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Optimal input signal design for fractional-order system identification

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Abstract. The optimal design of excitation signal is a procedure of generating an informative input signal to extract the model parameters with maximum pertinence during the identification process. The fractional calculus provides many new possibilities for system modeling based on the definition of a derivative of noninteger-order. A novel optimal input design methodology for fractional-order systems identification is presented in the paper. The Oustaloup recursive approximation (ORA) method is used to obtain the fractional-order differentiation in an integer order state-space representation. Then, the presented methodology is utilized to solve optimal input design problem for fractional-order system identification. The fundamental objective of this approach is to design an input signal that yields maximum information on the value of the fractional-order model parameters to be estimated. The method described in this paper was verified using a numerical example, and the computational results were discussed.

Key words: fractional calculus, optimal input design, Oustaloup recursive method, parameter identification.

1. Introduction

Taking into consideration a derivative of noninteger-order, the fractional calculus provides a new modeling tool for precise system identification and controller tuning purposes [1, 2]. It was shown that fractional models guarantee a more precise specification of the system dynamics than models obtained using ordinary differential equations [3–5]. Moreover, fractional calculus is a generalization of the ordinary differential equations by partial order differentiation [6]. Many reports have been created to examine in details the accuracy of fractional calculus, applied to solving problems in various domains, such as bioengineering [7], physics [8, 9], chaos theory [10], control systems [11, 12] and fractional processes [13].

Implementation of the noninteger-order identification methods to real-life industrial problems should cause performance improvement and finally cost minimization. Thus, fractional calculus is mostly helpful in mechatronics and control engineering areas where fractional identification methods are applied to more accurate control strategies development and control loops improvements [2]. It was shown that fractional-order controllers behaviour varies from integer-order controllers, and in some applications, fractional PID controllers have a better precision in comparison with classic PID's [14].

Identification is usually performed by perturbing processes or plants during an identification experiment in order to model parameters extraction. The selection of an informative input utilized for plant excitation is an essential step in the task of an unknown model parameters extraction.

However, the fundamental goal of identification experiment is to minimize or maximize selected criterion based on Fisher information matrix, in the presence of disturbances affecting a plant around its operating point [15]. Identification is carried out by disturbing the system using an optimal input signal and use an output data to extract a plant model using minimal cost and resources [16]. The improper experimental conditions can lead to performance degradation of the control loop, and therefore the cost of the experiment (i.e., departure from the nominal operating point) should be quantified. The objective of input signal design is minimizing the cost of identification experiment, which still delivers acceptable application performance. It was shown that more than 60% of the investigated control processes are not ensuring the appropriate quality [17].

Studies show that model development contains over 75% of the cost related to industrial control loops design [18]. The expenses of the system identification are also defined by efficiency degradation of the closed-loop system, during perturbation, subject to the closed-loop system under optimal operational policy [19].

Plant-friendly input design is based on the framework of application-oriented system identification. The idea is to find a trade-off between minimal disruption to the operating conditions and expected properties of the model to be identified [20]. The plant friendliness assumptions are often inconsistent with demands for precise model parameters estimation [21]. That is why, more safe perturbations which are providing a good accuracy of parameter estimates, should be considered [22]. In [23] a robust formulation of a plant friendly input design problem with the power of the excitation trajectory and output fitting constraints was proposed. This kind of research is a combination of solving the sequential and robust problems. The methodology for solving input design in the economic, plant friendly and application-oriented setup, where the purpose is to minimize the departure from the standard operating conditions was outlined in [24–26].

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In previous papers, the theory of optimal control methods for linear and nonlinear integer order systems was reviewed. In this paper, a novel optimal input design formulation and the numerical scheme for fractional-order system identification are presented. The Oustaloup recursive approximation (ORA) method provides a precise approximation of the fractional operator, which is then converted into zero-pole transfer function. The affine problem can also be described by the state-space formulation to solve a fractional-order input design problem. Numerical examples for fractional-order linear time-invariant model identification to verify pertinence of the method are presented. The issues of the optimal input design, in the classic framework, are considered in earlier works of the author [27, 28].

2. Fractional-order optimal input problem formulation

Fractional calculus is an idea as old as the ordinary differential equations calculus description where differentiation and integration are applied to fractional index operator. The continuous operator of the fractional order α is featured as follows:

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \Re(\alpha) > 0 \\ 1 & \Re(\alpha) = 0 \\ \int_a^t (d\tau)^{-\alpha} & \Re(\alpha) < 0 \end{cases}, \tag{1} $$

where: a, t – signify the integration limits and α is the operator order. Presently many different forms of fractional derivation and integration operators were developed [1].

The Riemann-Liouville differentiation ($\alpha > 0$) of a function f at t is commonly utilized definition:

$$ {}_aD_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dt}\right)^m \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{2} $$

where: $\Gamma(\cdot)$ is a gamma function defined by Euler limit, and $m - 1 < \alpha \le m$, $m \in \mathrm{N}$, $\alpha \in (0, 1\rangle$. The above expression could be reformulated into the following equation:

$$ {}_aD_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau, \tag{3} $$

Another fractional-order derivative ($\alpha > 0$) is called the Caputo's fractional derivative of order α given by:

$$ {}_aD_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(p)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \tag{4} $$

where: $f^{(m)}(t) = \left(d^m / d^{tm}\right) f(t)$, $m - 1 < \alpha \le m$. In like manner, Caputo's fractional integral for $\alpha \in (0, 1\rangle$ can be converted to the Caputo fractional derivative of order α as follows:

$$ {}_aD_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau. \tag{5} $$

It has been proven that for real values, the fractional derivatives defined by Riemann-Liouville and Grünwald-Letnikov are the same [11].

Eventually, the Grünwald-Letnikov fractional formula for $\alpha \in \mathrm{R}$ should be noticed:

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f(t - jh), \tag{6} $$

where: $\omega_j^\alpha = (-1)^j \binom{\alpha}{j}$ represents polynomial coefficients, which are calculated recursively from:

$$ \omega_0^\alpha = 1, \omega_j^\alpha = \left(1 - \frac{\alpha+1}{j}\right)\omega_{j-1}^\alpha, j = 1,2,\ldots, \tag{7} $$

Based on the equation (7) the fractional formula (6) should be obtained from:

$$ {}_aD_t^\alpha f(t) \approx \frac{1}{h^\alpha} \sum_{j=0}^{k} \omega_j^\alpha f(t - jh), \tag{8} $$

where $a = 0$, $t = kh$ denotes step index and the step duration h.

The fractional calculus is the special case of traditional ordinary differential equations approach (ODE). Linear noninteger SISO continuous-time dynamic model is comensature-order if all powers of derivative are integer multiples of the order q in such a way that α_k, $\beta_k = kq$, $q \in \mathrm{R}^+$ as follows [1, 2]:

$$ \sum_{k=0}^{n} a_k D_t^{\alpha_k} y(t) = \sum_{k=0}^{m} b_k D_t^{\beta_k} u(t), \tag{9} $$

where: a_k, b_k are systems constant coefficients. The description for a discrete signal and different degrees could be discovered from [29]. The LTI system is the rational-order if $q = r^{-1}$, and $q \in \mathrm{R}^+$. Using Laplace law to equation (9) with zero initial conditions the fractional-order linear time-invariant model can be described by a transfer function written as:

$$ G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}}. \tag{10} $$

The quantity of fractional poles in transfer function (10) is referred to as the pseudo-order of the system. The system with commensurate order q can be reformulated to obtain the pseudo-rational transfer formula $H(\lambda)$ given by:

$$ H(\lambda) = \frac{\sum_{k=0}^{m} b_k \lambda^k}{\sum_{k=0}^{n} a_k \lambda^k}, \tag{11} $$

where: $\lambda = s^q$. On the basis of this idea, the state space representation of a fractional-order linear and nonlinear dynamic models can be written as:

$$
\begin{aligned}
{}_0D_t^{\alpha} x(t) &= A_F x(t) + B_F u(t), \\
y(t) &= C_F x(t) + D_F u(t).
\end{aligned} \tag{12}
$$

Such a description is called an internal system representation because all the state variables are included in state space notation.

From any of the presented above fractional-order derivative definitions, we can specify the problem of optimal excitation signal design for the fractional system identification. The objective function to be minimized can be written as:

$$
J = g\big(x(t_0), x(t_f)\big) + \int_{t_0}^{t_f} l(x,u,t)\,dt, \tag{13}
$$

in respect to the plant dynamics with the initial condition:

$$
{}_{t_0}D_t^{\alpha} x(t) = h(x,u,t),\; x(t_0) = x_{t_0},\; t \in [t_0, t_f], \tag{14}
$$

subject to the trajectory constraints:

$$
u(t) \in \langle u_{\min}(t), u_{\max}(t) \rangle,\; t \in \lfloor t_0, t_f \rfloor, \tag{15}
$$

$$
x(t_0) \in \langle x_{\min}(t_0), x_{\max}(t_0) \rangle, \tag{16}
$$

$$
l_{tc}^{\varsigma}\big(t, x(t), u(t)\big) \le 0,\; \varsigma \in \mathrm{q}_{\mathrm{tc}},\; t \in [t_0, t_f], \tag{17}
$$

$$
g_{eic}^{\varsigma}\big(x(t_0), x(t_f)\big) \le 0,\; \varsigma \in \mathrm{q}_{\mathrm{tc}}, \tag{18}
$$

$$
g_{eec}^{\varsigma}\big(x(t_0), x(t_f)\big) = 0,\; \varsigma \in \mathrm{q}_{\mathrm{eec}}. \tag{19}
$$

where x is the state-space vector, $t \in [t_0, t_f]$ denotes time duration, $\mathrm{q} = \{1, \ldots, q\}$ and l, g, h are *a priori* linear or nonlinear functions. The functions $g(\cdot,\cdot)$ and $l(\cdot,\cdot,\cdot)$ with indexes tc, eec, and eic are trajectory constraint, endpoint equality constraint and endpoint inequality constraint, respectively.

3. Fractional-order operator approximation

Some continuous filters, which provide a possibility of approximating the fractional operators have been discussed in [1]. The Oustaloup methods, which have an excellent matching to the noninteger-order zero-pole transfer functions approximation, are often utilized in practical implementations. We tend to take into the consideration the recursive Oustaloup approximation method (ORA) during an experiment. Assuming the expected frequency fitting range, the fractional operators could be approximated by following formulas:

$$
s^{\alpha} \approx K \prod_{k=-N}^{N} \frac{s + \omega_k'}{s + \omega_k}, \tag{20}
$$

where: gain, zeros, and poles of the filter can be obtained from:

$$
\omega_k' = \omega_b \left(\frac{\omega_h}{\omega_b} \right)^{\frac{k+N+1/2(1-\alpha)}{2N+1}}, \tag{21}
$$

$$
\omega_k = \omega_b \left(\frac{\omega_h}{\omega_b} \right)^{\frac{k+N+1/2(1+\alpha)}{2N+1}}, \tag{22}
$$

$$
K = \omega_h^{\alpha}, \tag{23}
$$

$$
\omega_u = \left(\frac{\omega_h}{\omega_b} \right)^{1/2}, \tag{24}
$$

where: N is the degree of the approximation and (ω_b, ω_h) is the given pulsation interval. The degree of the approximation is N and order of the filter is $2N+1$, considering successive values of N the parameters of the filter became more precise. Application of higher order N increases the amount of the evaluations.

The Oustaloup approximation method provides a pretty accurate approximation of fractional operators in an established wide fitting range. Applying ORA filter for the fractional operators where $\alpha \ge 1$ one should separate fractional orders using the following strategy:

$$
s^{\alpha} = s^n s^{\gamma}, \tag{25}
$$

where α signifies the order of the differentiation and s^{γ} was approximated based on (20) utilizing ORA filter method. The transfer function obtained from Oustaloup filter method is used for converting external model representation into the integer-order internal state-space formulation. For a general n-th order transfer function obtained from pole-zero formulation is given by:

$$
H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}, \tag{26}
$$

where a and b specify the coefficients of the polynomials in descending powers of s, and $a_0 = 1$. Then, it is possible to solve optimal input design problem for fractional-order system identification using integer state-space representation [30].

Because the choice of the state factors can vary, the transfer function implementation can also be different. Regarding publication [31], the fractional-order operator ${}_{t_0}D_t^{\alpha}$ can be obtained from:

$$
{}_{t_0}D_t^{\alpha} x(t) \approx {}_{t_0}D_t^{\alpha} x(t) \approx \left\{ \begin{aligned} \dot{z} &= A_F z + B_F u \\ x &= C_F z + D_F u \end{aligned} \right\}, \tag{27}
$$

where the compatible matrices are respectively:

$$A_F = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{28}$$

$$B_F = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{29}$$

$$C_F = \begin{bmatrix} (b_n - a_n b_0)(b_{n-1} - a_{n-1} b_0)\cdots \\ \cdots(b_2 - a_2 b_0)(b_1 - a_1 b_0) \end{bmatrix}, \tag{30}$$

$$D_F = b_0 = d. \tag{31}$$

To design an optimal input signal for fractional system identification it is required to approximate the noninteger-order operator, and convert our problem to be solved using one of the accessible programs for optimal control tasks.

4. Optimal input design problem reformulation

For this case study, the following equations (13–19) were reformulated to provide the optimal input problem solution. The optimal input for fractional order dynamical system that minimizes performance index is as follows:

$$J = g\big(C_F z(t_0) + D_F u(t_0), C_F z(t_f) + D_F u(t_f)\big) + \int_{t_0}^{t_f} l(C_F z + D_F u, u, t)\, dt, \tag{32}$$

according to the fractional system dynamics:

$$\dot{z}(t) = A_F z + B_F\big(h(C_F z + D_F u, u, t)\big), \tag{33}$$

and initial condition

$$z(t_0) = \frac{x_{t_0} T}{C_F T}, \tag{34}$$

The original state-space variable $x(t)$ is:

$$x(t) = C_F z(t) + D_F u(t), \tag{35}$$

with a set of possible constraints:

$$u(t) \in \langle u_{\min}(t), u_{\max}(t)\rangle, \tag{36}$$

$$\big(C_F z(t_0) + D_F u(t_0)\big) \in \langle x_{\min}(t_0), x_{\max}(t_0)\rangle, \tag{37}$$

$$l_{tc}^{\varsigma}\big(t, C_F z(t) + D_F u(t), u(t)\big) \leq 0, \tag{38}$$

$$g_{eic}^{\gamma}\big(C_F z(t_0) + D_F u(t_0), C_F z(t_f) + D_F u(t_f)\big) \leq 0, \tag{39}$$

$$g_{eec}^{\gamma}\big(C_F z(t_0) + D_F u(t_0), C_F z(t_f) + D_F u(t_f)\big) = 0. \tag{40}$$

where x is the state-space vector, $t \in [t_0, t_f]$ denotes time duration, $q = \{1, \ldots, q\}$ and l, g, h are *a priori* linear or nonlinear functions. The functions $g(\cdot,\cdot)$ and $l(\cdot,\cdot,\cdot)$ with indexes tc, eec, and eic are: trajectory constraint, endpoint equality constraint and endpoint inequality constraint, respectively.

The convergence of optimization depends on the selection of vector T. Concerning vector B, which is given by the equation (29), vector T should have the following form:

$$T = [1 \;\; 0 \;\; 0 \;\; \cdots \;\; 0]^{\mathrm{T}}. \tag{41}$$

The bandwidth for the Oustaloup recursive approximation was chosen as [0.01, 100] rad/s. The choice of the frequency range was related to the digitization of the control duration imposed by a package for solving OCP, a wide fitting range causes a significant calculation effort. The assumed termination time is $t_f = 1.5$ [s], and the choice of Oustaloup filter N was expressed as:

$$N = \log(\omega_h) - \log(\omega_b). \tag{42}$$

The choice of frequency fitting range for the ORA method is especially important as a narrow bandwidth lead to incorrect results.

5. Optimal input design for fractional system identification

In this section, the problem of synthesizing of an optimal input for fractional-order time-invariant system identification is considered. The matrix computation method for fractional variable order time-invariant control models in state-space representation was presented in [32].

Our aim is to transcript the problem of optimal input design using the Lagrange method with the suitable set of restrictions. In order to verify the suitability of this technique to the model identification, a fractional inertial object was used:

$$G(s) = \frac{k}{s^{\alpha} T + 1}, \quad 0.5 \leq \alpha \leq 1.0, \tag{43}$$

where $k = 1$ is the gain of the system, and $T = a_1/a_0 = 1$ is the time constant.

In a general case, the fractional-order linear or nonlinear time-invariant system can be then expressed by the state-space formulation denoted by:

$$ {}_0D_t^{\alpha} x(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t) + v(t), \tag{44} $$

where $u(t)$, $y(t)$ are the input and output vectors, $x(t)$ is the state vector, A, B, C, D are the state-space matrices describing the system dynamics, and $v(t)$ is a stationary random Gaussian process with zero mean:

$$ E[v(t)] = 0, \\ E[v(t)v^{\mathrm{T}}(\tau)] = R\delta(t-\tau) = \sigma_n^2 \delta(t-\tau). \tag{45} $$

The fundamental principle of system parameter estimation is to maximize the sensitivity of the state variable to the unidentified parameters [15]. The motivation for such an experiment is the Cramer-Rao definition, which gives a lower bound for the variance of an unbiased parameter to be estimated. Applying the above definition to input design purposes, we calculate the parameter estimate which has a tendency to getting lower for optimal input:

$$ \mathrm{cov}(A, B, C, D) \geq M^{-1}. \tag{46} $$

In the considerations which follow it was assumed that $\sigma_n = 1$ to obtain an optimal excitation signal for model parameter estimation where measurements do not include additive white noise.

In this paragraph, the optimal input design for fractional inertial model identification is considered. According to the above definition, the sensitivity of the state variable $x(t, d)$ to the coefficient d (i.e., the gain of the open circuit) was maximized. The performance index is as follows:

$$ J_{\alpha}(u) = \int_0^{t_f} x_d^2(t,d)\,dt, \tag{47} $$

where

$$ x_d(t,d) = \frac{\partial x(t,d)}{\partial d}, \tag{48} $$

in respect to constraint

$$ \int_0^{t_f} u(t)^{\mathrm{T}} u(t)dt \leq E, \tag{49} $$

The problem of minimum control energy for the fractional positive system with constraint inputs was presented in [33, 34].

In general the task of designing an optimal input signal to the fractional linear inertial model is presented in the form:

$$ {}_0D_t^{\alpha} x = Ax + Bu, \\ y = Cx, \tag{50} $$

where $A = -1$, $B = 1$, and $C = 1$ are constant model parameters (with reference to (43)), with the initial condition:

$$ x(0) = 5. \tag{51} $$

Utilizing the presented method for the reformulated problem described by the equations (32–41), which maximizes the performance index is:

$$ J_{\alpha}(u) = \int_0^{t_f} \left(C_F x_d(t) + u\right)^2 dt, \tag{52} $$

in respect to constraints:

$$ -1 \leq u(t) \leq 1,\ t \in [0, t_f], \\ \int_0^{t_f} (t_f - t)^{2(1-\alpha)} u(t)^{\mathrm{T}} u(t)dt \leq 1,\ t \in [0, t_f], \tag{53} $$

The controllability Gramian [35] of fractional order α, from the duration interval $[0, t_f]$ is responsible for the system constraint. The term $(t_f - t)^{2(1-\alpha)}$ under the integral (53) is called a neutralizer of the singularity at $t = t_f$. It is needed to ensure the convergence of the integral. The reformulated system dynamics (according to (33), and (50)) is:

$$ \dot{z} = A_F z + B_F\left(-\left(C_F z + D_F u\right) + u\right), \tag{54} $$

with the initial conditions

$$ z(0) = [5 \quad 0 \quad \cdots \quad 0]^{\mathrm{T}}. \tag{55} $$

Solving the optimal input problem, the system's dynamics was discretized according to chosen grid interval using Runge-Kutta method.

6. Simulation results for linear time-invariant problem

The bandwidth for the Oustaloup recursive approximation was chosen as $[10^{-2}, 10^{2}]$ rad/s. Therefore, the order of filter was chosen according to equation (42) as $N = 4$. Such a narrow Oustaloup filter fitting range was established using trial and error method and was related to the discretization of the integration algorithm adopted in Riots_95 [36]. This software is implemented in Matlab package as a separate module and has tools for solving constrained optimal problems of control including fixed or free terminal conditions.

Constrained optimal inputs for fractional inertial system identification were then computed for the assumed preliminary parameters value (50): $A = -1$, $B = 1$, $C = 1$, and selected time interval $t = [0, 1.5]$ seconds, using sequential quadratic programming (SQP) routine. The grater terminal time t_f causes significant computational burden. The initial state conditions of the fractional time-invariant system were selected according to (55), and the excitation initial value was chosen as $u(0) = 1$. It should be noted that an optimal control $u(t)$ has been restricted to the interval $[-1, +1]$ to prevent rapid changes of the input signal. The computational results were obtained utilizing the Runge-Kutta 4th order technique with mesh interval of 0.01 sec.

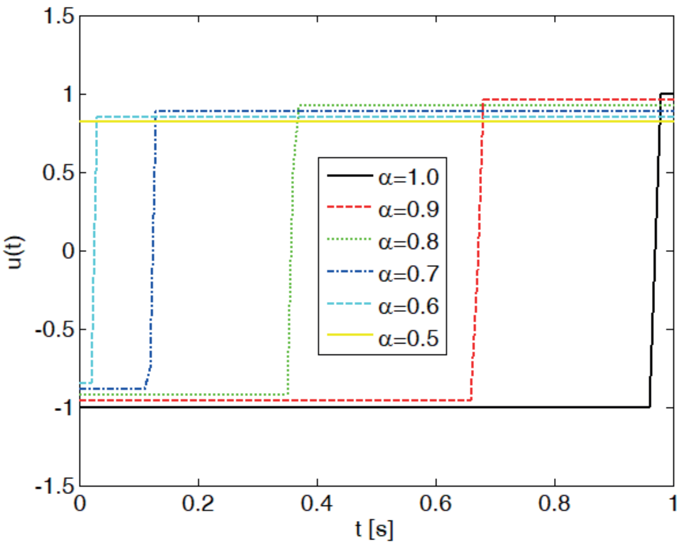

Fig. 1. The optimal input signals $u(t)$ to the fractional inertial system as function of time t for various orders of α

Table 1
Comparison of the performance index rates for different values of α

α	1.00	0.98	0.96	0.94	0.92	0.90	0.80	0.70	0.60	0.50
$J_\alpha(u)$	10.18	8.13	6.54	5.27	4.25	3.60	1.52	0.72	0.41	0.28

Figure 1 shows the optimal control signal for fractional inertial system received for different values of alpha (i.e., from the interval $\alpha = <0.5, 1.0>$ in the equation (50)). As has been seen, the excitation signal is appreciably different when the corresponding order of α decreases. For the orders of $\alpha \leq 1.0$ the control signal switching time reduces its duration, while the optimal signal received for $\alpha = 0.5$ is an almost constant value (i.e., step input signal-yellow solid line). Comparison of the performance indexes for different orders of α to the fractional inertial system is displayed in Table 1. As it could be noticed based on the presented method (Table 1), when the desired order of the factor α increases, the ratio of the optimal excitation signal performance index also increases (according to (52)).

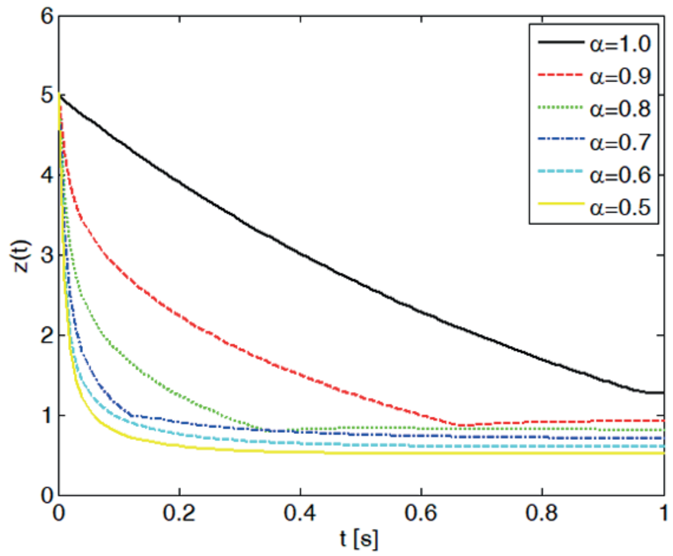

Fig. 2. The state variable $z(t)$ to the fractional inertial system as function of time t for various orders of α

Figure 3 shows the state variable $x_d(t)$ sensitivity to the coefficient d of the system as a function of duration for different orders of α. The parameter d is the gain of the noninteger model (27) after the process of Oustaloup approximation.

The visualization of the performance index component obtained for increasing values of α subject to controllability Gramian of fractional-order is shown in Fig. 4. The ratio of the objective function, which determines the interpolation

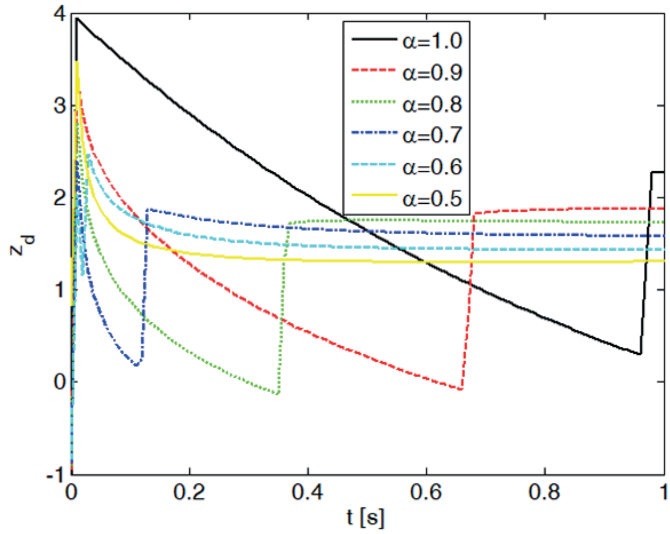

Fig. 3. The sensitivity $z_d(t)$ of the state variable to the coefficient d for the fractional inertial system as function of time t for various orders of α

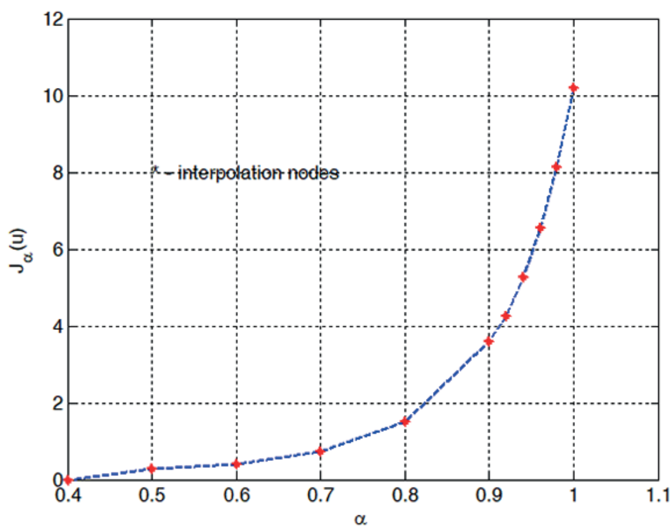

Fig. 4. Interpolation of the performance index component for various orders of α

nodes is indicated as red stars. As we can see (Fig. 4) the most considerable loss in the performance index ratio was obtained from the interval $0.9 \leq \alpha \leq 1.0$. Such a significant loss of the performance index value was the result of the fractional-order differentiator transition into the integer index form.

The optimal input design method for fractional system identification, presented in this paper, does not appear to be correct for $|\alpha| > 1$. If the order of the approximated numerator is higher than the order of the denominator, it becomes impossible to convert from a zero-pole transfer function to a state space model. However, this problem could be solved by incorporation the fractional-order system dynamics with one extra state.

7. Conclusions

In this work, a novel optimal input design methodology for fractional-order system identification has been presented. The methodology for the issue solution was verified using the numerical example. The formulation is based on appropriate Oustaloup's estimation and was then utilized for modeling the fractional-order operator in the form of integer transfer function. If the numerator order is equal to the denominator order, it becomes possible to convert a transfer function to a state-space form. Investigations described in this paper present a solution of the fractional-order optimal input signal task, where the sensitivity of the state variable to the fractional model parameter d (gain of the system) was maximized, at the same time providing a set of constraints on input signal design. Increasing the gain makes the system underdamped, and further increases lead to instability of the open-loop dynamical system. Therefore the accurate gain parameter estimation during fractional-order system identification is very important.

One of the essential steps in the presented methodology was to reformulate the problem into a similar fractional optimal input design problem described using Lagrange formulation with the series of constraints. The optimal excitation signal was then calculated utilizing one of the existing toolboxes for solving optimal control tasks. Numerical experiments show that solution for the integer order case (i.e., for $\alpha = 1$) is similar to the results of the fractional-order optimal input design problem. However, the choice of frequency fitting range for the Oustaloup filter method is especially important.

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