies for $t = d+1, \ldots, T$ with $P_{t|t-1} = P_{*,t|t-1}$.

The marginal log-likelihood function is evaluated by the diffuse Kalman filter together with an additional recursion described in Francke, Koopman and de Vos (2010). This function and the resulting estimator do not depend on initial states.

Figures shown in Section 4 exclude twelve initial observations of state variables which were necessary for the convergence of the diffuse filter. Following Holston, Laubach and Williams (2017a), the standard errors of state variables are computed using the procedure developed in Hamilton (1986).

## 3.4 Data Description and Preliminary Analysis

The data are collected from the OECD.Stat database. The real quarterly seasonally-adjusted GDP series span from 1977:2 to 2017:1. The quarterly time series of price indices span from 1968:2 to 2017:1. The nominal 3-month interest rates are available from 1978:4 to 2017:1. For Germany all time series include estimates based on the West-German data before 1991.

A measure of global output is obtained from the Area-wide Model Database of the Euro Area Business Cycle Network. The global output is computed as a weighted sum of the GDPs of the main trading partners of the Euro Area (for more details see EABCN 2017). For the purpose of consistency with the filtering approach implemented in Laubach and Williams (2003) and Holston, Laubach and Williams (2017a), a measure of global output gap is computed using the univariate latent variable model described in Watson (1986), of which the model described in Laubach and Williams (2003) is a multivariate extension.

Two consumer price indices are considered: the CPI excluding food and energy prices and the total CPI including all items. Since food and energy prices are most sensitive to international price fluctuations, the measure excluding these prices is considered as a proxy for domestic price index. (Holston, Laubach and Williams (2017a) also use the CPI excluding food and energy prices as a proxy for domestic prices). The real interest rate determined using domestic inflation, is not directly affected by international price fluctuations and, under some additional assumptions (see Galí 2008), the Phillips curve describing domestic inflation in a small open economy is isomorphic to the Phillips curve describing inflation in a closed economy. The inflation rate based on the CPI excluding food and energy prices is used in the baseline model. For robustness analysis, the model is also estimated using the total CPI.

In order to compute the real rate of interest $r_t$, a measure of inflation expectations $\pi_t^e$ is produced as a one-quarter-ahead forecast of the rolling autoregressive model with a window size of 40 quarters. The first forecast which is used in the model, is computed in 1978:4 using the data from 1969:1 to 1978:4. The last forecast is computed in 2017:1 using the data from 2008:2 to 2017:1.

Since inflation is modeled by an integrated process, the approximate mean-unbiased estimator of Roy and Fuller (2001) is applied with the number of lags selected on the basis of the Akaike Information Criterion. This model is not intended to be a model of expectation formation, but a tool for inferring inflation expectations assuming that these expectations are unbiased and the inflation can be described by an integrated process (see a discussion in Burmeister, Wall and Hamilton 1986). The real rate of interest $r_t$ is computed by subtracting inflation forecasts $\hat{\pi}_t^e$ from the nominal rate of interest $i_t$: $r_t = i_t - \pi_t^e$.

As an alternative, a simple moving average of inflation over previous four quarters is considered (as in Holston, Laubach and Williams 2017a) and the results for both measures are reported in the next section.

The initial measure of output gap is obtained by computing least-squares deviations of the log-output from the quadratic deterministic trend and the initial estimates of parameters in equations (3)-(4), (3′)-(4′) and (3″)-(4″) are obtained by the ordinary least-squares regressions.

# 4 Results

Parameter estimates for the baseline model and the alternatives which are used for robustness analysis, are reported in Table 1. It can be seen that for both economies the persistence of the "headwinds" $z_t$ which is obtained using the approximately

median-unbiased estimator, is smaller than one in all models. The estimated value of $\rho_z$ is smaller for Germany than for Italy; however, the estimated value of the standard deviation $\sigma_z$ is smaller for Italy: discrepancies between the natural rate of interest and the growth rate of potential output are less persistent in Germany, but shocks causing these discrepancies, have greater variance.
The lags of foreign output gap have a positive and significant effect on the output gap both in Germany and Italy: assuming 5-percent level of significance zero restrictions on the corresponding coefficients are jointly rejected.
The estimated slope coefficient $a_r$ measuring the effect of the lagged real rate gap $(r_t - \hat{r}_{t|t})$ onto the output gap, has an expected sign (negative) and it is significantly smaller than zero assuming 5-percent level of significance (for the baseline model as well as for the alternatives). The absolute value of the estimated $a_r$ is greater for Germany indicating a stronger response of the output gap to the real rate gap. The slope parameter $b_y$ measuring the effect of the lagged output gap onto the inflation rate, is significantly greater than zero assuming 5-percent level of significance. These estimates suggest that both the output gap and the real rate gap are well identified.

Table 1: Parameter Estimates

| | Baseline Model | | Moving-Average Expectations | | All-Item CPI | |
|---|---|---|---|---|---|---|
| | Germany | Italy | Germany | Italy | Germany | Italy |
| $\lambda_g$ | 0.057 | 0.081 | 0.057 | 0.081 | 0.059 | 0.081 |
| $\lambda_z$ | 0.025 | 0.017 | 0.041 | 0.021 | 0.019 | 0.002 |
| $\rho_z$ | 0.855 | 0.954 | 0.891 | 0.952 | 0.894 | 0.960 |
| $a_{y,1} + a_{y,2}$ | 0.935 | 0.909 | 0.931 | 0.919 | 0.945 | 0.900 |
| $a_{x,1} + a_{x,2}$ | 0.258 | 0.264 | 0.294 | 0.251 | 0.270 | 0.291 |
| (p-value) | (0.032) | (0.000) | (0.027) | (0.002) | (0.036) | (0.020) |
| $a_r$ | -0.070 | -0.040 | -0.063 | -0.032 | -0.078 | -0.045 |
| (p-value) | (0.036) | (0.000) | (0.058) | (0.000) | (0.050) | (0.000) |
| $b_y$ | 0.081 | 0.082 | 0.076 | 0.089 | 0.058 | 0.081 |
| (p-value) | (0.028) | (0.000) | (0.026) | (0.000) | (0.050) | (0.005) |
| $b_\pi$ | 0.466 | 0.731 | 0.430 | 0.723 | 0.417 | 0.740 |
| $\sigma_{y^*}$ | 0.403 | 0.454 | 0.371 | 0.468 | 0.291 | 0.449 |
| $\sigma_{\tilde{y}}$ | 0.762 | 0.349 | 0.773 | 0.338 | 0.803 | 0.351 |
| $\sigma_\pi$ | 1.187 | 1.259 | 1.162 | 1.248 | 1.444 | 1.434 |
| $\sigma_{g^{(a)}}$ | 0.092 | 0.147 | 0.085 | 0.145 | 0.048 | 0.113 |
| $\sigma_z$ | 0.375 | 0.120 | 0.723 | 0.170 | 0.272 | 0.098 |
| $\sigma_{r^*} = \sqrt{\sigma^2_{g^{(a)}} + \sigma^2_z}$ | 0.386 | 0.190 | 0.727 | 0.223 | 0.276 | 0.192 |
| MSE $(\hat{r}^*)$ | 0.972 | 0.876 | 1.553 | 1.121 | 0.659 | 0.411 |
| Log-Likelihood | -635.464 | -592.829 | -635.822 | -593.610 | -667.048 | -608.733 |

Notes: p-values for testing zero restrictions on corresponding parameters are in parenthesis

The mean standard errors (MSE) of the estimated natural rates of interest reported in Table 1 are high: it is a pertaining problem for the empirical models of the natural rate (see Holston, Laubach and Williams 2017a or Fries *et al.* 2018). For this reason, any statement about point estimates of the natural rates should be taken with precaution. However, the comparative cross-country analysis based on the estimated trends can be revealing.

Figures 1 and 2 show the dynamics of the smoothed potential growth rates $\hat{g}^{(a)}_{t|T}$ and the smoothed natural interest rates $\hat{r}^{*}_{t|T}$ obtained using the baseline model together with 68 percent confidence intervals (plus/minus one standard error) for the estimated natural interest rates. Consistently with other research concerning advanced industrial economies (see Garnier and Wilhelmsen 2009, and Holston, Laubach and Williams 2017a), the growth rate of potential output decreases over 1982-2017 both in Germany and Italy. However, the estimated decrease in the potential growth rate is much larger in Italy (from about 3 percents annually in 1982 to just below zero in 2017) than in Germany (from about 2.3 percent in 1982 to 1.1 percent in 2017). The negative values of the growth rate attained in Italy in 2009-2017, are consistent with the hypothesis of secular stagnation.

The lower bound of the confidence band for the natural rate of interest in Germany is above zero over the full sample. However, the confidence band for the natural rate of interest in Italy includes zero since 2005: the estimated natural rate of interest in Italy is not significantly different from zero since 2005 with confidence probability of 68 percent.

Figure 1: Germany, Smoothed Trend Growth ($\hat{g}^{(a)}_{t|T}$) and Natural Rate ($\hat{r}^{*}_{t|T}$)

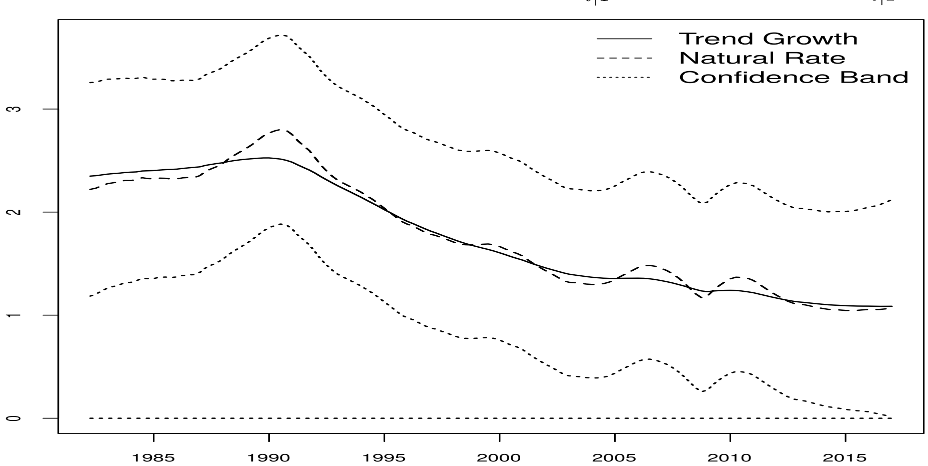
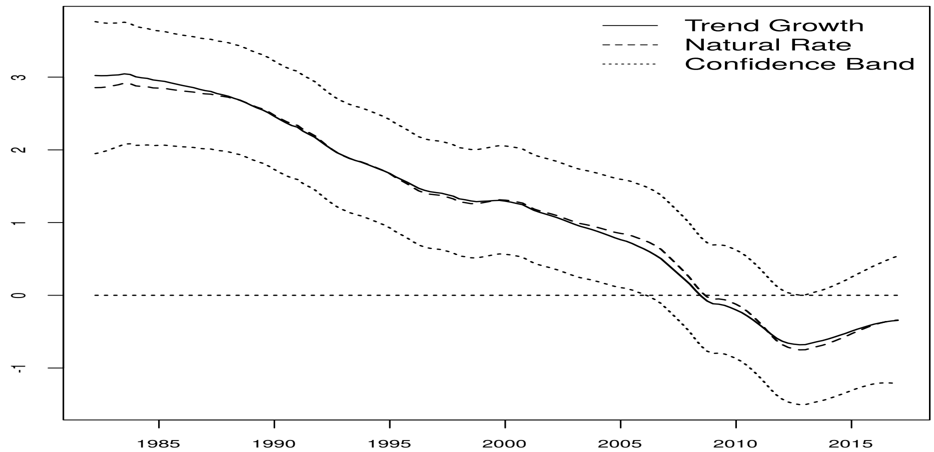


Figure 2: Italy, Smoothed Trend Growth ($\hat{g}^{(a)}_{t|T}$) and Natural Rate ($\hat{r}^{*}_{t|T}$)




For both economies the estimated natural rate of interest $\hat{r}^{*}_{t|T}$ does not diverge from the growth rate of potential output $\hat{g}^{(a)}_{t|T}$, as the "headwinds" $z_t$ are modeled by a stationary (although persistent) process. The common dynamics is consistent with stable long-term relations between the growth rate of potential output and the natural rate of interest implied by the definition of the latter. This result differs from the results obtained for the Euro Area in Holston, Laubach and Williams (2017a) where the "headwinds" $z_t$ are described by a random walk and the growth rate of potential output and the natural rate of interest diverge during the Great Recession.

Figures 1 and 2 demonstrate differences between Germany and Italy during the Great Recession: both the growth rate of potential output and the natural rate of interest remained above one percent annually in Germany since 2008; however, in Italy these rates reached negative values. These differences have implications for the effectiveness of a single monetary policy.

A stance of monetary policy can be described by a real rate gap, the difference between the actual real rate of interest and the natural rate: a positive real rate gap means a contractionary policy stance and a negative real rate gap means an expansionary stance. The smoothed real rate gaps and output gaps in Germany and Italy are plotted respectively in Figures 3 and 4. These figures can be used to interpret the effectiveness of a single monetary policy, which is implemented in the Euro Area since 1999, in each economy: as the natural rates of interest differ across economies, the real rate gaps and policy stances differ as well.

The real rate gap in Germany had been growing since 2005 and reached 3 percent in the fourth quarter of 2008 (see Figure 3). This was followed by a sharp recession with the output gap falling below 5 percent in the first quarter of 2009. However, the

Figure 3: Germany, Smoothed Real Rate Gap ($r_t - \hat{r}_{t|T}$) and Output Gap ($\tilde{y}_{t|T}$)

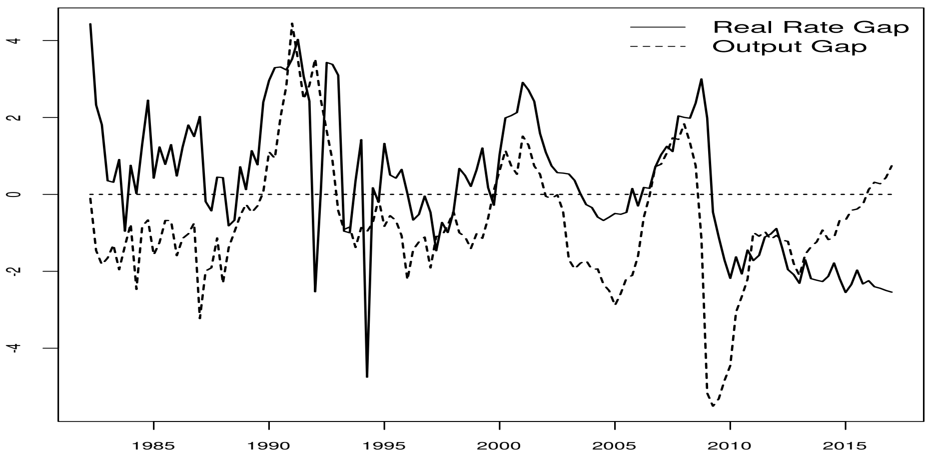
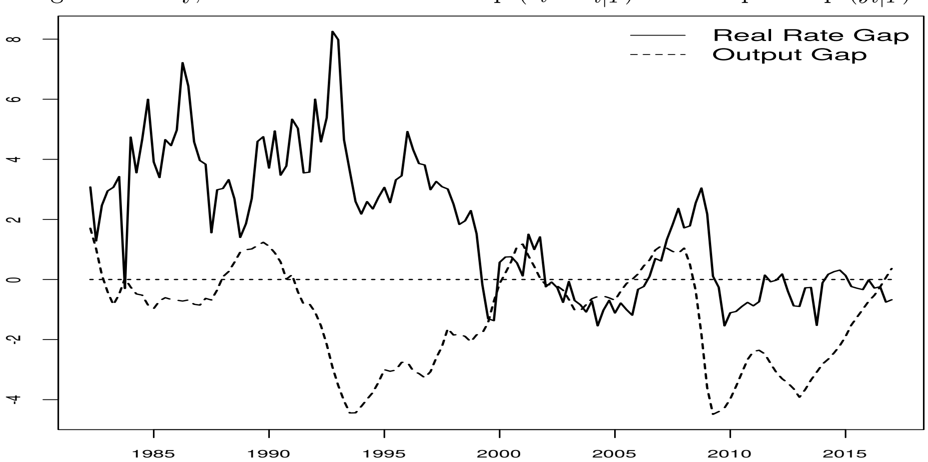



Figure 4: Italy, Smoothed Real Rate Gap ($r_t - \hat{r}_{t|T}$) and Output Gap ($\tilde{y}_{t|T}$)




monetary policy response had brought the real rate gap to the negative area by the second quarter of 2009 and the real rate gap was steadily below zero and decreasing ever since. Hence, the economy returned to the growth path in 2010 and the output gap converged to zero and reached positive values in 2016.
Figure 4 shows the evolution of interest and output gaps in Italy. Over the 80s – 90s Italy followed dis-inflationary policies with high real rates of interest: the real rate gap was high and the output gap was negative. After the transition to lower inflation rates in the late 90s and the accession to the Euro Area in 1999, the real rate gap decreased and stayed low until 2007. During 2007-2008 the real rate gap reached a

level above 2 percent and the output gap reached -4 percent. The prolonged recession caused a decline in the estimated potential output. The monetary policy response in 2009-2010 brought the real rate gap to the negative area. However, the real rate gap in Italy did not reach as low a level as the real rate gap in Germany: the monetary policy stance has been less expansionary in Italy than in Germany for the whole period after the financial crisis of 2007-2008.

These results differ from the results reported in Fries *et al.* (2018) who find that the natural rates and interest rate gaps in the largest Euro-Area economies converged over 2014-2016. A potential explanation of the differences in the results may lay in the different assumptions about the processes generating the natural rate of interest and the growth rate of potential output. Fries *et al.* (2018) assume that both rates are generated by a stationary process and the differences between these rates can be described by a white noise. It implies the eventual convergence of the natural rates and the real rate gaps. The estimated output gap is highly persistent and its dynamics suggests a prolonged boom in the Italian economy over 2000-2007 (see Table 3 and Figure 3 in Fries *et al.* 2018) which is at odds with a well-documented slowdown in Italy over this period.

It should be mentioned that the quantitative easing implemented by the Eurosystem in the aftermath of the sovereign debt crisis did not have a direct effect on short-term nominal interest rates and the estimated model does not account for for these policy measures. However, as it is argued in Fries *et al.* (2018), the real rate gaps can account for the effects of these measures indirectly, as these gaps are computed using current and expected inflation rates.

The approach implemented in this paper allows explaining the observed slowdown of output growth rates by a slowdown of the growth rates of potential output and demonstrates differences in the natural rates of interest and the real rate gaps which imply different effectiveness of a single monetary policy in Germany and Italy. These results are robust with respect to an alternative choice of expectations model (see Table 1). The alternative choice of the expectations model changes the parameter estimates and the value of the log-likelihood function only marginally.

For the model using all-item CPI, the log-likelihood is not directly comparable to the log-likelihood for the baseline model as the explained variable in the Phillips curve equation is different in these two models. In the model using all-item CPI, the variance of "headwind" shocks is much smaller than in the baseline model while the effect of the real rate gap onto the output gap is stronger (especially, in case of Italy). This can be attributed to the fact that in the model using all-item CPI, the real interest rate is affected by external inflationary shocks which are not controlled for in this model. As a result, the variance of "headwinds", including external inflationary shocks, is underestimated in this model.

## 5 Conclusions

In this paper, the natural rates of interest and the growth rates of potential output are jointly estimated for two large economies of the Euro Area: Germany and Italy. The estimation is based on a modification of the semi-structural model described in Holston, Laubach and Williams (2017a).

In order to control for international business cycle fluctuations, the IS equation of the model is augmented by a measure of foreign output gap. The natural rates are modeled as non-stationary processes, which allows explaining the long-term slowdown of the output growth rates in both economies. However, the deviations of the natural rate of interest from the growth rate of potential output are modeled as a stationary process, which is consistent with the long-run equilibrium relation between the growth rate of potential output and the natural rate of interest. Additional methodological innovations introduced, including diffuse initialization of non-stationary state variables and marginal likelihood estimation, are aimed to add credibility to the results reported in this paper.

The estimates of the growth rates of potential output and the natural rate of interest indicate a larger slowdown of the potential growth and a greater decrease of the natural rate in Italy compared with Germany in the aftermath of the financial crisis of 2007-2008. The estimated real rate gap which measures the monetary policy stance, is smaller in Germany: the common monetary policy was more expansionary in Germany over the whole after-crisis period. These results can contribute to the explanation of relatively fast recovery in Germany and a prolonged stagnation in Italy. Conditionally on these estimates a more aggressive expansionary policy had been necessary in Italy in order to bring the actual output closer to potential.

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# A State-Space Representations and Initial Conditions

The model estimated at each stage has a state-space representation (9)-(10). Specifications of vectors and matrices are given below.

**Stage 1:**

$$\mathbf{y}_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}, \mathbf{x}_t = \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ x_{t-1} \\ x_{t-2} \\ \pi_{t-1} \\ \bar{\pi}_{t-2,4} \end{bmatrix}, \xi_t = \begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_t \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ -a_{y,1} & -b_y \\ -a_{y,2} & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} a_{y,1} & b_y \\ a_{y,2} & 0 \\ a_{x,1} & 0 \\ a_{x,2} & 0 \\ 0 & b_\pi \\ 0 & 1-b_\pi \end{bmatrix}, R = \begin{bmatrix} \sigma_{\tilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, P_* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad P_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Stage 2:**

$$\mathbf{y}_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}, \mathbf{x}_t = \begin{bmatrix} 1 \\ y_{t-1} \\ y_{t-2} \\ x_{t-1} \\ x_{t-2} \\ r_{t-1} \\ r_{t-2} \\ \pi_{t-1} \\ \bar{\pi}_{t-2,4} \end{bmatrix}, \xi_t = \begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ -a_{y,1} & -b_y \\ -a_{y,2} & 0 \\ -a_r/2 & 0 \\ -a_r/2 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} a_0 & 0 \\ a_{y,1} & b_y \\ a_{y,2} & 0 \\ a_{x,1} & 0 \\ a_{x,2} & 0 \\ a_r/2 & 0 \\ a_r/2 & 0 \\ 0 & b_\pi \\ 0 & 1-b_\pi \end{bmatrix}, R = \begin{bmatrix} \sigma_{\tilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} \sigma_{y^*}^2 + \lambda_g^2\sigma_{y^*}^2 & 0 & 0 & \lambda_g^2\sigma_{y^*}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \lambda_g^2\sigma_{y^*}^2 & 0 & 0 & \lambda_g^2\sigma_{y^*}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, P_* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**Stage 3:**

$$\mathbf{y}_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}, \mathbf{x}_t = \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ x_{t-1} \\ x_{t-2} \\ r_{t-1} \\ r_{t-2} \\ \pi_{t-1} \\ \pi_{t-2,4} \end{bmatrix}, \xi_t = \begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ -a_{y,1} & -b_y \\ -a_{y,2} & 0 \\ -a_r/2 & 0 \\ -a_r/2 & 0 \\ -a_r/2 & 0 \\ -a_r/2 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} a_{y,1} & b_y \\ a_{y,2} & 0 \\ a_{x,1} & 0 \\ a_{x,2} & 0 \\ a_r/2 & 0 \\ a_r/2 & 0 \\ 0 & b_\pi \\ 0 & 1-b_\pi \end{bmatrix}, R = \begin{bmatrix} \sigma_{\tilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} \sigma_{y^*}^2 + \lambda_g^2\sigma_{y^*}^2 & 0 & 0 & \lambda_g^2\sigma_{y^*}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_g^2\sigma_{y^*}^2 & 0 & 0 & \lambda_g^2\sigma_{y^*}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\lambda_z^2\sigma_{\tilde{y}}^2/a_r^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{2\lambda_z^2\sigma_{\tilde{y}}^2}{a_r^2(1-\rho_z^2)} & \dfrac{2\rho\lambda_z^2\sigma_{\tilde{y}}^2}{a_r^2(1-\rho_z^2)} \\ 0 & 0 & 0 & 0 & 0 & \dfrac{2\rho_z\lambda_z^2\sigma_{\tilde{y}}^2}{a_r^2(1-\rho_z^2)} & \dfrac{2\lambda_z^2\sigma_{\tilde{y}}^2}{a_r^2(1-\rho_z^2)} \end{bmatrix}, P_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$