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Positive fractional 2D hybrid linear systems

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Abstract. A new class of positive fractional 2D hybrid linear systems is introduced. The solution of the hybrid system is derived. The classical Cayley-Hamilton theorem is extended for fractional 2D hybrid systems. Necessary and sufficient conditions for the positivity are established.

Key words: positive, partial realization, existence, computation, linear, discrete-time, system.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1, 2]. Recent developments in positive discrete-time and continuous-time systems without and with delays was considered in [1, 2–9].

The reachability, controllability and minimum energy control of positive linear discrete-time systems with delays have been considered in [10].

The relative controllability of stationary hybrid systems has been investigated in [11] and the observability of linear differential-algebraic systems with delays has been considered in [12].

The positive 2D hybrid linear systems have been investigated in [13].

The main purpose of this paper is to introduce a class of fractional 2D hybrid systems. A solution to the hybrid system will be derived. The classical Cayley-Hamilton theorem will be extended for fractional hybrid systems. Necessary and sufficient conditions for the positivity will be established.

To the best knowledge of the author the positive fractional 2D hybrid linear systems have not been considered yet.

2. Equations of the fractional 2D hybrid systems and their solutions

Let $\Re^{n\times m}$ be the set of $n\times m$ real matrices with entries from the real number $\Re$ and Z_+ be the set of nonnegative integers. The $n\times m$ identity matrix will be denoted by I_n.

Consider a hybrid fractional 2D system described by the equations

$$\frac{d^\alpha x_1(t,i)}{dt^\alpha} = A_{11}x_1(t,i) + A_{12}x_2(t,i) + B_1u(t,i), \quad t\in\Re_+ = [0,\ +\infty] \tag{1a}$$

$$\Delta^\beta x_2(t,i+1) = A_{21}x_1(t,i) + A_{22}x_2(t,i) + B_2u(t,i), \quad i\in Z_+ \tag{1b}$$

$$y(t,i) = C_1x_1(t,i) + C_2x_2(t,i) + Du(t,i) \tag{1c}$$

where α $(0<\alpha<1)$ is the order of fractional derivative, β $(0<\beta<1)$ is the order of fractional difference, $x_1(t,i)\in\Re^{n_1}$, $x_2(t,i)\in\Re^{n_2}$, $u(t,i)\in\Re^m$, $y(t,i)\in\Re^p$ and A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D are real matrices with appropriate dimensions.

Boundary conditions for (1a) and (1b) have the form

$$x_1(0,i) = x_1(i), \quad i\in Z_+ \quad \text{and} \quad x_2(t,0) = x_2(i), \quad t\in\Re_+ \tag{2}$$

Note that fractional 2D hybrid system (1) has a similar structure as the Roesser model [2, 13, 14].
The Caputo definition of the fractional derivative [14]

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau$$
$$\left(x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n}\right) \quad n-1<\alpha<n\in N=\{1,2,...\} \tag{3}$$

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where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \tag{4}$$

is the gamma function will be used.

The fractional difference of the order β of x_i will be defined by

$$\Delta^\beta x_i = \sum_{k=0}^{i} (-1)^k \begin{pmatrix} \beta \\ k \end{pmatrix} x_{i-k}, \quad 0 < \beta < 1, \ i \in Z_+ \tag{5}$$

where

$$\begin{pmatrix} \beta \\ k \end{pmatrix} = \begin{cases} 1 & \text{for} \quad k = 0 \\ \dfrac{\beta(\beta-1)...(\beta-k+1)}{k!} & \text{for} \quad k = 1, 2, ... \end{cases}. \tag{6}$$

Using (5) we may write the equation (1b) in the form

$$x_2(t, i+1) = A_{21}x_1(t, i) + A_{22}x_2(t, i) + \sum_{k=1}^{i+1} c_k x_2(t, i-k+1) + B_2 u(t, i), \quad i \in Z_+ \tag{7}$$

where

$$c_k = c_k(\beta) = (-1)^{k-1} \begin{pmatrix} \beta \\ k \end{pmatrix}, \quad k = 1, 2, \dots . \tag{8}$$

Remark. From (6) and (8) it follows that coefficients c_k strongly decrease when k increases. Therefore, in practical problems it is assumed that i is bounded by a natural number L and $\sum_{k=1}^{i+1} c_k x_2(t, i-k+1) = \sum_{k=1}^{L+1} c_k x_2(t, i-k+1)$.

Theorem 1. Solutions to the equations (1a) and (7) with given boundary conditions (2) have the forms

$$\begin{aligned} \begin{bmatrix} x_1(t,i) \\ x_2(t,i) \end{bmatrix} = & \sum_{p=0}^{\infty}\sum_{q=0}^{i} \frac{T_{p,i-q}B_{10}}{\Gamma[(p+1)\alpha]} \int_0^t (t-\tau)^{(p+1)\alpha-1} u(\tau, q) d\tau \\ & + \sum_{p=0}^{\infty}\sum_{q=0}^{i} \frac{T_{p,i-q-1}B_{01}}{\Gamma(\alpha p - 1)} \int_0^t (t-\tau)^{\alpha p-1} u(\tau, q) d\tau \\ & + \sum_{p=0}^{\infty}\sum_{q=0}^{i} \frac{T_{p,i-q}}{\Gamma(\alpha p + 1)} (t-\tau)^{p\alpha} \begin{bmatrix} x_1(0,q) \\ 0 \end{bmatrix} \\ & + \sum_{p=0}^{\infty} \frac{T_{p,i}}{\Gamma(\alpha p)} \int_0^t (t-\tau)^{p\alpha} \begin{bmatrix} 0 \\ x_2(t,0) \end{bmatrix} d\tau \end{aligned} \tag{9}$$

where

$$T_{pq} = \begin{cases} I_{n_1+n_2} & \text{for } p = q = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for } p = 1, \ q = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} + I_{n_2}c_1 \end{bmatrix} & \text{for } p = 0, \ q = 1 \\ T_{10}T_{p-1,q} + T_{01}T_{p,q-1} & \text{for } \ p \geq 0, \ q \geq 0, \ p+q > 1 \\ 0 & \text{for } p < 0, \ \text{or/and } q < 0 \end{cases} \tag{10a}$$

and

$$T_{oq} = T_{01}^q + \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2}c_q \end{bmatrix}, \quad q = 2, 3, \dots . \tag{10b}$$

Proof. The solutions will be derived using the Laplace transform ($\mathcal{L}$) with respect to t and the Z transform ($\mathcal{Z}$) with respect to i.

Taking into account that

$$\begin{aligned} \mathcal{L}\left[\frac{d^\alpha x_1(t,i)}{dt^\alpha}\right] &= s^\alpha X_1(s,i) - s^{\alpha-1}x_1(0,i) \\ \mathcal{Z}\mathcal{L}\left[\frac{d^\alpha x_1(t,i)}{dt^\alpha}\right] &= Z\left[s^\alpha X_1(s,i) - s^{\alpha-1}x_1(0,i)\right] \\ &= s^\alpha X_1(s,z) - s^{\alpha-1}X_1(0,z) \\ \mathcal{Z}\mathcal{L}\left[x_2(t, i-k+1)\right] &= z^{1-k}X_2(s,z) \quad (k \geq 1) \end{aligned} \tag{11}$$

where

$$\mathcal{L}\left[x(t,i)\right] = \int_0^\infty x(t,i)e^{-st}dt, \qquad \mathcal{Z}\left[x(t,i)\right] = \sum_{i=0}^{\infty} x(t,i)z^{-i} \tag{12}$$

from (1a) and (7) we obtain

$$\begin{aligned} & s^\alpha X_1(s,z) - s^{\alpha-1}X_1(0,z) = A_{11}X_1(s,z) + A_{12}X_2(s,z) + B_1U(s,z) \\ & zX_2(s,z) - zX_2(s,0) = A_{21}X_1(s,z) + A_{22}X_2(s,z) + \sum_{k=1}^{i+1} c_k z^{1-k} X_2(s,z) + B_2U(s,z) \end{aligned} \tag{13}$$

where $U(s,z) = \mathcal{Z}\mathcal{L}\left[u(i,t)\right]$.

From (13) we have

$$\begin{bmatrix} X_1(s,z) \\ X_2(s,z) \end{bmatrix} =$$

$$= \begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix}^{-1}$$

$$\left\{ \begin{bmatrix} B_1 s^{-\alpha} \\ B_2 z^{-1} \end{bmatrix} U(s,z) + \begin{bmatrix} s^{-1}X_1(0,z) \\ X_2(s,0) \end{bmatrix} \right\}. \tag{14}$$

Let

$$\begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix}^{-1} = \\ = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}s^{-\alpha p}z^{-q}. \tag{15}$$

From definition of inverse matrix and (15) we have

$$\begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix}^{-1}$$

$$\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}s^{-\alpha p}z^{-q}\right) = \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}s^{-\alpha p}z^{-q}\right)$$

$$\begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix}^{-1}$$

$$= I_{n_1+n_2}. \tag{16}$$

Comparison of the coefficients at the same powers of s and z of the equality (16) yields (10).

Substituting (15) into (14) we obtain

$$\begin{bmatrix} X_1(s,z) \\ X_2(s,z) \end{bmatrix} = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}s^{-\alpha p}z^{-q}$$

$$\left\{ \begin{bmatrix} B_1 s^{-\alpha} \\ B_2 z^{-1} \end{bmatrix} U(s,z) + \begin{bmatrix} s^{-1}X_1(0,z) \\ X_2(s,0) \end{bmatrix} \right\} =$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left(T_{pq}s^{-\alpha(p+1)}z^{-q}B_{10} + T_{pq}s^{-\alpha p}z^{-(q+1)}B_{01}\right)$$

$$U(s,z) + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left(T_{pq}s^{-(\alpha p+1)}z^{-q}\begin{bmatrix} X_1(0,z) \\ 0 \end{bmatrix}\right.$$

$$\left. + T_{pq}s^{-\alpha p}z^{-q}\begin{bmatrix} 0 \\ X_2(s,0) \end{bmatrix}\right) \tag{17}$$

where

$$B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \qquad B_{01} = \begin{bmatrix} 0 \\ \mathbf{B}_2 \end{bmatrix}.$$

Applying the inverse transforms to (17) and taking into account that

$$\mathcal{L}\left[t^{\alpha}\right] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \tag{18}$$

we obtain (9). □

3. Extension of the Cayley-Hamilton theorem for the fractional 2D hybrid systems

Taking into account Remark we may write

$$\det\begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix} \tag{19}$$

$$= \sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{N_1-k,N_2-l}s^{-\alpha k}z^{-l}$$

for the same natural numbers N_1, N_2.

Theorem 2. Let (19) be the characteristic polynomial of the fractional 2D hybrid system. Then the matrices T_{pq} defined by (10) satisfy the equation

$$\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{kl}T_{kl} = 0. \tag{20}$$

Proof. From the definition of inverse matrix and (19), (15) we have

$$\mathrm{Adj}\begin{bmatrix} I_{n_1} - A_{11}s^{-\alpha} & -A_{12}s^{-\alpha} \\ -A_{21}z^{-1} & I_{n_1} - A_{22}z^{-1} - \sum_{k=1}^{i+1} I_{n_2}c_k z^{-k} \end{bmatrix}$$

$$= \left(\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{N_1-k,N_2-l}s^{-\alpha k}z^{-l}\right)\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{pq}s^{-\alpha p}z^{-q}\right) \tag{21}$$

where Adj M denotes the adjoint matrix of M.

Comparison of the coefficients at the same powers $s^{-\alpha N_1}z^{-N_2}$ of the equality (21) yields (20) since the degrees of the left-hand side are less than αN_1 and N_2.

Theorem 2 is an extension of the well-known classical Cayley-Hamilton theorem for the fractional 2D hybrid systems.

4. Positive fractional 2D hybrid systems

Let $\Re_+^{n\times m}$ be the set of $n \times m$ real matrices with nonnegative entries $\Re_+^n = \Re_+^{n\times 1}$.

Definition 1. The fractional 2D hybrid system (1) is called (internally) positive if and only if $x_1(t,i) \in \Re_+^{n_1}$, $x_2(t,i) \in \Re_+^{n_2}$, $t \geq 0$, $i \in Z_+$ for any boundary conditions

$$x_1(0,i) = x_1(i) \in \Re_+^{n_1}, \quad i \in Z_+ \quad \text{and} \quad x_2(t,0) = x_2(i) \in \Re_+^{n_2}, \quad t \geq 0 \tag{22}$$

and all inputs $u(t,i) \in \Re_+^m$, $t \geq 0$, $i \in Z_+$.

Lemma 1. If $0 < \beta < 1$ then

$$c_k = c_k(\beta) > 0 \quad \text{for} \quad k = 1, 2, \ldots \tag{23}$$

Proof. The hypothesis is true for $k = 1$ since from (8) and (6) we have $c_1 = \begin{pmatrix} \beta \\ 1 \end{pmatrix} = \beta$. Assuming that the hypothesis is true for $k \geq 1$ we shall show that it is also true for $k+1$. From (8) and (6) we obtain

$$c_{k+1} = (-1)^k \begin{pmatrix} \beta \\ k+1 \end{pmatrix}$$

$$= (-1)^{k-1} \frac{\beta(\beta-1)\ldots(\beta-k+1)(k-\beta)}{k!\,(k+1)}$$

$$= (-1)^{k-1} \begin{pmatrix} \beta \\ k \end{pmatrix} \frac{k-\beta}{k+1} = c_k \frac{k-\beta}{k+1} > 0$$

since $c_k > 0$ for $k \geq 1$.

Lemma 2. If $0 < \beta < 1$ and $A_{21} \in \Re_+^{n_2 \times n_1}$, $A_{22} \in \Re_+^{n_2 \times n_2}$ then

$$T_{0q} \in \Re_+^{(n_1+n_2)\times(n_1+n_2)} \quad \text{for} \quad q = 2, 3, \ldots \tag{24}$$

Proof. From (10b) and Lemma 1 we have

$$T_{0q} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} + I_{n_2} c_1 \end{bmatrix}^q$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} c_q \end{bmatrix} \in \Re_+^{(n_1+n_2)\times(n_1+n_2)} \quad \text{for} \quad q = 2, 3, \ldots$$

since $c_k > 0$ for $k = 1, 2, \ldots$.

Let M_n be the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries).

Theorem 3. The fractional 2D hybrid system (1) is (internally) positive if and only if

$$\begin{gathered} A_{11} \in M_{n_1}, \quad A_{12} \in \Re_+^{n_1 \times n_2}, \quad A_{21} \in \Re_+^{n_2 \times n_1}, \\ A_{22} \in \Re_+^{n_2 \times n_1}, \quad B_1 \in \Re_+^{n_1 \times m}, \quad B_2 \in \Re_+^{n_2 \times m}, \\ C_1 \in \Re_+^{p \times n_1}, \quad C_2 \in \Re_+^{p \times n_2}, \quad D \in \Re_+^{p \times m}. \end{gathered} \tag{25}$$

Proof. From (1a) for and (22) we have [14]

$$\frac{d^\alpha x_1(t,0)}{dt^\alpha} = A_{11} x_1(t,0) + A_{12} x_2(t) + B_1 u(t,0) \tag{26}$$

and

$$x_1(t,0) = \Phi_0(t) x_1(0) + \int_0^t \Phi(t-\tau)[A_{12} x_2(\tau) + B_1 u(\tau,0)] d\tau \tag{27}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A_{11}^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{28a}$$

and

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A_{11}^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{28b}$$

In [14] was shown that $\Phi_0(t) \in \Re_+^{n_1 \times n_1}$ and $\Phi(t) \in \Re_+^{n_1 \times n_1}$ for $t \geq 0$ if and only if A_{11} is a Metzler matrix. From (7) for $i = 0$ we obtain

$$x_2(t,1) = A_{21} x_1(t,0) + (A_{22} + I_{n_2} c_1) x_2(t) + B_2 u(t,0) \tag{29}$$

and after substitution of (27) into (29)

$$\begin{gathered} x_2(t,1) = A_{21} \Phi_0(t) x_1(0) + A_{21} \int_0^t \Phi(t-\tau) \\ [A_{12} x_2(\tau) + B_1 u(\tau,0)] d\tau \\ + (A_{22} + I_{n_2} c_1) x_2(t) + B_2 u(t,0). \end{gathered} \tag{30}$$

From (30) for (22) it follows that $x_2(t,1) \in \Re_+^{n_2}$ if and only if $A_{11} \in M_{n_1}$, $A_{21} \in \Re_+^{n_2 \times n_1}$, $A_{22} \in \Re_+^{n_2 \times n_2}$, $B_1 \in \Re_+^{n_1 \times m}$, $B_2 \in \Re_+^{n_2 \times m}$ and $u(t,0) \in \Re_+^m$, $t \geq 0$. Likewise from (1a) for we obtain

$$\frac{d^\alpha x_1(t,1)}{dt^\alpha} = A_{11} x_1(t,1) + A_{12} x_2(t,1) + B_1 u(t,1) \tag{31}$$

and

$$x_1(t,1) = \Phi_0(t) x_1(0,1) + \int_0^t \Phi(t-\tau)[A_{12} x_2(\tau,1) + B_1 u(\tau,1)] d\tau. \tag{32}$$

From (32) it follows that $x_1(t,1) \in \Re_+^{n_1}$, $t \geq 0$ if and only if $A_{11} \in M_{n_1}$ since $x_1(0,1) \in \Re_+^{n_1}$ and $x_2(t,1) \in \Re_+^{n_2}$, $t \geq 0$.

Continuing this procedure it can be shown that the fractional 2D hybrid system is positive if and only if the conditions (25) are met.

5. Concluding remarks

A new class of fractional 2D hybrid positive linear systems has been introduced. Necessary and sufficient conditions for the positivity of the hybrid 2D linear systems has been established. The classical Cayley-Hamilton theorem has been extended for the hybrid systems. Following [14] the sufficient conditions for the reachability and controllability [15, 16] can be extended for the fractional 2D positive hybrid systems.

An open problem is extension of the considerations for 2D hybrid systems described by models with structure similar to the Kurek model [2, 17, 18].

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